Non-commutative spacetime and the anomalous magnetic moment

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Abstract

In the framework of a non-commutative space-time model obtained from deformation theory, the QED correction to the anomalous magnetic moment, arising from non-commutativity, is computed.

1 Introduction

Several authors (see for example [1] [2]) have emphasized that the transition from nonrelativistic to relativistic theory as well as the from classical to quantum theory correspond to the stabilization of two unstable theories. In fact, because experimental parameters cannot be known with absolute precision, it makes good sense, when constructing physical theories, to choose those that do not change in a qualitative manner for a small change of parameters. This is a principle that extends well beyond the fundamental theories of Nature [3]. In the transition from nonrelativistic to relativistic theory one stabilizes the Galilean algebra by deforming it to the Lorentz algebra and in the transition from classic to quantum by deforming the Poisson to the Moyal algebra or, equivalently, to the Heisenberg algebra. It turns out that the full algebra of non-relativistic quantum theory, the Poincaré-Heisenberg algebra, is also not stable [4] [5] [6]. One way to stabilize it, introduces two new small parameters ℓ and ϕ, of dimensions L and L^{-1}, associated to the commutators

$$[\widehat{x}_\mu, \widehat{x}_\nu] = -i\epsilon\ell^2 \widehat{M}_{\mu\nu} \tag{1}$$

and

$$[\widehat{p}_\mu, \widehat{p}_\nu] = -i\epsilon'\phi^2 \widehat{M}_{\mu\nu} \tag{2}$$

ϵ and ϵ' being ± 1.

Other authors in the past have suggested non-commutativity of the space-time coordinates, mostly in connection with gravity at short distances (see [5]

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and references therein), leading to linear or nonlinear extensions of the space-time algebra. A novel feature in the approach of stabilization by deformation is the emergence of two independent fundamental length scales. Non-commutativity of momenta being associated to gravity, the fundamental length scale associated to the commutator (2) might be the Planck length. However the parameter ℓ associated to the commutator (1) is an independent parameter and if, for example, $\ell \simeq 10^{-18} - 10^{-19}$cm (or $\tau = \frac{1}{\ell} \simeq 3.3 \times 10^{-29} - 3.3 \times 10^{-30}$ s) the space-time non-commutative effects might already be observable in the laboratory [7] [8].

2 Non-commutativity and star product

For the QED calculations in this letter only the subalgebra generated by $\left\{\widehat{x}_\mu, \widehat{M}_{\mu\nu}\right\}$ will be needed, namely

$$[\widehat{x}_\mu, \widehat{x}_\nu] = -i\epsilon\ell^2 \widehat{M}_{\mu\nu} \tag{3}$$

as well as the usual commutator of the Lorentz group generators $M_{\mu\nu}$. A way to deal with the non-commuting variables $\left\{\widehat{x}_\mu, \widehat{M}_{\mu\nu}\right\}$ is to map them into a commutative space $\{x_\mu, M_{\mu\nu}\}$ with a modified $*-$product. If the right-hand side of the commutator were a $c-$number $\theta_{\mu\nu}$, instead of $\widehat{M}_{\mu\nu}$, the Moyal $*-$product might be used

$$f(x_\mu) * h(x_\nu) = f(x_\mu)\, e^{-i\frac{\epsilon\ell^2}{2}\overleftarrow{\partial^\mu}\theta_{\mu\nu}\overrightarrow{\partial^\nu}}\, h(x_\nu) \tag{4}$$

This reproduces the commutator $[\widehat{x}_\mu, \widehat{x}_\nu]$ but not necessarily the commutators of arbitrary elements of the enveloping algebra. Because $\widehat{M}_{\mu\nu}$ is a non-trivial operator a more careful definition should be done. A simple way to do it is to associate each arbitrary polynomial on the commuting variables x_μ and $M_{\mu\nu}$ to the corresponding symmetrized element of the enveloping algebra of $\left\{\widehat{x}_\mu, \widehat{M}_{\mu\nu}\right\}$. In this way the association of the non-commutative space $\{x_\mu, M_{\mu\nu}\}$ is not to the full enveloping algebra of $\left\{\widehat{x}_\mu, \widehat{M}_{\mu\nu}\right\}$ but to the symmetrized subalgebra. Finally on obtains a $*-$product, correct to order ℓ^2[1]

$$f(x_\mu) * h(x_\nu) =|_{O(\ell^2)} f(x_\mu)\left(1 - i\frac{\epsilon\ell^2}{2}\overleftarrow{\partial_{x_\mu}} M_{\mu\nu}\overrightarrow{\partial_{x_\nu}}\right) h(x_\nu) \tag{5}$$

The association of the commuting space, with a $*-$product, to a subset of the non-commuting space rather than to its full enveloping algebra arises from the fact that one wants to maintain the usual variables $x_\mu, M_{\mu\nu}$ in the commuting space.

A more general construction, correct to all ℓ^2 orders is obtained as follows. Represent the operators $\left\{\widehat{x}_\mu, \widehat{M}_{\mu\nu}\right\}$ by operators $\{\xi^a, \pi^a\}$ in a $5-$dimensional

[1] The notation $f(x)\overleftarrow{\partial_x}\overrightarrow{\partial_x}g(x)$ means $\lim_{x\to y}\partial_x\partial_y f(x)\,g(y)$

space with metric $g^{ab}=(1,-1,-1,-1,\epsilon)$, $a\in\{0,1,2,3,4\}$; $\mu\in\{0,1,2,3\}$

$$\begin{aligned} M_{\mu\nu} &= \xi_\mu\pi_\nu-\xi_\nu\pi_\mu \\ x_\mu &= \xi_\mu+\ell\left(\xi_\mu\pi_4-\xi_4\pi_\mu\right) \end{aligned} \tag{6}$$

with the $\{\xi^a,\pi^a\}$ obeying commutation relations

$$\left[\pi^a,\xi^b\right]=ig^{ab} \tag{7}$$

Because $\{\xi^a,\pi^a\}$ have Heisenberg algebra commutation relations, one may use for them a Moyal-type star product, which is known to be well defined and associative for polynomials.

$$f(\xi,\pi)*h(\xi,\pi)=f(\xi,\pi)\,e^{\frac{i}{2}\left(\overleftarrow{\partial_{\pi^a}}g^{ab}\overrightarrow{\partial_{\xi^b}}-\overleftarrow{\partial_{\xi^b}}g^{ba}\overrightarrow{\partial_{\pi^a}}\right)}h(\xi,\pi) \tag{8}$$

Computing the star product of two functions of $\{x_\mu\}$, using this representation and star product, one obtains the same result as with (5) in ℓ^2order.

3 Vertex corrections from non-commutativity

The Dirac equation for a spin $\frac{1}{2}$ fermion interacting with an external electromagnetic field is

$$i\frac{\partial}{\partial t}\psi=H\psi=\left\{\gamma^0\overrightarrow{\gamma}\cdot\left(\frac{1}{i}\bigtriangledown-e\overrightarrow{A}\right)+\gamma^0m+eA^0\right\}\psi \tag{9}$$

The magnetic moment contribution in the Hamiltonian is extracted by writing $\psi=\begin{pmatrix}\Phi\\ \chi\end{pmatrix}$ and taking the non-relativistic limit [9]

$$i\frac{\partial}{\partial t}\Phi\simeq\left\{\frac{\left(\frac{1}{i}\bigtriangledown-e\overrightarrow{A}\right)^2}{m}-\frac{e}{2m}\overrightarrow{\sigma}\cdot\overrightarrow{B}+eA^0\right\}\Phi \tag{10}$$

from which the magnetic moment $\overrightarrow{\mu}=2\left(\frac{e}{2mc}\right)\frac{\hbar\overrightarrow{\sigma}}{2}$ is obtained. Hence the QED gyromagnetic ratio $g=2$.

With the $*-$product

$$A_\mu(x)\,\psi(x)\rightarrow A_\mu(x)*\psi(x)=A_\mu(x)\left(1-i\frac{\epsilon\ell^2}{2}\overleftarrow{\partial^\alpha}M_{\alpha\beta}\overrightarrow{\partial^\beta}\right)\psi(x)+O\left(\ell^4\right) \tag{11}$$

the Hamiltonian gains an additional term

$$H_{NC}\psi(x)=-i\frac{\epsilon\ell^2}{2}\gamma^\mu\partial^\alpha A_\mu(x)\,M_{\alpha\beta}\partial^\beta\psi(x) \tag{12}$$

which, p^0 being the energy of a positive energy spinor, is

$$H_{NC}\psi(x) = \left(-\eta\gamma^{\mu}\partial^{\alpha}A_{\mu}(x)\,\sigma_{\alpha\beta}p^{\beta}\right)\psi(x) \tag{13}$$

with $\eta = \frac{e\epsilon\ell^2}{4}$. Expanding this term for a constant external field, one obtains

$$H_{NC}\psi(x) = \left\{-\eta E^k\left(\sigma_{k0}p^0 + \sigma_{kl}p^l\right) - \eta p^0\overrightarrow{B}\cdot\overrightarrow{\sigma} + \eta\left(\bigtriangledown\cdot\overrightarrow{A}\overrightarrow{\alpha}\cdot\overrightarrow{p} + \overrightarrow{p}\cdot\partial^k\overrightarrow{A}\alpha^k\right)\right\}\psi(x) \tag{14}$$

It is the second term in (14) that leads to an additional contribution to the gyromagnetic ratio

$$g_{NC} = 2\left(1 + \frac{mp^0\epsilon\ell^2}{2}\right)$$

If $\ell = 4.3\times10^{-18}$cm, $\epsilon = +1$, $m = 105$ MeV and $p^0 = 3.1$ GeV one would obtain a positive correction of order 2.5×10^{-9}. This would be in a range relevant to the current theory-experiment disagreement in the measurement of the muon magnetic moment.

Estimates of ℓ based on the spectral lags of gamma ray bursts [8] suffer from great uncertainties concerning the intrinsic lags at the source and the large error bars on the delayed correlation measurements. Nevertheless these estimates rather suggest a value of ℓ in the range of 10^{-19} cm. With this value the muon anomalous magnetic moment correction would be 1.3×10^{-12}, much too small to be experimentally relevant.

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