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ABSTRACT

A probability space is a pair $(\mathcal{A}, \phi)$ where $\mathcal{A}$ is an algebra and ϕ is a state on the algebra. In classical probability, $\mathcal{A}$ is the algebra of linear combinations of indicator functions on the sample space, and in quantum probability, $\mathcal{A}$ is the Heisenberg or Clifford algebra. However, other algebras are of interest in noncommutative probability. After a short review of the framework of classical and quantum probability, other noncommutative probability spaces are discussed, in particular those associated with noncommutative space-time.

I. INTRODUCTION

Probability plays a prominent role in physics and in all other natural sciences. This is significant not only for the working scientist but also for foundational questions. Most important issues are the interpretation of probability results as well as the existence of one or more probability frameworks. Quantum mechanics has called our attention to the need to generalize the classical probability framework, the main structuring role there being played by the Heisenberg algebra representations. This clearly suggests that other probability structures might be associated with other algebras relevant to phenomena in the natural sciences. In this paper, after a short introduction to the formulation of noncommutative probability and the particular case of the (Heisenberg algebra) quantum probability, a construction is made of probability structures associated with *iso*(1, 1) and *iso*(3, 1) algebras, which arise in some formulations of noncommutative spacetime.

A. Noncommutative probability

In classical probability theory, a *probability space* is a triple (Ω, F, P), where Ω (the sample space) is the set of all possible outcomes, F (the set of events) is a $\sigma-$ algebra of subsets of Ω, and P is a countably additive function from F to $[0,1]$ assigning probabilities to events.

Let $H = L^2(\Omega, P)$ be the Hilbert space of square-integrable functions on Ω. In H, consider the set of *indicator functions* $X_B = I_{\{x\in B\}}$, with $x \in \Omega$ and $B \in F$. Given a unit vector $\mathbf{1}$ in H, the *law* of X_B is the probability measure,

$$B \to (1, X_B 1) = \int X_B(x) dP(x) = P(B). \tag{1}$$

X_B's form a set of *countably additive projections* on functions that only depend on the events B. $\{X_B\}$ with *involution* (by complex conjugation) is a $*-$ *algebra* $\mathcal{A}$, the algebra of random variables. The algebra may be identified (by homomorphism) with the $*-$ algebra $\mathcal{L}(H)$ of bounded operators in H.

A *spectral measure* X over a measurable space E is a countably additive mapping from measurable sets to projections in some Hilbert space. In this way, classical probability is reframed as a spectral measure over Ω. When a non-negative real number is assigned to each positive element in the algebra $\mathcal{A}$ (such a functional is called a *state* in $\mathcal{A}$), one obtains von Neuman's view of probability theory.

In classical probability theory, $\mathcal{A}$ is a commutative algebra. Replacing it by a noncommutative $*-$ algebra and keeping the prescription of obtaining the law of $X \in \mathcal{A}$ by computation in a state as in (1), one obtains *noncommutative probability*.

B. Noncommutative processes

Going from probability spaces to stochastic processes, a few more steps are required. Let us revisit Kolmogorov's extension theorem. It states that given some interval T in the real line, a finite sequence of points $t_1, t_2, \ldots, t_n$ in this interval, a probability measure $\nu_{t_1,\ldots,t_n}$ in $\mathbb{R}^n$ and measurable sets $B_1, \ldots, B_n$ satisfying

1. $\nu_{\pi(t_1),\ldots,\pi(t_n)}\left(B_{\pi(t_1)} \times \cdots \times B_{\pi(t_n)}\right) = \nu_{t_1,\ldots,t_n}\left(B_{t_1} \times \cdots \times B_{t_n}\right)$ for any permutation π
2. $\nu_{t_1,\ldots,t_n}\left(B_{t_1} \times \cdots \times B_{t_n}\right) = \nu_{t_1,\ldots,t_n,t_{n+1},\ldots,t_{n+k}}\left(B_{t_1} \times \cdots \times B_{t_n} \times \underbrace{\mathbb{R} \cdots \times \mathbb{R}}_{k}\right)$, then a probability space $(\Omega, \mathcal{F}, P)$ exists, as well as a stochastic process $X_t : T \times \Omega \to \mathbb{R}$ such that

$$\nu_{t_1,\ldots,t_n}\left(B_{t_1} \times \cdots \times B_{t_n}\right) = P(X_{t_1} \in B_1, \ldots, X_{t_n} \in B_n).$$

The index set T may be thought of as a part of the spectrum of an operator. In the commutative case, it is a subset of a multiplicative operator $\widehat{T}$. When going to noncommutative probability if the $\widehat{T}$ operator commutes with the elements of the probability algebra $\mathcal{A}$, then the construction of the noncommutative process proceeds in a way similar to the Kolmogorov extension theorem. For each t_i, a noncommutative probability space $(\mathcal{A}_i, P_i)$, P_i being a state, is constructed and then compatibility conditions as in the Kolmogorov's theorem are imposed. Processes of this type, of which *quantum probability* is an example, will be called *noncommutative processes of type I*.

If, however, the index set operator $\widehat{T}$ has nontrivial commutation relations with the elements of the probability algebra $\mathcal{A}$, the construction of the process will be different. Processes where $\widehat{T}$ does not commute with $\mathcal{A}$ will be called *noncommutative processes of type II*.

In Secs. II–IV, after a short review of the quantum probability setting, noncommutative processes for more general algebras will be discussed as well as the construction of noncommutative processes of type II. Some results in this context were reported in Ref. 1. Here, a more systematic presentation is given as well as some new results.

II. QUANTUM PROBABILITY: NONCOMMUTATIVE PROCESSES AND THE HEISENBERG ALGEBRA

The main motivation to extend probability theory to the noncommutative setting came from the application of probabilistic concepts such as independence and noise to quantum mechanics. For this reason, the class of developments in noncommutative probability inspired by the structure of quantum mechanics carries the names of *quantum probability* or *quantum stochastic processes* or, more generally, *quantum stochastic analysis*.[2–5]

The dynamics of a particle in classical mechanics is described in phase space by functions of its coordinate q and momentum p. Functions $f(q,p)$ form a commutative algebra, and a classical probabilistic description of the particle dynamics is an assignment of a probability P_f to each function $f(q,p)$ by

$$P_f = \int f(p,q) d\mu(p,q), \tag{2}$$

the measure $\mu(p,q)$, with $\int d\mu(p,q) = 1$, being the state.

In quantum mechanics, such functions do not commute. In particular, for the phase-space coordinate functions, one has the canonical commutation relations (CCRs),

$$[q,p] = i\hbar, \tag{3}$$

with $\hbar = 1.054 \times 10^{-34}\, J \times s = 1.054 \times 10^{-27}\, g \times cm^2 \times s^{-1}$. For phenomena at the scale of cm (q) and $g \times cm \times s^{-1}$ (p), $\hbar$ is a small quantity. Nevertheless, being nonzero, it entirely changes the structure. For n particle species, it would be

$$\begin{aligned}&[q_i, p_j] = i\hbar\delta_{ij}\mathbb{I}\\ &[q_i, q_j] = [p_i, p_j] = [p_i, \mathbb{I}] = [x_i, \mathbb{I}] = 0.\end{aligned} \tag{4}$$

This is the Lie algebra of the Heisenberg group $\mathcal{H}(n)$, the maximal nilpotent subgroup in $U(n+1,1)$. *Quantum probability* is the particular case of noncommutative probability associated with this Heisenberg algebra. Let $n = 1$. In the unitary representations of $\mathcal{H}(1)$ in $L^2(\mathbb{R})$, the Lie algebra operators are (Ref. 6, Chap. 12)

$$\begin{aligned} qf(x) &= xf(x), \\ pf(x) &= -i\lambda \frac{d}{dx} f(x), \\ \mathbb{I}f(x) &= \lambda f(x), \end{aligned} \tag{5}$$

the representations being irreducible for $\lambda \neq 0$. Defining creation and annihilation operators

$$\begin{aligned} a &= \frac{1}{\sqrt{2}}(q + ip), \\ a^\dagger &= \frac{1}{\sqrt{2}}(q - ip), \end{aligned} \tag{6}$$

one obtains a representation of the operators in Fock space, a convenient dense set in Fock space being the set of exponential vectors,

$$\psi(f) = 1 \oplus f \oplus \cdots \oplus \frac{f^{(n)}}{\sqrt{n!}} \oplus \cdots, \tag{7}$$

where $f^{(n)}$ is the $n-$ fold tensor product of f.

To define processes, one considers an index set $[0, T)$ and a family of operators $\{A_t, A_t^\dagger, \Lambda_t\}$ indexed by the characteristic functions $[0, t)$. These operators have the following action on exponential vectors:

$$\begin{aligned} A_t \psi(f) &= \int_0^t ds f(s) \psi(f), \\ A_t^\dagger \psi(f) &= \frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} \psi\big(f + \varepsilon \chi_{[0,t]}\big), \\ \Lambda_t \psi(f) &= \frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} \psi\big(e^{\varepsilon \chi_{[0,t]}} f\big). \end{aligned} \tag{8}$$

Quantum stochastic differentials are defined by

$$\begin{aligned} dA_t &= A_{t+dt} - A_t, \\ dA_t^\dagger &= A_{t+dt}^\dagger - A_t^\dagger, \\ d\Lambda_t &= \Lambda_{t+dt} - \Lambda_t, \end{aligned}$$

and quantum stochastic calculus is, in practice, an application of the rules

$$dA_t dA_t^\dagger = dt;\ dA_t d\Lambda_t = dA_t;\ d\Lambda_t dA_t^\dagger = dA_t^\dagger;\ d\Lambda_t d\Lambda_t = d\Lambda_t,$$

all other products vanishing.

Many deep results have been obtained in the quantum stochastic processes field[24] of practical importance for physical systems perturbed by quantum noise. Also, nonlinear extensions have been obtained.[7–9]

In quantum mechanics, bosons satisfy CCR, whereas fermions have canonical anticommutation relations (CAR). Based on the fermion Clifford algebra, a noncommutative stochastic calculus has also been developed.[10–12] Nevertheless, boson and fermion stochastic calculus may be unified in a single theory.[13]

Other extensions, closely related to the Heisenberg algebra, have been made by Boukas[14] who deals with a discrete analog of the Heisenberg algebra and Privault[15] for Lévy processes on finite difference algebras. Generalized Gaussian processes have also been constructed associated with the q-deformed Fock space[16] or with extensions of this concept.[17–19]

III. NONCOMMUTATIVE PROCESSES BEYOND THE HEISENBERG AND CLIFFORD ALGEBRAS

A. Noncommutative processes for the *iso*(1,1) algebra

This probability space was first discussed in Ref. 1, and the main results are recalled here. The algebra is

$$\begin{aligned} [P, Q] &= -i\mathfrak{I}, \\ [Q, \mathfrak{I}] &= -iP, \\ [P, \mathfrak{I}] &= 0. \end{aligned} \tag{9}$$

This is the algebra of one-dimensional noncommutative phase-space with the correspondence to the physical variables p, x established by $P = \ell p$ and $Q = \frac{x}{\ell}$, with ℓ being a fundamental length.

As in quantum probability, the first step is to obtain the representations of the algebra. The algebra (9) is the Lie algebra of the group of motions of pseudo-Euclidean space $iso(1, 1)$. The group contains hyperbolic rotations in the plane and two translations, being isomorphic to the group of 3 × 3 matrices,

$$\begin{pmatrix} \cosh\mu & \sinh\mu & a_1 \\ \sinh\mu & \cosh\mu & a_2 \\ 0 & 0 & 1 \end{pmatrix}, \tag{10}$$

its elements being labeled by (μ, a_1, a_2). The representations of the group in the space of functions in the hyperbola $f(\cosh v, \sinh v)$ are

$$T_R(\mu, a_1, a_2) f(v) = e^{R(-a_1 \cosh\mu + a_2 \sinh\mu)} f(v - \mu). \tag{11}$$

The T_R representation is unitary if $R = ir$ is purely imaginary and irreducible if $R \neq 0$. For the Lie algebra, one obtains

$$\begin{aligned} Q &= i\frac{d}{d\mu}, \\ P &= \sinh\mu, \\ \mathfrak{I} &= \cosh\mu, \end{aligned} \tag{12}$$

acting on the space V_1 of functions on the hyperbola. The $r = 1$ representation has been chosen. A quite similar construction applies to all $r \neq 0$ cases. It is convenient to define the operators

$$\begin{aligned} A_+ &= \frac{1}{\sqrt{2}}(Q - iP), \\ A_- &= \frac{1}{\sqrt{2}}(Q + iP). \end{aligned} \tag{13}$$

Using the representation (12), the state in V_1 that is annihilated by A_- is

$$\psi_0 = \frac{1}{\sqrt{N}} e^{-\cosh\mu}, \tag{14}$$

with N being a normalization factor $N = 2K_0(2)$ and K_0 being a modified Bessel function of the 2nd kind.

With the algebra of the operators $\{A_+, A_-, \mathfrak{I}\}$ and the state defined by ψ_0- expectation values, the probability space is defined. The rest of the construction is standard.

With the scalar product

$$(\psi, \phi)_\mu = \int_R d\mu \psi^*(\mu)\phi(\mu), \tag{15}$$

the space V_1 of square integrable functions in the hyperbola becomes, by completion, a Hilbert space, and let $h = L^2(R_+)$ be the Hilbert space of square integrable functions on the half-line $R_+ = [0, \infty)$. One now constructs an infinite set of operators labeled by functions on h, with the algebra

$$\begin{aligned} &[A_-(f), A_+(g)] = \mathfrak{I}(fg) \\ &[A_-(f), \mathfrak{I}(g)] = [A_+(f), \mathfrak{I}(g)] = \frac{1}{2}(A_-(fg) - A_+(fg)). \end{aligned}$$

These operators act on $H_1 = h \otimes V_1$ by

$$\begin{aligned} A_-(f)\psi(g) &= \frac{i}{\sqrt{2}}\left(\left(\frac{d}{d\mu} + \sinh\mu\right)\psi\right)(fg), \\ A_+(f)\psi(g) &= \frac{i}{\sqrt{2}}\left(\left(\frac{d}{d\mu} - \sinh\mu\right)\psi\right)(fg), \\ \mathfrak{I}(f)\psi(g) &= (\cosh\mu \times \psi)(fg), \end{aligned} \tag{16}$$

$f, g \in h$, $\psi \in V_1$, $\psi(g) \in H_1$.

Adapted processes are associated with the splitting

$$h = L^2(0, t) \oplus L^2(t, \infty) = h^t \oplus h^{(t}, \tag{17}$$

with the corresponding

$$H_1 = H_1^t \oplus H_1^{(t} = h^t \otimes V_1 \oplus h^{(t} \otimes V_1. \tag{18}$$

An adapted process is a family $O = (O(t), t \geq 0)$ of operators such that $O(t) = O^t \otimes 1$. The basic adapted processes are

$$\begin{aligned} \Im(t) &= \Im(\chi_{[0,t]}), \\ A_+(t) &= A_+(\chi_{[0,t]}), \\ A_-(t) &= A_-(\chi_{[0,t]}). \end{aligned} \tag{19}$$

For each adapted process $O(t)$, define $dO(t) = O(t + dt) - O(t)$. Given adapted processes E_0, E_+, E_-, a stochastic integral

$$N(t) = \int_0^t E_0 d\Im + E_+ dA_+ + E_- dA_- \tag{20}$$

is defined as the limit of the family of operators $(N(s); s \geq 0)$ such that $N(0) = 0$ and for $t_n < t \leq t_n + dt$,

$$N(t) = N(t_n) + E_0^{(n)}(\Im(t) - \Im(t_n)) + E_+^{(n)}(A_+(t) - A_+(t_n)) + E_-^{(n)}(A_-(t) - A_-(t_n)), \tag{21}$$

with E_0, E_+, E_- assumed to be simple processes,

$$\begin{aligned} E_0 &= \sum_{n=0}^{\infty} E_0^{(n)} \chi_{[t_n, t_{n+1})}, \\ E_+ &= \sum_{n=0}^{\infty} E_+^{(n)} \chi_{[t_n, t_{n+1})}, \\ E_- &= \sum_{n=0}^{\infty} E_-^{(n)} \chi_{[t_n, t_{n+1})}, \end{aligned} \tag{22}$$

Then, from (21) and (22), it follows that in $H_1 = h \otimes V_1$, one has

$$\begin{aligned} \langle \phi(f), N(t)\psi(g) \rangle = \int_0^t \langle \phi(f), & E_0(s)(\cosh\mu \times \psi)(g\chi_{ds}) \\ & + E_-(s)\frac{i}{\sqrt{2}}\left(\left(\frac{d}{d\mu} + \sinh\mu\right)\psi\right)(g\chi_{ds}) \\ & + E_+(s)\frac{i}{\sqrt{2}}\left(\left(\frac{d}{d\mu} - \sinh\mu\right)\psi\right)(g\chi_{ds}) \Big\rangle. \end{aligned}$$

Multiplication rules for the stochastic differentials,

$$\begin{aligned} dA_-(t) &= A_-(t + dt) - A_-(t), \\ dA_+(t) &= A_+(t + dt) - A_+(t), \\ d\Im(t) &= \Im(t + dt) - \Im(t), \end{aligned} \tag{23}$$

are obtained taking into account the commutation relations and expectation values in the ψ_0 state. Nonvanishing products are

$$\begin{aligned} dA_-(t)dA_+(t) &= \Im(dt), \\ d\Im(t)dA_+(t) &= -dA_-(t)d\Im(t) = -\frac{1}{2}(A_-(dt) - A_+(dt)). \end{aligned} \tag{24}$$

Summarizing, this proves the following:

Proposition 1: Associated with each irreducible unitary representation of the iso(1, 1) *algebra, there is a stochastic process characterized by the operator set* $\{A_+(f), A_-(f), \Im(g)\}$ *acting on* $H_1 = h \otimes V_1$ *with adapted processes and stochastic integrals satisfying properties (19)–(24).*

Of interest is also the characterization of the coordinate process $Q(t) = \frac{1}{\sqrt{2}}(A_-(t) + A_+(t))$. The characteristic functional is

$$C_{Q(t)}(f) = \frac{K_0\left(2\cos\left(\frac{1}{2}\int_0^t f(s)ds\right)\right)}{K_0(2)}. \tag{25}$$

B. Noncommutative processes for the *iso*(3,1) algebra

The algebra here is

$$\begin{aligned}
\left[Q^i, Q^j\right] &= -iM^{ij},\\
\left[P^i, Q^j\right] &= -i\Im\delta^{ij},\\
\left[Q^i, \Im\right] &= iP^i,\\
\left[M^{ij}, Q^k\right] &= i\left(Q^i\delta^{jk} - Q^j\delta^{ik}\right),\\
\left[M^{ij}, P^k\right] &= i\left(P^i\delta^{jk} - P^j\delta^{ik}\right),\\
\left[P^i, \Im\right] &= \left[M^{ij}, \Im\right] = \left[P^i, P^j\right] = 0.
\end{aligned} \tag{26}$$

This is, in three dimensions, the phase space algebra of noncommutative space-time. It is the algebra of the Poincaré group. For those familiar with the use of this group in relativistic quantum mechanics, notice that here the three Q^i's play the role of the boost generators and $\Im$ the role of the generator of time translations.

The group $iso(3,1)$ is isomorphic to the group of matrices $\begin{pmatrix} M & a \\ 0 & 1 \end{pmatrix}$, where $M \in SO(3,1)$ and $a \in R^4$.

A representation of the $iso(3,1)$ algebra that generalizes the representation (12) is obtained in the space V_3 of functions in the upper sheet of the hyperboloid $\mathcal{H}_+^3$ with coordinates

$$\begin{aligned}
\xi_1 &= \sinh\mu\sin\theta_1\cos\theta_2,\\
\xi_2 &= \sinh\mu\sin\theta_1\sin\theta_2,\\
\xi_3 &= \sinh\mu\cos\theta_1,\\
\xi_4 &= \cosh\mu.
\end{aligned} \tag{27}$$

The representation is

$$\begin{aligned}
\Im &= \cosh\mu,\\
P^1 &= \sinh\mu\sin\theta_1\cos\theta_2,\\
P^2 &= \sinh\mu\sin\theta_1\sin\theta_2,\\
P^3 &= \sinh\mu\cos\theta_1,\\
M^{12} &= i\frac{\partial}{\partial\theta_2},\\
M^{31} &= i\left(\cos\theta_2\frac{\partial}{\partial\theta_1} - \cot\theta_1\sin\theta_2\frac{\partial}{\partial\theta_2}\right),\\
M^{23} &= -i\left(\sin\theta_2\frac{\partial}{\partial\theta_1} + \cot\theta_1\cos\theta_2\frac{\partial}{\partial\theta_2}\right),\\
Q^1 &= i\left(\sin\theta_1\cos\theta_2\frac{\partial}{\partial\mu} + \coth\mu\left(\cos\theta_1\cos\theta_2\frac{\partial}{\partial\theta_1} - \frac{\sin\theta_2}{\sin\theta_1}\frac{\partial}{\partial\theta_2}\right)\right),\\
Q^2 &= i\left(\sin\theta_1\sin\theta_2\frac{\partial}{\partial\mu} + \coth\mu\left(\cos\theta_1\sin\theta_2\frac{\partial}{\partial\theta_1} + \frac{\cos\theta_2}{\sin\theta_1}\frac{\partial}{\partial\theta_2}\right)\right),\\
Q^3 &= i\left(\cos\theta_1\frac{\partial}{\partial\mu} - \coth\mu\sin\theta_1\frac{\partial}{\partial\theta_1}\right).
\end{aligned} \tag{28}$$

Define the following operators

$$A_-^k = \frac{1}{\sqrt{2}}\left(Q^k + iP^k\right),$$
$$A_+^k = \frac{1}{\sqrt{2}}\left(Q^k - iP^k\right), \tag{29}$$

to which in $H_3 = h \otimes V_3$ correspond the operators $A_-^k(t) \doteq A_-^k\left(\chi_{[0,t]}\right), A_+^k(t) \doteq A_-^k\left(\chi_{[0,t]}\right)$ as well as $\Im(t) \doteq \Im\left(\chi_{[0,t]}\right), M^{kj}(t) \doteq M^{kj}\left(\chi_{[0,t]}\right)$ with algebra

$$\left[A_-^k(t), A_+^j(s)\right] = \delta^{kj}\Im(t\wedge s) - \frac{i}{2}M^{kj}(t\wedge s),$$
$$\left[A_-^k(t), \Im(s)\right] = \left[A_+^k(t), \Im(s)\right] = \frac{i}{\sqrt{2}}P^k(t\wedge s),$$
$$\left[A_\pm^k(t), M^{jn}(s)\right] = i\left(\delta^{kj}A_\pm^n(t\wedge s) - \delta^{kn}A_\pm^j(t\wedge s)\right),$$
$$\left[M^{kj}(t), M^{nl}(s)\right] = -i\left(\delta^{jn}M^{kl} + \delta^{kl}M^{jn} - \delta^{kn}M^{jl} - \delta^{jl}M^{kn}\right)(t\wedge s), \tag{30}$$

which follows from the commutation relations (26). The normalized spherical symmetric state $\psi_0 = \frac{1}{\sqrt{8\pi K_0(2)}}\exp(-\cosh\mu)$ is annihilated by all $A_-^k(t)$,

$$A_-^k(t)\psi_0 = 0;\ k = 1, 2, 3. \tag{31}$$

In conclusion,

Proposition 2: The operators $\left\{A_-^k(t), A_+^j(t), \Im(t), M^{kj}(t)\right\}$, *the representation (28), and expectations on the* ψ_0 *state define a probability space. Multiplication rules for the stochastic differentials* $dA_-^k(t) = A_-^k(t+dt) - A_-^k(t)$, etc., *are*

$$dA_-^k(t)dA_+^j(t) = \delta^{kj}\ \Im(dt),$$
$$dA_+^k(t)dA_-^j(t) = 0,$$
$$dA_-^k(t)d\Im(t) = -d\Im(t)dA_-^k(t) = \frac{1}{2}\left(A_-^k(dt) - A_+^k(dt)\right). \tag{32}$$

The multiplication rules for the stochastic differentials are obtained, as before, by taking into account the commutation relations and expectation values in the ψ_0 state.

The characteristic functionals of the coordinate processes $Q^k(t) = \frac{1}{\sqrt{2}}\left(A_-^k(t) + A_+^k(t)\right)$ are

$$C_{Q^k(t)}(f) = \frac{1}{4\pi K_0(2)}\int d\Omega K_0\left(2\cos\left(\frac{g_k(\theta_1,\theta_2)}{2}\int_0^t f(s)ds\right)\right), \tag{33}$$

with $g_1(\theta_1,\theta_2) = \sin\theta_1\cos\theta_2;\ g_2(\theta_1,\theta_2) = \sin\theta_1\sin\theta_2;\ g_3(\theta_1,\theta_2) = \cos\theta_1$ and $d\Omega = \sin\theta_1 d\theta_1 d\theta_2$. Notice, however, that these processes are not independent being related by the rotation process $M^{kj}(t)$,

$$\left[Q^k(t), Q^j(s)\right] = -iM^{kj}(t\wedge s). \tag{34}$$

IV. NONCOMMUTATIVE PROCESSES OF TYPE II

In Secs. III A and III B, the index set of the stochastic processes (or the $h-$ space) may be looked at as the continuous spectrum of an operator that commutes with all other operators in the algebra. Here, the case where such operator does not commute with the algebra operators will be analyzed.

In this case, the H space, where the stochastic process operators act, can no longer be a direct product as before; rather, it is a representation space for the whole set of operators, including the one that generates the index set. An example is the following algebra:

$$
\begin{aligned}
[E,T] &= i\Im,\\
[T,\Im] &= iE,\\
[Q,P] &= i\Im,\\
[Q,\Im] &= iP,\\
[P,\Im] &= [P,T] = [M^{10},\Im] = 0,\\
[Q,T] &= -iM^{10},\\
[M^{10},Q] &= iT,\\
[M^{10},P] &= iE,\\
[M^{10},T] &= iQ,\\
[M^{10},E] &= iP.
\end{aligned}
\tag{35}
$$

This $iso(2,1)$ algebra is the algebra of two-dimensional noncommutative space-time. Notice that as a real algebra, it is different from the algebra studied in Ref. 1 because a different metric is used in the deformed space. For consistency with the physical interpretation of T as the time operator, T will be the operator that generates the index set of the stochastic process. A representation of this algebra is obtained on functions $f(\sigma,\theta)$ in the C^2 cone,

$$
\begin{aligned}
T &= -i\frac{\partial}{\partial\theta},\\
E &= \sigma\sin\theta,\\
\Im &= \sigma\cos\theta,\\
P &= \sigma,\\
Q &= i\left(\sigma\cos\theta\frac{\partial}{\partial\sigma} - \sin\theta\frac{\partial}{\partial\theta}\right),\\
M^{10} &= i\left(\sigma\sin\theta\frac{\partial}{\partial\sigma} + \cos\theta\frac{\partial}{\partial\theta}\right).
\end{aligned}
\tag{36}
$$

T has discrete spectrum, with eigenfunctions

$$
T\left(e^{in\theta}f(\sigma)\right) = ne^{in\theta}f(\sigma),
\tag{37}
$$

with $f(\sigma)$ being an arbitrary function of σ.

In the cases studied before, the space H, where the stochastic process operators act, is a direct product $h\otimes V$, that is, the space V is the same for all times. Here, however, once a normalized cyclic vector [$\exp(-\sigma^2/2)/(\pi^{1/4})$ for example] of σ is chosen to generate a v_σ space, the space V_n at time n is $V_n = e^{in\theta}v_\sigma$ and

$$
H = \oplus V_n
\tag{38}
$$

An adapted, discrete time, stochastic process is

$$
O(t) = \oplus_{n=0}^{t} O_n,
\tag{39}
$$

with O_n being one of the operators in (36) acting on V_n. The fact that in this case, the space H is not a direct product is the main difference from the previous cases. Otherwise, the construction is similar with (discrete) stochastic differentials defined by

$$
dO(n) = O_{n+1} - O_n,
$$

with O_{n+1} operating in V_{n+1} and O_n on V_n.

For example, for functionals $\mathcal{F}(Q(t))$ of the coordinate process $Q(t)$, expectation values are obtained by

$$
\langle\mathcal{F}(Q(t))\rangle = \sum_{0}^{t}\frac{1}{2\pi}\int d\sigma d\theta\phi^*(\sigma)e^{-in\theta}\mathcal{F}(Q(\sigma,\theta))e^{in\theta}\phi(\sigma).
\tag{40}
$$

V. REMARKS AND CONCLUSIONS

1. von Neuman's view of probability theory as a pair $(\mathcal{A},\phi)$, where $\mathcal{A}$ is an algebra and ϕ is a state, is a powerful insight with far reaching implications. The main purpose of this paper was to emphasize that in addition to the many results already obtained in quantum probability, where the Heisenberg or the Clifford algebras play the central role, other algebras are of potential interest, in particular in the noncommutative space-time context.[1,20,21]

In addition, it has been pointed out that in some cases, the index set of the stochastic process might be generated by a nontrivial operator in the algebra, the consequence being that the space where the stochastic process operators act is no longer a direct product.

Most results on the processes associated with the *iso*(1,1)-algebra were previously reported in Ref. 1 but not those related to the *iso*(3,1)-algebra. Also, concerning the processes of type II, a different metric structure is used, which might be physically more relevant.

2 In the past, a central role is played in quantum probability by the algebra of the second quantization operators, the creation and annihilation operators. In 1977, Palev[22] had shown that the basic requirements of the second quantization formulation may also be obtained by operators that lead to generalized quantum statistics.[23] The algebras of these generalized quantum statistics operators would also provide new examples of noncommutative processes.

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[24] See, for example, the proceedings of the series of conferences "Quantum Probability and Infinite Dimensional Analysis" and references therein.