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# Current algebra, statistical mechanics and quantum models

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**Abstract.** Results obtained in the past for free boson systems at zero and nonzero temperatures are revisited to clarify the physical meaning of current algebra reducible functionals which are associated to systems with density fluctuations, leading to observable effects on phase transitions.

To use current algebra as a tool for the formulation of quantum statistical mechanics amounts to the construction of unitary representations of diffeomorphism groups. Two mathematical equivalent procedures exist for this purpose. One searches for quasi-invariant measures on configuration spaces, the other for a cyclic vector in Hilbert space. Here, one argues that the second approach is closer to the physical intuition when modelling complex systems. An example of application of the current algebra methodology to the pairing phenomenon in two-dimensional fermion systems is discussed.

**Keywords:** algebraic structures of integrable models, Hubbard and related model

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## Contents



## 1. Introduction

All representations of canonical fields with a finite number of degrees of freedom are equivalent to the Fock representation. However, for an infinite number of degrees of freedom there are, in addition to the Fock representation, infinitely many inequivalent representations of the canonical commutation relations. In relativistic quantum field theory, Haag's theorem states that, with a space-invariant vacuum, any representation equivalent to Fock can only describe a free system. Therefore, to obtain a non-trivial theory, one either works with a non-Fock representation or with a Fock representation in a finite volume. In the latter case one considers $N$ particles in a finite volume $V$. Calculations are then carried out in the Fock representation, but in the end one may take $N, V \to \infty$ with the $N/V = \rho$ ratio fixed. The $N/V$ limit thus provides a way to deal with non-trivial infinite systems using the Fock representation. However, by the very nature of the fixed $\rho$ density limit, it is unable to deal with systems with density fluctuations. This shortcoming might be solved by the use of the reducible functionals to be described later on.

In the field theory description of matter, the field operators $\psi(x)$ and $\psi^{\dagger}(x)$ do not represent actual physical observables. This, together with the strong uniqueness results on the representation of the (finite-dimensional) canonical commutation relations, were the original motivations for the proposal by Dashen, Sharp, Callan and Sugawara

[1–4] to use local density and current operators as descriptors of quantum observables. Despite some early successes, mostly in the derivation of sum rules, relativistic current algebra in space-time dimensions higher than $1+1$ faced serious difficulties related to the non-finiteness of Schwinger terms. By contrast, no such problem occurs for non-relativistic current algebras which, already at a very early stage, have been proposed as a tool for statistical mechanics [5, 6]. Nonrelativistic current algebra was then extensively studied by Goldin and collaborators [7–9]. A relation with the classical Bogolubov generating functional has also been established, in particular as a tool for constructing the irreducible current algebra representations [10, 11]. From a mathematical point of view, the early considerations related to the $N/V$ limit have found a rigorous interpretation in the framework of the infinite-dimensional Poisson analysis in configuration spaces ([12] and references therein).

In this paper results obtained in the past for free boson systems at zero and nonzero temperature are revisited with a view to clarify the physical meaning and potential usefulness of current algebra reducible functionals. Reducible functionals are associated to systems with density fluctuations, which may lead to observable effects on phase transitions.

Using current algebra as a tool for the formulation of quantum statistical mechanics is closely related to the problem of construction of unitary representations of diffeomorphism groups. Two mathematical equivalent procedures exist for this purpose. One searches for quasi-invariant measures on configuration spaces, the other for a cyclic vector in Hilbert space. Here, one argues that the second approach is closer to the physical intuition when modelling complex systems. An example of application of the current algebra methodology to the pairing phenomenon in two-dimensional fermion systems is included.

## 2. Boson gas, the infinite-dimensional Poisson measure and reducible functionals

### 2.1. Infinite-dimensional Poisson measures and free Boson gases

The framework of non-relativistic current algebra of many-body systems is a particularly convenient way to establish the connection of the Boson gas functional with infinite-dimensional measures, as well as to explore generalizations. The basic variables of the many-body system are the smeared currents [6, 7] (see also [8, 9] and references therein)

$$\begin{aligned} \varrho(f) &= \int \mathrm{d}^3x f(x)\varrho(x) \\ \mathbf{J}(\mathbf{g}) &= \int \mathrm{d}^3x \mathbf{J}(x) \bullet \mathbf{g}(x) \end{aligned} \tag{1}$$

$f(x)$ and $\mathbf{g}(x)$ being respectively smooth compactly supported functions and smooth vector fields. The smeared currents satisfy the infinite-dimensional Lie algebra,

$$\begin{aligned} [\varrho(f), \varrho(h)] &= 0 \\ [\varrho(f), \mathbf{J}(\mathbf{g})] &= \mathrm{i}\varrho(\mathbf{g} \bullet \nabla f) \\ [\mathbf{J}(\mathbf{g}), \mathbf{J}(\mathbf{k})] &= \mathrm{i}\mathbf{J}(\mathbf{k} \bullet \nabla \mathbf{g} - \mathbf{g} \bullet \nabla \mathbf{k}) \end{aligned} \tag{2}$$

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each particular physical system corresponding to a different Hilbert space representation of this algebra or of the semidirect product group generated by the exponentiated currents

$$U\,(\,f) = \mathrm{e}^{\mathrm{i}\varrho(\,f)}$$
$$V\,(\phi_t^{\mathbf{g}}) = \mathrm{e}^{\mathrm{i}t\mathbf{J}(\mathbf{g})} \tag{3}$$

$\phi_t^{\mathbf{g}}$ being the flow of the vector field $\mathbf{g}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_t^{\mathbf{g}}(x) = \mathbf{g}\,(\phi_t^{\mathbf{g}}(x))\,. \tag{4}$$

For a system of $N$ free bosons in a box of volume $V$, the normalized ground state is

$$\Omega_{N,V}\,(x_1, \cdots, x_N) = \left(\frac{1}{\sqrt{V}}\right)^N \tag{5}$$

and the ground state functional is

$$L_{N,V}\,(\,f) = (\Omega_{N,V}, U_{N,V}\,(\,f)\,\Omega_{N,V})$$
$$= \left(\frac{1}{V}\int_V \mathrm{d}^3x\mathrm{e}^{\mathrm{i}f(x)}\right)^N. \tag{6}$$

Coupled with an equation of continuity relating $\varrho$ and $\mathbf{J}$, this functional determines not only the representation of $U\,(\,f)$ but also that of $V\,(\phi_t^{\mathbf{g}})$, up to a complex phase multiplier that satisfies a cocycle condition[1].

In the $N \to \infty$ limit with constant average density $\rho = \frac{N}{V}$ (also called the $N/V$ limit) one obtains

$$L\,(\,f) = \lim_{N\to\infty}\left(1 + \frac{\rho}{N}\int\left(\mathrm{e}^{\mathrm{i}f(x)} - 1\right)\mathrm{d}^3x\right)^N$$
$$= \exp\left(\rho\int\left(\mathrm{e}^{\mathrm{i}f(x)} - 1\right)\mathrm{d}^3x\right) \tag{7}$$

which one recognizes as the characteristic functional of the infinite-dimensional Poisson measure (see the appendix).

Likewise the functional

$$L_{N/V}\,(\,f, \mathbf{g}) = \left(\Omega_{N/V}, \mathrm{e}^{\mathrm{i}\varrho(\,f)}\mathrm{e}^{\mathrm{i}\mathbf{J}(\mathbf{g})}\Omega_{N/V}\right)$$

is [13] in the $N/V$ limit

$$L\,(\,f, \mathbf{g}) = \exp\left(\rho\int\left\{\mathrm{e}^{\mathrm{i}f(x)}\left(\det\partial_m\phi_n^{\mathbf{g}}(x)\right)^{1/2} - 1\right\}\mathrm{d}^3x\right)$$

where $\det\partial_m\phi_n^{\mathbf{g}}(x)$ stands for the Jacobian of the transformation $x \to \phi^{\mathbf{g}}(x)$.

Identifying $\rho\mathrm{d}^3x$ in (7) with the measure $\mathrm{d}\mu$ in the configuration spaces discussed in the appendix, the $L\,(\,f)$ functional may also be written as a vacuum expectation functional. Expanding the exponential in (7)

[1] See equation (21).



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$$L\left(f\right)=\sum_{n=0}^{\infty}\frac{\mathrm{e}^{-\int \mathrm{d}\mu}}{n!}\left(\int \mathrm{e}^{\mathrm{i}f(x)}\mathrm{d}\mu\right)^{n} \tag{8}$$

one may write

$$L\left(f\right)=\left(\Omega,U\left(f\right)\Omega\right) \tag{9}$$

for

$$\Omega=\bigoplus_{n}\mathrm{e}^{-\frac{1}{2}\int \mathrm{d}\mu}\mathbf{1}_{n} \tag{10}$$

$\mathbf{1}_n$ denoting the identity function in the $n$-particle subspace of a direct sum Hilbert space, the $\frac{1}{n!}$ factor in (8) being recovered by the symmetrization operation.

However (7) is not the most general consistent representation of the nonrelativistic current algebra of a free boson gas, a more general one being [7], the reducible functional

$$L\left(f\right)=\int_{0}^{\infty}\exp\left(\rho\int\left(\mathrm{e}^{\mathrm{i}f(x)}-1\right)\mathrm{d}^{3}x\right)\mathrm{d}\xi\left(\rho\right) \tag{11}$$

with $\xi$ a positive measure on $[0,\infty)$ normalized so that $\int_0^\infty \mathrm{d}\xi\left(\rho\right)=1$. This inifinite-dimensional compound Poisson measure may represent a boson gas with density fluctuations. As pointed out in [15], among the many possible reducible functionals consistent with (11) there is a fractional generalization of (8), namely

$$L_{\alpha}\left(f\right)=\sum_{n=0}^{\infty}\frac{E_{\alpha}^{(n)}\left(-\int \mathrm{d}\mu\right)}{n!}\left(\int \mathrm{e}^{\mathrm{i}f(x)}\mathrm{d}\mu\right)^{n} \tag{12}$$

$(0<\alpha\leqslant 1)$, which corresponds to a vacuum state

$$\Omega_{\alpha}=\bigoplus_{n}\sqrt{E_{\alpha}^{(n)}\left(-\int \mathrm{d}\mu\right)}\mathbf{1}_{n} \tag{13}$$

$E_{\alpha}^{(n)}$ denoting the $n$th derivative of the Mittag-Leffler function [16]. $\Omega_\alpha$ differs from $\Omega$ in the weight given to each one of the $n$-particle spaces. The measure associated to the functional (12) was called *the infinite-dimensional fractional Poisson measure* and the corresponding physical system *the fractional boson gas*.

The reducible functional associated to the infinite-dimensional fractional Poisson measure was introduced because the Mittag-Leffler is a very natural analytic generalization of the exponential function. The main interest in studying such an example is the possibility to analyse rigorously its support properties as well as the Hilbert space structure, in particular the nature of the $n$-particle subspaces. This is the mathematical motivation for the study of the fractional boson gases. Of course it also suggests that similar support and Hilbert space modifications would occur for other reducible functionals.

The study of the fractional boson gas has been carried out elsewhere [15] and, for the convenience of the reader, the main results are summarized in the appendix. The meaning and relevance of the reducible functionals of type (11) becomes clear when

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finite temperature functionals are computed. These were computed by Girard [17] in the current algebra framework.

### 2.2. The zero temperature limit of finite-temperature functionals

For $T \neq 0$, instead of the matrix element (6), one computes

$$L_{N,V}^{(T)}(f) = \frac{\mathrm{Tr}\left(\mathrm{e}^{-\beta H}\mathrm{e}^{\mathrm{i}\varrho(f)}\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta H}\right)} \tag{14}$$

for the canonical ensemble and

$$L_{\mu,V}^{(T)}(f) = \frac{\mathrm{Tr}\left(\mathrm{e}^{\beta\mu N}\mathrm{e}^{-\beta H}\mathrm{e}^{\mathrm{i}\varrho(f)}\right)}{\mathrm{Tr}\left(\mathrm{e}^{\beta\mu N}\mathrm{e}^{-\beta H}\right)} \tag{15}$$

for the grand canonical ensemble. $H$ is the free particle Hamiltonian, $\beta = \frac{1}{kT}$, $\mu$ is the chemical potential and $N$ the particle number operator. Girard [17] obtains for the grand canonical functional

$$L_{\mu,V}^{(T)}(f) = \det[I - (\mathrm{e}^{\mathrm{i}f(x)} - l)/(\mathrm{e}^{\beta(H-\mu)} - I)]^{-1}. \tag{16}$$

However, taking the zero temperature limit of (16) one does not recover the infinite dimensional Poisson measure of (7). Instead of (7) the following functional is obtained [17]

$$L_0(\overline{\rho}) = \left(1 - \overline{\rho}\int\left(\mathrm{e}^{\mathrm{i}f(x)} - 1\right)\mathrm{d}x\right)^{-1} \tag{17}$$

which is seen to be a reducible functional, as in (11), with density

$$\mathrm{d}\xi(\rho) = (\frac{1}{\overline{\rho}})\mathrm{e}^{-\rho/\overline{\rho}}\mathrm{d}\rho. \tag{18}$$

Physically this makes sense, because since the grand canonical ensemble only fixes the particle number in average, it is reasonable that the corresponding ground state be a state with density fluctuations. The zero temperature limit of the grand canonical Boson gas is therefore a ground state with density fluctuations defined by (18). A natural question to ask is what is the physical meaning of all other reducible functionals. One possible answer is the following result:

*# All reducible functionals of type (11) (infinite-dimensional compound Poisson measures) may be obtained as zero-temperature limits of superpositions of grand canonical free boson gases with different chemical potentials.*

Consider a superposition of grand canonical free boson gases with different chemical potentials, hence with different average densities $\overline{\rho}$. Let the superposition be described by the measure $\nu(\overline{\rho})$ with

$$\int \mathrm{d}\overline{\rho}\nu(\overline{\rho}) = 1.$$

Then the corresponding reducible functional would be

$$L_\nu(f) = \int_0^\infty \exp\left(\rho \int \left(\mathrm{e}^{\mathrm{i}f(x)} - 1\right) \mathrm{d}^3x\right) \Gamma(\rho)\, \mathrm{d}\rho.$$

with

$$\Gamma(\rho) = \int_0^\infty \nu(\overline{\rho})\, (\frac{1}{\overline{\rho}}) \mathrm{e}^{-\rho/\overline{\rho}} \mathrm{d}\overline{\rho}.$$

Changing variables $t = \frac{1}{\overline{\rho}}$

$$\Gamma(\rho) = \int_0^\infty \frac{\nu\left(\frac{1}{t}\right)}{t} \mathrm{e}^{-\rho t} \mathrm{d}t$$

$\Gamma(\rho)$ is seen to be the Laplace transform of $\frac{\nu\left(\frac{1}{t}\right)}{t}$. Therefore, invertibility of the Laplace transform implies that given a $\Gamma(\rho)$ one may find a $\nu(\overline{\rho})$-superposition of grand canonical free boson gases with that particular reducible functional.

This is one possible physical interpretation of the meaning of the reducible functionals. Alternatively we might consider the reducible functionals in (11) simply as zero-temperature limits of statistical ensembles with density fluctuations. In favor of this alternative interpretation is the fact that particles with different chemical potentials would be different particles, but for example both the infinite dimensional Poisson measure and the infinite dimensional fractional Poisson measure have the same support, the configuration spaces of locally finite point measures without any additional labelling (see the appendix).

The study of the support of the measures associated to the irreducible and the reducible cases gives some hints on their role as far as physical modeling is concerned. For example, although the support for the infinite dimensional Poisson measure and the fractional one (fractional boson gas) are the same, the weights given to the $n$-particle states are different. The grand canonical ensemble might not be the only useful particle number fluctuation ensemble and different types of particle density fluctuations might imply different low-temperature phase transition behaviors.

Here we explore this possibility by computing the modifications introduced on the thermodynamic functions near the Bose–Einstein condensation temperature when, instead of the usual grand canonical ensemble, we have other types of particle number fluctuations, which would correspond in the zero-temperature limit to general classes of reducible functionals. Based on the equivalence result proved above this may be obtained by considering the superposition of grand-canonical free boson gases with different chemical potentials. For the grand-canonical free boson gas the number density $\frac{\langle N \rangle}{V}$ is

$$\frac{N}{V} = \frac{1}{V}\frac{z}{1-z} + \frac{1}{\lambda^3} g_{3/2}(z)$$

the first term being the fraction of particles condensed in the ground state, $z = \mathrm{e}^{\beta\mu}$, $\lambda = \sqrt{\frac{2\pi\hbar^2}{mkT}}$ and $g_{3/2}(z)$ is the function

$$g_{3/2}(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^{3/2}}.$$

For a superposition of grand-canonical free boson gases with different chemical potentials we replace $e^{\beta\mu}$ by $e^{\beta\mu x}$ and integrate over $x$ with a measure $\nu$ such that $\int dx\nu(x) = 1$. Then

$$\frac{N_\nu}{V} = \int_0^\infty dx\nu(x) \left\{ \frac{1}{V} \frac{z^x}{1-z^x} + \frac{1}{\lambda^3} g_{3/2}\left(z^x\right) \right\}.$$

Likewise the free energy becomes

$$\frac{U}{N} = \begin{cases} \frac{3}{2}\frac{kTV}{N\lambda^3} \int_0^\infty dx\nu(x) g_{5/2}\left(z^x\right) & T > T_c \\ \\ \frac{3}{2}\frac{kTV}{N\lambda^3} g_{5/2}(1) & T < T_c \end{cases}$$

with

$$g_{5/2}(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^{5/2}}.$$

For $T > T_c$, $z\left(T\right)$ is obtained from

$$\int_0^\infty dx\nu(x) g_{3/2}\left(z^x\right) = \frac{N_\nu}{V} \left(\frac{2\pi\hbar^2}{mk}\right)^{3/2} T^{-3/2}$$

and the specific heat $C_V = \frac{\partial U}{\partial T}$ for $T > T_c$

$$C_V = \frac{15}{4} kVT^{3/2} \left(\frac{mk}{2\pi\hbar^2}\right)^{3/2} \int_0^\infty dx\nu(x) g_{5/2}\left(z^x\right) + \frac{3}{2}\frac{kTV}{\lambda^3} \int_0^\infty dx\nu(x) g_{3/2}\left(z^x\right) xz^{-1} \frac{dz\left(T\right)}{dt}.$$

Let, as an example, $\nu(x)$ be a lognormal distribution peaked at $x = 1$

$$\nu(x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{\left(\ln x - \sigma^2\right)}{2\sigma^2}}.$$

Computing $C_V$ from the equations above for several values of $\sigma$ one obtains the results plotted in figure 1, where

$$T^* = \frac{mk}{2\pi\hbar^2 \rho^{2/3}} T.$$

One sees that as $\sigma$ becomes larger the specific heat behavior, above the condensation point, becomes sharper, more $\lambda$-like than the grand canonical Bose condensation transition. Physically a larger $\sigma$ means that the particle number fluctuations are larger than in the grand canonical ensemble. Notice that this is a purely statistical effect associated to the number fluctuations, no interaction being assumed in the Bose gas.

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## 3. Current algebra of many-body interacting systems

### 3.1. Representations of nonrelativistic currents, quasi-invariant measures and the ground state formulation

The search for representations of the current commutators (2) or of the semidirect product group generated by the exponentiated currents (3) is a very general method to characterize many-body quantum systems. In particular the current algebra structure is independent of whether one deals with Boson or Fermi or even other exotic statistics.

If the relevant configuration space is $R^d$, a unitary representation of the exponentiated currents in (3)

$$U\left(f\right)=\mathrm{e}^{\mathrm{i}\varrho(f)}$$
$$V\left(\phi_t^{\mathbf{g}}\right)=\mathrm{e}^{\mathrm{i}t\mathbf{J}(\mathbf{g})}$$

is a unitary representation of a semidirect product of infinite dimensional Lie groups

$$G=\mathcal{D}\wedge\mathrm{Diff}\left(\mathbb{R}^d\right) \tag{19}$$

$\mathcal{D}$ being the commutative multiplicative group of Schwartz functions $f\in\mathbb{C}_0^\infty\left(\mathbb{R}^d\right)$ and $\mathrm{Diff}\left(\mathbb{R}^d\right)$ the group of smooth diffeomorphisms of $\mathbb{R}^d$. Of special concern here is the restriction to the connected component of the identity $\mathrm{Diff}_0\left(\mathbb{R}^d\right)$. The group composition laws are

$$\begin{aligned}U\left(f_1\right)U\left(f_2\right)&=U\left(f_1+f_2\right)\\V\left(\phi\right)U\left(f\right)&=U\left(f\circ\phi\right)V\left(\phi\right)\\V\left(\phi_1\right)V\left(\phi_2\right)&=V\left(\phi_2\circ\phi_1\right).\end{aligned} \tag{20}$$

Taking the currents as the fundamental structures of quantum mechanics, all physical models of (nonrelativistic) quantum mechanics should be obtained as the unitary representations of the group $G$. A very general formulation for the representations of this group starts from a space of square-integrable functions $\mathcal{H}=\mathcal{L}_\mu^2\left(\Delta,\mathcal{W}\right)$ where $\Delta$ is a configuration space, $\mathcal{W}$ an inner product space and $\mu$ a measure on $\Delta$ quasi-invariant for the diffeomorphisms action $V\left(\phi\right)$. Then

$$\left(V\left(\phi\right)\Psi\right)\left(\gamma\right)=\chi_\phi\left(\gamma\right)\Psi\left(\phi\gamma\right)\sqrt{\frac{\mathrm{d}\mu_\phi}{\mathrm{d}\mu}\left(\gamma\right)} \tag{21}$$

where $\gamma\in\Delta$, $\Psi\in\mathcal{H}$ and $\chi_\phi\left(\gamma\right):\mathcal{W}\rightarrow\mathcal{W}$ is a family of unitary operators in $\mathcal{W}$ satisfying the cocycle condition

$$\chi_{\phi_1}\left(\gamma\right)\chi_{\phi_2}\left(\phi_1\gamma\right)=\chi_{\phi_1\circ\phi_2}\left(\gamma\right). \tag{22}$$

Quasi-invariance of the measure $\mu$ is essential to insure the existence of the Radon–Nikodym derivative in (21). On the other hand the unitary operators $U\left(f\right)$ are assumed to act by multiplication

$$\left(U\left(f\right)\Psi\right)\left(\gamma\right)=\mathrm{e}^{\mathrm{i}\langle\gamma,f\rangle}\Psi\left(\gamma\right) \tag{23}$$



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the meaning of $\langle \gamma, f \rangle$ depending on the particular choice of configuration space.

A popular configuration space for statistical mechanics applications has been the space of locally finite configurations, for which the Poisson measure and some Gibbs measures have been extensively studied [12]. However other, more general, configuration spaces have been proposed, the space of closed subsets of a manifold [18], the space of distributions $\mathcal{D}'$ or $\mathcal{S}'$, the space of embeddings and immersions and the space of countable subsets of $\mathbb{R}^d$ (see for example [8, 19]). Other configuration space worth to explore, when dealing with accumulation points of infinite cardinality, is the space of ultradistributions or ultradistributions of compact support, which have been found useful in another context [20].

The characterization of quantum systems through the construction of quasi-invariant measures on configuration spaces is quite general. However, a basic difficulty with this approach is that once a quasi-invariant measure is obtained it might not be easy to figure out what is the physical interaction (potential) that originates such measure. An alternative constructive approach, already foreshadowed in [7, 14], is suggested by the following construction.

Let for definiteness the configuration space be $\mathcal{S}'$ and assume the representation to be cyclic, that is, there is a normalized vector $\Omega \in \mathcal{H}$ such that the set $\{U(f)\Omega | f \in \mathcal{S}\}$ is dense in $\mathcal{H}$. Then the functional

$$L(f) = (\Omega, U(f)\Omega) \tag{24}$$

with $L(0) = 1$ is positive definite and continuous. By the Bochner–Minlos theorem it is the characteristic funtional of a measure on $\mathcal{H}$. The cyclic vector $\Omega$ becomes the central ingredient of the construction and, as will be seen later on, it relates in an easy manner to the interactions of the system.

In this spirit, Menikoff [14] proposed a set of axioms for the construction of (non-relativistic) quantum models: let $\mathcal{H}$ be a Hilbert space and $H$ a positive self-adjoint operator,

(i) There is a normalized state $\Omega$ of lowest energy. Then, by eventually subtracting a constant from $H$

$$H\Omega = 0. \tag{25}$$

(ii) $D = \mathrm{Span}\{U(f)\Omega; f \in \mathcal{S}\}$ is dense in $\mathcal{H}$ and $D$ is in the domain of $\mathcal{H}$.

(iii) Current conservation

$$[H, \rho(f)] = -i\mathbf{J}(\nabla f). \tag{26}$$

(iv) There is an antiunitary time reversal operator $\mathcal{T}$

$$\mathcal{T}\rho(f)\mathcal{T}^{-1} = \rho(f);\ \mathcal{T}\ \ \mathbf{J}(\mathbf{g})\mathcal{T}^{-1} = -\mathbf{J}(\mathbf{g});\ \mathcal{T}\ \ \Omega = \Omega. \tag{27}$$

In this framework it is proved [14] that the matrix elements of $\mathbf{J}(\mathbf{g})$ and $H$ are expressed in terms of those of $\rho(f)$, namely

$$\langle e\,(\,f_1)\,|\mathbf{J}\,(\mathbf{g})|\,e\,(\,f_2)\rangle = \frac{1}{2}\,\langle e\,(\,f_1)\,|\rho\,(\mathbf{g}\cdot\nabla\,(\,f_1+f_2))|\,e\,(\,f_2)\rangle$$
$$\langle e\,(\,f_1)\,|H|\,e\,(\,f_2)\rangle = \frac{1}{2}\,\langle e\,(\,f_1)\,|\rho\,(\boldsymbol{\nabla} f_1\cdot\nabla f_2)|\,e\,(\,f_2)\rangle \tag{28}$$

with $e\,(\,f) = \exp\,(\mathrm{i}\rho\,(\,f)\,\Omega)$. With time reversal invariance the equation (28) follow from the commutation relations

$$[\exp\,(\mathrm{i}\rho\,(\,f))\,,\mathbf{J}\,(\mathbf{g})] = -\rho\,(\mathbf{g}\cdot\nabla\mathbf{f})\exp\,(\mathrm{i}\rho\,(\,f))$$
$$[\exp\,(\mathrm{i}\rho\,(\,f))\,,H] = \left(-\mathbf{J}\,(\nabla f) + \frac{1}{2}\rho\,(\nabla f\cdot\nabla f)\right)\exp\,(\mathrm{i}\rho\,(\,f))$$

easily obtained from (2). A Hermitian form on a dense set of states does not uniquely determine an unbounded operator. Nevertheless, equation (28) show the central role played by the density operator $\rho\,(\,f)$ and the ground state $\Omega$ in the formulation of a quantum theory. This information is summarized in the generating functional

$$L\,(\,f) = (\Omega, U\,(\,f)\,\Omega)\,.$$

Many-body quantum systems are usually explored by postulating a interparticle potential and then obtaining the spectrum and eigenfunctions of the corresponding Hamiltonian. What the above current algebra characterization suggests is that a more natural (and complete) specification of the system is through a guess to the ground state which may be easier to infer from the macroscopic properties of the system than the microscopic potential that leads to such behavior. The idea of '*quantum mechanics from the ground state*' traces its origin to the papers of Coester and Haag [21] and Araki [22]. It has been further developed for single particle nonrelativistic quantum mechanics in several papers [23–25]. In this setting situations that would correspond to singular or nonlocal potentials are easily handled. The current algebra formulation now suggests that such an approach should also be carried out for many-body statistical mechanics.

Once a ground state function $\Omega = \exp\,(-W)$ without nodes is defined, by adding a constant to the Hamiltonian

$$H = -\triangle + V$$

such that

$$H\Omega = 0$$

the corresponding potential is

$$V = \frac{\triangle\Omega}{\Omega} = -\triangle W + \nabla W \bullet \nabla W.$$

Whereas in the approach through potentials, one usually restricts to a sum of two body interactions, if an arbitrary ground state function is postulated, it will in general correspond to potentials involving more than two particles. Some exceptions are the harmonic interaction ground state in arbitrary dimensions

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$$W_1 = \frac{\omega}{2} \sum_{i,j=1}^{N} (x_i - x_j)^2$$

and also

$$W_2 = \frac{\omega}{2} \sum_{i,j=1}^{N} (x_i - x_j)^2 + \frac{\lambda}{2} \sum_{i \neq j}^{N} \log |x_i - x_j|$$

in one dimension [26, 27].

In the following subsection one shows how the search for the ground state,inspired by an algebraic structure of the currents may shed light on the relevant physical problem of pairing in two-dimensional fermion systems.

### 3.2. Hole pairing and current quivers

Here, using currents, one attempts a formulation of a model for pairing as is required in the high-temperature superconductor phenomenon. First a short outline of the most relevant phenomenological facts which inspire the search for the elements of the model.

> First: the role of hole carriers and antiferromagnetic interactions
> Experiments have shown that the charge carriers have hole character for all classes of high-temperature superconductors and the superconducting state arises near the antiferromagnetic phase, experiments on the inelastic magnetic scattering of neutrons indicating the existence of strong magnetic fluctuations in the doped region, even beyond the limits of the antiferromagnetic phase. Though the long-range order disappears in the metallic and the superconducting phases, strong fluctuations with a wide spectrum of excitations are conserved, suggesting at least some local antiferromagnetic order. The closeness of the superconducting to the antiferromagnetic transition emphasizes the important role of spin fluctuations.
> Second: the dual role of a gap and the phase coherence
> In high-temperature superconductors, a gap is present even in the absence of phase coherence, i.e. in nonsuperconducting specimens. It appears at temperatures less than some characteristic temperature which depends on the doping. The (pseudo)gap is related to the appearance of coupled pairs, even before the onset of the phase coherence responsible for the change of the resistance.
> Therefore a key question is the nature of the mechanism of pairing of the carriers. Many different models were proposed, among which the following ones: the magnon model, the exciton model, the resonant valence bond, bipolaronic model, bisoliton model, anharmonic model, local pairs model, plasmon model, etc. All these models use the concept of pairing with the subsequent formation of a Bose-condensation at the superconducting transition. Pairing is therefore the central physical mechanism to be explained.

All this experimental information led to relate high-temperature superconductivity to the class of strongly correlated systems, the Hubbard model, the $t-J$ model, the antiferromagnetic Heisenberg model, etc. At the basis of these models are two simple ideas: first that in a regular array of lattice positions, the dominant positive contribution to the energy is the Coulomb repulsion when two (opposite spin) electrons occupy the same site, modelled by a term

$$\sum_a c_{a\uparrow}^{\dagger} c_{a\uparrow} c_{a\downarrow}^{\dagger} c_{a\downarrow} \tag{29}$$

and second that the energy decreases when the electrons are allowed to hop between closeby sites, modelled by

$$-\sum_{(a,b),\sigma} c_{a\sigma}^{\dagger} c_{b\sigma} \tag{30}$$

$(a, b)$ meaning closeby sites, nearest-neighbors or next-to-nearest neighbors. $c_{a\sigma}$ and $c_{a\sigma}^{\dagger}$ destruction and creation electron operators at the site $a$ and $\sigma$ is the spin orientation $(\uparrow, \downarrow)$. The interacting terms (29) and (30) form the basis of the Hubbard model. However, there is some evidence (see for example [28]) that by itself the Hubbard model is not sufficient to provide an hole pairing mechanism and that extra interactions must be called into play. We discuss this matter in terms of currents.

The interaction terms may be expressed in physical variables, that is, currents and densities. Notice however that the most appropriate algebraic structure for these physical variables might not be a Lie algebra. Consider a 2D square lattice with the atoms at the lattice vertices. The physical variables are the densities at each site $a$

$$\rho_\sigma(a) = c_{a\sigma}^{\dagger} c_{a\sigma} \tag{31}$$

and the currents

$$J_\sigma(a, b) = -\mathrm{i}\left(c_{b\sigma}^{\dagger} c_{a\sigma} - c_{a\sigma}^{\dagger} c_{b\sigma}\right) \tag{32}$$

corresponding to electron fluxes between the sites $a$ and $b$. The commutation relations are

$$\begin{aligned}
\left[\rho_\sigma(a), J_{\sigma'}(m, n)\right] &= -\mathrm{i}\left(\delta_{a,n} - \delta_{a,m}\right) K(m, n)\, \delta_{\sigma\sigma'} \\
\left[\rho_\sigma(a), K_{\sigma'}(m, n)\right] &= \mathrm{i}\left(\delta_{a,n} - \delta_{a,m}\right) J(m, n)\, \delta_{\sigma\sigma'} \\
\left[J_\sigma(a, b), J_{\sigma'}(m, n)\right] &= \mathrm{i}\left(-\delta_{a,m} J(b, n) + \delta_{a,n} J(b, m) - \delta_{b,n} J(a, m) + \delta_{b,m} J(a, n)\right) \delta_{\sigma\sigma'} \\
\left[J_\sigma(a, b), K_{\sigma'}(m, n)\right] &= \mathrm{i}\left(-\delta_{a,m} K(n, b) - \delta_{a,n} K(m, b) + \delta_{b,n} K(m, a) + \delta_{b,m} K(n, a)\right) \delta_{\sigma\sigma'} \\
\left[K_\sigma(a, b), K_{\sigma'}(m, n)\right] &= \mathrm{i}\left(\delta_{a,m} J(n, b) + \delta_{a,n} J(m, b) + \delta_{b,n} J(m, a) + \delta_{b,m} J(n, a)\right) \delta_{\sigma\sigma'}
\end{aligned} \tag{33}$$

$K_\sigma(m, n)$ being the operator

$$K_\sigma(m, n) = c_{n\sigma}^{\dagger} c_{m\sigma} + c_{m\sigma}^{\dagger} c_{n\sigma}. \tag{34}$$

This is the operator that in the continuum case leads to the term $\varrho(\mathbf{g} \bullet \nabla f)$ in right hand side of (2). Notice that $K_\sigma(m, m) \equiv 2\rho_\sigma(m)$.

One sees from the commutation relation of the currents that starting from currents connecting close neighbors one obtains, by successive commutators, currents involving direct hoppings between all sites in the lattice which, for the strongly correlated systems, are of no immediate physical interest. Therefore a Lie algebra is not an useful algebraic structure for these currents. Instead, restricting to nearest neighbor and next-to-nearest neighbor hoppings one obtains the following quiver (figure 2). A quiver is a directed graph. A representation of a quiver assigns a vector space $\mathcal{N}$ to each vertex, and a linear map to each edge (arrow). In the *current quiver* of figure 2 the arrows connecting the vertices to themselves are charge density contributions $\rho_\sigma(a)$ and those connecting different vertices correspond to the operators

$$V_\sigma(a,b) = \frac{1}{2}\left(K_\sigma(a,b) + \mathrm{i}J_\sigma(a,b)\right) \tag{35}$$

$V_\sigma(a,b)$ being a directed map corresponding to an hop from site $a$ to site $b$. Notice that $V_\sigma(a,a) \equiv \rho_\sigma(a)$.

To each vertex one assigns a four-dimensional space corresponding to the electron configurations (↑↓,↑,↓,◯), respectively double occupancy, spin up, spin down and a hole. The directed hop maps $V_\sigma(a,b)$ are represented by $4\times4$ matrices with elements

$$V_\uparrow(a,b) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; V_\downarrow(a,b) = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{36}$$

each element of the matrices accounting for the possible hopping contributions from vertex $a$ to $b$. For the arrows connecting one vertex to itself, the representation maps are

$$\rho_\uparrow(a) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \rho_\downarrow(a) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{37}$$

The relevant algebraic framework is therefore the quiver with maps $\rho_\sigma(a)$ and $V_\sigma(a,b)$ and composition laws

$$V_\sigma(a,b)\,V_\sigma(m,n) = \delta_{an}V_\sigma(m,b) + \delta_{mn}V(a,b) - V_\sigma(m,b)\,V_\sigma(a,n). \tag{38}$$

In particular from $V_\sigma(a,a) \equiv \rho_\sigma(a)$ it follows that

$$V_\sigma(a,b)\,V_\sigma(b,a) = \rho_\sigma(b)\left(1-\rho_\sigma(a)\right). \tag{39}$$

The state $\Psi$ of the system is the tensor product of the states $\psi_i \in \mathcal{N}$ for each vertex. Stationary states of the quiver are states that are invariant for some iteration of the quiver. Collecting the simplest quiver operations that leave a state $\Psi$ invariant, a general form for the stationary energy associated to the quiver is

$$\begin{aligned} E = {} & U\sum_a \rho_\uparrow(a)\rho_\downarrow(a) - t\sum_{\langle a,b\rangle,\sigma} V_\sigma(a,b)\,V_\sigma(b,a) \\ & + k\sum_{\langle a,b\rangle,\sigma}\left(1-\rho_\sigma(a)\right)\left(1-\rho_{-\sigma}(a)\right)\left(1-\rho_\sigma(b)\right)\left(1-\rho_{-\sigma}(b)\right) \\ & - J\sum_{a,\sigma}\left\{\alpha + \beta\left(1-\rho_\sigma(a)\right)\left(1-\rho_{-\sigma}(a)\right)\right\}\sum_{[n_a n'_\sigma],\sigma'} V_{\sigma'}(n_a,n'_a)\,V_{\sigma'}(n'_a,n_a) \end{aligned} \tag{40}$$

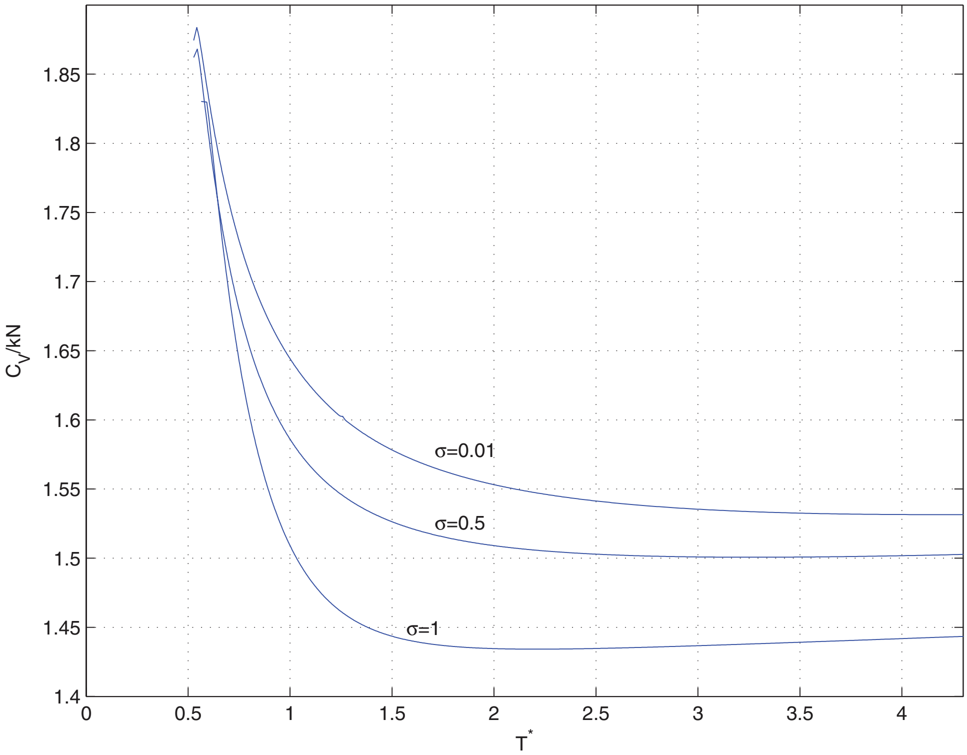


**Figure 1.** Specific heat behavior, above the condensation point, for different particle number fluctuations.

where $\langle a, b\rangle$ denotes nearest-neighbors and $[a, b]$ next-to-nearest-neighbors.

The first term is a positive contribution from Coulomb repulsion of two electrons in the same lattice site. The second is a symmetric hopping term between nearest-neighbor sites. The third is a hole repulsion term for holes at nearest-neighbor sites and finally the last term accounts for the hopping contributions between the neighbors of site $a$, which in the square lattice are next-to-nearest neighbors. In this last term two main possibilities are considered. If $\alpha = 1$ and $\beta = 0$ one has an unconditional next-to-nearest hopping contribution of intensity $-J$. However, if $\alpha = 0$ and $\beta = 1$, hopping between the neighbors of site $a$ only takes place if there is an hole in this site. The physical idea behind this possibility is that the hole distorts the orbitals in its neighborhood increasing the overlap of the wave functions of its neighbors.

Notice that our definition of the quiver energy does differ from similar operators derived from the Hubbard model by canonical transformations and leading order truncations. Using (39) the quiver energy is rewritten

$$
\begin{aligned}
E =& U\sum_{a}\rho_{\uparrow}(a)\rho_{\downarrow}(a) - t\sum_{\langle a,b\rangle,\sigma}\rho_{\sigma}(b)\left(1-\rho_{\sigma}(a)\right) \\
&+ k\sum_{\langle a,b\rangle,\sigma}\left(1-\rho_{\sigma}(a)\right)\left(1-\rho_{-\sigma}(a)\right)\left(1-\rho_{\sigma}(b)\right)\left(1-\rho_{-\sigma}(b)\right) \\
&- J\sum_{a,\sigma}\left\{\alpha+\beta\left(1-\rho_{\sigma}(a)\right)\left(1-\rho_{-\sigma}(a)\right)\right\}\sum_{[n_a n'_\sigma],\sigma'}\rho_{\sigma'}\left(n'_a\right)\left(1-\rho_{\sigma'}\left(n_a\right)\right).
\end{aligned}
$$

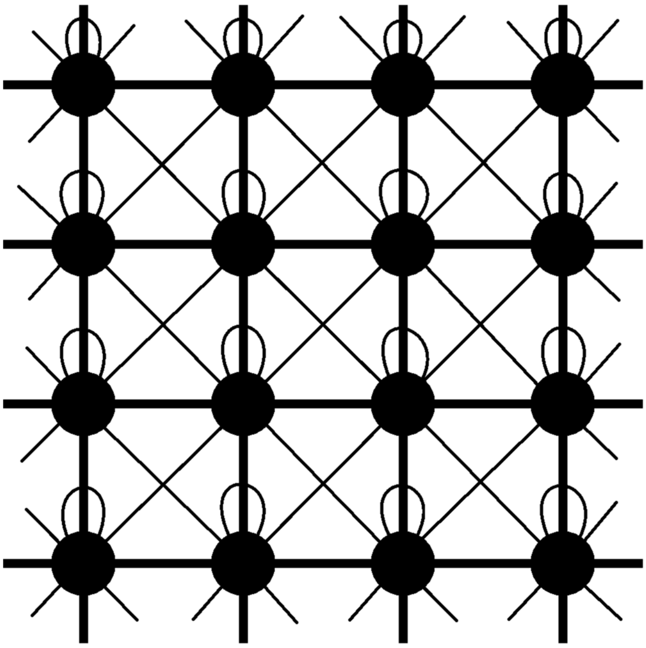

**Figure 2.** A current quiver.

Having the stationary energy fully expressed in number operators, the search for minimum energy states becomes a simple counting matter. Consider a 2D square lattice with N sites, $N-H$ electrons and $H$ holes ($H \ll N$) and

$$U \gg t,\ t > J,\ 4J > k.$$

To lower the energy, the large $U$ value implies single occupancy of the lattice sites and $t > J$ (local) antiferromagnetic order. In the case $\alpha = 1$, $\beta = 0$ a lowest energy estimate yields

$$E_{1,0} \simeq -t\frac{(N-H)(N-H-1)}{2} - 4JH$$

the holes being spread over the lattice without any special correlation among them. Any hole pairing would imply a $k$ positive contribution. In contrast for the case $\alpha = 0$, $\beta = 1$ the minimum energy estimate is

$$E_{0,1} \simeq -t\frac{(N-H)(N-H-1)}{2} - 2JH + k\frac{H}{2}.$$

The physical mechanism is clear. Although the wave function overlap in the neighborhood of an hole facilitates hopping between the neighbors of the hole, the local antiferromagnetic order frustrates this hopping. Hence, to lower the energy, another hole must be attracted to the neighborhood of the first hole and all holes are paired. Larger hole clusters will be avoided if $2k > 5J$.

Hole pairing is a precondition to the latter formation of the coherent state leading to superconductivity. The hole-induced hopping described here is a plausible mechanism for a possible hole pairing mechanism.

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## 4. Conclusions

1 In contrast with the quantum fields of canonical quantization, local currents are directly related to physical observables. In addition, whereas there are strong uniqueness results for the representation of finite-dimensional canonical commutation relations, the algebra of nonrelativistic currents has many non-equivalent represenations, each particular physical system corresponding to a different one. These two facts make (non-relativistic) current algebra a candidate of choice for the formulation of the statistical mechanics of many-body systems.

2 The construction of representations of the current algebra may be carried out either by defining quasi-invariant measures on configuration spaces or by a generating functional obtained from a (ground state) cyclic vector. It is argued in this paper that the second approach is more appropriate as a modeling tool for physical systems. An extensive application of this approach was done in the construction of a hole pairing model. It has also been found that for some models, instead of the full current algebra, a subset of operators is sufficient. A current quiver is used in the hole pairing model.

3 For boson systems, in addition to the ground state of the fixed density N/V limit, other reducible functionals might be useful to describe systems with number density fluctuations. A reducible functional is already implicit in the use of the grand canonical ensemble, but other functionals provide alternative phase transition behaviors.

## Acknowledgments

The author is grateful to Eric Carlen for an enlightening discussion on the physical meaning of reducible functionals.

## Appendix. The support of the infinite-dimensional Poisson and fractional Poisson measures

Here, for the reader's convenience and in particular as a background to section 2, a short summary is given of the properties of the infinite-dimensional Poisson measure, its support on configuration spaces [29–35] as well as of a fractional generalization [15, 36].

### A.1. The infinite-dimensional Poisson measure

The Poisson measure $\pi$ in $\mathbb{R}$ (or $\mathbb{N}$) is

$$\pi\left(A\right)=\mathrm{e}^{-s}\sum_{n\in A}\frac{s^{n}}{n!} \tag{A.1}$$

the parameter $s$ being called the *intensity*. The Laplace transform of $\pi$ is

$$l_\pi(\lambda) = \mathbb{E}\left(\mathrm{e}^{\lambda\cdot}\right) = \mathrm{e}^{-s}\sum_{n=0}^{\infty}\frac{s^n}{n!}\mathrm{e}^{\lambda n} = \mathrm{e}^{s\left(\mathrm{e}^{\lambda}-1\right)}$$

and for $n$-tuples of independent Poisson variables one would have the Laplace transform

$$l_\pi(\boldsymbol{\lambda}) = \mathrm{e}^{\sum s_k\left(\mathrm{e}^{\lambda_k}-1\right)}, \quad \boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n).$$

Continuing $\lambda_k$ to imaginary arguments $\lambda_k = if_k$, yields the characteristic function,

$$C_\pi(\lambda) = \mathrm{e}^{\sum s_k\left(\mathrm{e}^{\mathrm{i}f_k}-1\right)}. \tag{A.2}$$

An infinite-dimensional generalization is obtained by generalizing (A.2) to

$$C(\varphi) = \mathrm{e}^{\int\left(\mathrm{e}^{\mathrm{i}\varphi(x)}-1\right)\mathrm{d}\mu(x)} \tag{A.3}$$

for test functions $\varphi \in \mathcal{D}(M)$, $M$ being the space of $C^\infty$-functions of compact support in a manifold $M$. It is easy to prove, using the Bochner–Minlos theorem, that $C(\varphi)$ is indeed the Fourier transform of a measure on the distribution space $\mathcal{D}'(M)$.

A support for this measure is obtained in the space of locally finite subsets. The *configuration space* $\Gamma := \Gamma_M$ over the manifold $M$ is defined as the set of all locally finite subsets of $M$ (simple configurations)

$$\Gamma := \{\gamma \subset M \,:\, |\gamma \cap K| < \infty \text{ for any compact } K \subset M\}. \tag{A.4}$$

Here $|A|$ denotes the cardinality of the set $A$. As usual one identifies each $\gamma \in \Gamma$ with a non-negative integer-valued Radon measure,

$$\Gamma \ni \gamma \mapsto \sum_{x\in\gamma}\delta_x \in \mathcal{M}(M)$$

where $\delta_x$ is the Dirac measure with unit mass at $x$ and $\mathcal{M}(M)$ denotes the set of all non-negative Radon measures on $M$. In this way the space $\Gamma$ can be endowed with the relative topology as a subset of the space of measures $\mathcal{M}(M)$ with the vague topology, i.e. the weakest topology on $\Gamma$ for which the mappings

$$\Gamma \ni \gamma \mapsto \langle\gamma, f\rangle := \int_M f(x)\mathrm{d}\gamma(x) = \sum_{x\in\gamma} f(x)$$

are continuous for all real-valued continuous functions $f$ on $M$ with compact support. Denote the corresponding Borel $\sigma$-algebra on $\Gamma$ by $\mathcal{B}(\Gamma)$.

For each $Y \in \mathcal{B}(M)$ let us consider the space $\Gamma_Y$ of all configurations contained in $Y$, $\Gamma_Y := \{\gamma \in \Gamma : |\gamma \cap (X\backslash Y)| = 0\}$, and the space $\Gamma_Y^{(n)}$ of $n$-point configurations,

$$\Gamma_Y^{(n)} := \{\gamma \in \Gamma_Y : |\gamma| = n\}, n \in \mathbb{N}, \quad \Gamma_Y^{(0)} := \{\emptyset\}.$$

A topological structure may be introduced on $\Gamma_Y^{(n)}$ through the natural surjective mapping of $\widetilde{Y^n} := \{(x_1, ..., x_n) : x_i \in Y, x_i \neq x_j \text{ if } i \neq j\}$ onto $\Gamma_Y^{(n)}$,

$$\begin{aligned} \mathrm{sym}_Y^n : \widetilde{Y^n} &\longrightarrow \Gamma_Y^{(n)} \\ (x_1, ..., x_n) &\longmapsto \{x_1, ..., x_n\} \end{aligned}$$



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which is at the origin of a bijection between $\Gamma_Y^{(n)}$ and the symmetrization $\widetilde{Y^n}/S_n$ of $\widetilde{Y^n}$, $S_n$ being the permutation group over $\{1, ..., n\}$. Thus, $\mathrm{sym}_Y^n$ induces a metric on $\Gamma_Y^{(n)}$ and the corresponding Borel $\sigma$-algebra $\mathcal{B}\left(\Gamma_Y^{(n)}\right)$ on $\Gamma_Y^{(n)}$.

For $\Lambda \in \mathcal{B}(M)$ with compact closure ($\Lambda \in \mathcal{B}_c(M)$), it clearly follows from (A.4) that

$$\Gamma_\Lambda = \bigsqcup_{n=0}^{\infty} \Gamma_\Lambda^{(n)}$$

the $\sigma$-algebra $\mathcal{B}(\Gamma_\Lambda)$ being defined by the disjoint union of the $\sigma$-algebras $\mathcal{B}\left(\Gamma_Y^{(n)}\right)$, $n \in \mathbb{N}_0$.

For each $\Lambda \in \mathcal{B}_c(M)$ there is a natural measurable mapping $p_\Lambda : \Gamma \to \Gamma_\Lambda$. Similarly, given any pair $\Lambda_1, \Lambda_2 \in \mathcal{B}_c(M)$ with $\Lambda_1 \subset \Lambda_2$ there is a natural mapping $p_{\Lambda_2,\Lambda_1} : \Gamma_{\Lambda_2} \to \Gamma_{\Lambda_1}$. They are defined, respectively, by

$$p_\Lambda : \begin{array}{l} \Gamma \longrightarrow \Gamma_\Lambda \\ \gamma \longmapsto \gamma_\Lambda := \gamma \cap \Lambda \end{array} \qquad p_{\Lambda_2,\Lambda_1} : \begin{array}{l} \Gamma_{\Lambda_2} \longrightarrow \Gamma_{\Lambda_1} \\ \gamma \longmapsto \gamma_{\Lambda_1} \end{array}.$$

It can be shown that $(\Gamma, \mathcal{B}(\Gamma))$ coincides (up to an isomorphism) with the projective limit of the measurable spaces $(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))$, $\Lambda \in \mathcal{B}_c(M)$, with respect to the projection $p_\Lambda$, i.e. $\mathcal{B}(\Gamma)$ is the smallest $\sigma$-algebra on $\Gamma$ with respect to which all projections $p_\Lambda$, $\Lambda \in \mathcal{B}_c(M)$, are measurable.

Let now $\mu$ be a measure on the underlying measurable space $(M, \mathcal{B}(M))$ and consider for each $n \in \mathbb{N}$ the product measure $\mu^{\otimes n}$ on $(M^n, \mathcal{B}(M^n))$. Since $\mu^{\otimes n}(M^n \backslash \widetilde{M^n}) = 0$, one may consider for each $\Lambda \in \mathcal{B}_c(M)$ the restriction of $\mu^{\otimes}$ to $(\widetilde{\Lambda^n}, \mathcal{B}(\widetilde{\Lambda^n}))$, which is a finite measure, and then the image measure $\mu_\Lambda^{(n)}$ on $(\Gamma_\Lambda^{(n)}, \mathcal{B}(\Gamma_\Lambda^{(n)}))$ under the mapping $\mathrm{sym}_\Lambda^n$,

$$\mu_\Lambda^{(n)} := \mu^{\otimes n} \circ (\mathrm{sym}_\Lambda^n)^{-1}.$$

For $n = 0$ we set $\mu_\Lambda^{(0)} := 1$. Now, one may define a probability measure $\pi_{\mu,\Lambda}$ on $(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))$ by

$$\pi_{\mu,\Lambda} := \sum_{n=0}^{\infty} \frac{\exp(-\mu(\Lambda))}{n!} \mu_\Lambda^{(n)}. \tag{A.5}$$

The family $\{\pi_{\mu,\Lambda} : \Lambda \in \mathcal{B}_c(M)\}$ of probability measures yields a probability measure on $(\Gamma, \mathcal{B}(\Gamma))$ with the $\pi_{\mu,\Lambda}$ as projections. This family is consistent, that is,

$$\pi_{\mu,\Lambda_1} = \pi_{\mu,\Lambda_2} \circ p_{\Lambda_2,\Lambda_1}^{-1}, \quad \forall\, \Lambda_1, \Lambda_2 \in \mathcal{B}_c(M), \Lambda_1 \subset \Lambda_2$$

and thus, by the version of Kolmogorov's theorem for the projective limit space $(\Gamma, \mathcal{B}(\Gamma))$, the family $\{\pi_{\mu,\Lambda} : \Lambda \in \mathcal{B}_c(M)\}$ determines uniquely a measure $\pi_\mu$ on $(\Gamma, \mathcal{B}(\Gamma))$ such that

$$\pi_{\mu,\Lambda} = \pi_\mu \circ p_\Lambda^{-1}, \quad \forall\, \Lambda \in \mathcal{B}_c(M).$$

The next step is to compute the characteristic functional of the measure $\pi_\mu$. Given a $\varphi \in \mathcal{D}(M)$ we have $\mathrm{supp}\varphi \subset \Lambda$ for some $\Lambda \in \mathcal{B}_c(M)$, meaning that

$$\langle \gamma, \varphi \rangle = \langle p_\Lambda(\gamma), \varphi \rangle, \quad \forall\, \gamma \in \Gamma.$$



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Thus

$$\int_{\Gamma} \mathrm{e}^{\mathrm{i}\langle\gamma,\varphi\rangle} \mathrm{d}\pi_{\mu}(\gamma) = \int_{\Gamma_\Lambda} \mathrm{e}^{\mathrm{i}\langle\gamma,\varphi\rangle} \mathrm{d}\pi_{\mu,\Lambda}(\gamma)$$

and the definition (A.5) of the measure $\pi_{\mu,\Lambda}$ yields for the right-hand side of the equality

$$\sum_{n=0}^{\infty} \frac{\exp(-\mu(\Lambda))}{n!} \int_{\Lambda^n} \mathrm{e}^{\mathrm{i}(\varphi(x_1)+\ldots+\varphi(x_n))} \mathrm{d}\mu^{\otimes n}(x) = \sum_{n=0}^{\infty} \frac{\exp(-\mu(\Lambda))}{n!} \left(\int_{\Lambda} \mathrm{e}^{\mathrm{i}\varphi(x)} \mathrm{d}\mu(x)\right)^n$$

which corresponds to the Taylor expansion of the characteristic function (A.3) of the infinite-dimensional Poisson measure

$$\exp\left(\int_{\Lambda} (\mathrm{e}^{\mathrm{i}\varphi(x)} - 1)\, \mathrm{d}\mu(x)\right).$$

This shows that the probability measure on $(\mathcal{D}'(M), \mathcal{C}_\sigma(\mathcal{D}'(M)))$ given by (A.3) is actually supported on generalized functions of the form $\sum_{x\in\gamma} \delta_x$, $\gamma \in \Gamma$. Thus, the inifinite-dimensional Poisson measure $\pi_\mu$ can either be considered as a measure on $(\Gamma, \mathcal{B}(\Gamma))$ or on $(\mathcal{D}', \mathcal{C}_\sigma(\mathcal{D}'(M)))$. Notice that, in contrast to $\Gamma$, $\mathcal{D}'(M) \supset \Gamma$ is a linear space. Since $\pi_\mu(\Gamma) = 1$, the measure space $(\mathcal{D}'(M), \mathcal{C}_\sigma(\mathcal{D}'(M)), \pi_\mu)$ can, in this way, be regarded as a linear extension of the Poisson space $(\Gamma, \mathcal{B}(\Gamma), \pi_\mu)$.

### A.2. The infinite-dimensional fractional Poisson measure

The Poisson process has a fractional generalization [37, 38], the probability of $n$ events being

$$P(X = n) = \frac{s^{\alpha n}}{n!} E_\alpha^{(n)}(-s^\alpha) \tag{A.6}$$

$E_\alpha^{(n)}$ denoting the $n$th derivative of the Mittag-Leffler function.

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad z \in \mathbb{C} \tag{A.7}$$

$(\alpha > 0)$. In contrast with the Poisson case $(\alpha = 1)$, this process has power law asymptotics rather than exponential, which implies that it is not longer Markovian. The characteristic function of this process is given by

$$C_\alpha(\lambda) = E_\alpha\left(s^\alpha\left(\mathrm{e}^{\mathrm{i}\lambda} - 1\right)\right). \tag{A.8}$$

By analogy with (A.3) an infinite-dimensional generalization is obtained by generalizing (A.8) to

$$C_\alpha(\varphi) := E_\alpha\left(\int (\mathrm{e}^{\mathrm{i}\varphi(x)} - 1)\, \mathrm{d}\mu(x)\right), \quad \varphi \in \mathcal{D}(M) \tag{A.9}$$

with $\mu$ a positive intensity measure fixed on the underlying manifold $M$. Using the Bochner–Minlos theorem and the complete monotonicity of the Mittag-Leffler function $C_\alpha$ is shown [36] to be the characteristic functional of a probability measure $\pi_\mu^\alpha$ on the distribution space $\mathcal{D}'(M)$

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It turns out that this measure is also supported in configuration spaces and the formulation in configuration spaces provides, through the Kolmogorov's theorem for projective limits, an alternative construction of the measure.

As in (A.5), for each $0 < \alpha < 1$ one defines a probability measure $\pi^{\alpha}_{\mu,\Lambda}$ on $(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))$ by

$$\pi^{\alpha}_{\mu,\Lambda} := \sum_{n=0}^{\infty} \frac{E_{\alpha}^{(n)}(-\mu(\Lambda))}{n!} \mu_{\Lambda}^{(n)}. \tag{A.10}$$

The family $\{\pi^{\alpha}_{\mu,\Lambda} : \Lambda \in \mathcal{B}_c(M)\}$ of probability measures yields a probability measure on $(\Gamma, \mathcal{B}(\Gamma))$ with the $\pi^{\alpha}_{\mu,\Lambda}$ as projections, which being consistent uniquely determines a measure $\pi^{\alpha}_{\mu}$ on $(\Gamma, \mathcal{B}(\Gamma))$ such that

$$\pi^{\alpha}_{\mu,\Lambda} = \pi^{\alpha}_{\mu} \circ p_{\Lambda}^{-1}, \quad \forall \Lambda \in \mathcal{B}_c(M).$$

For the characteristic functional of the measure $\pi^{\alpha}_{\mu}$ one obtains

$$\begin{aligned} C_{\alpha}(\varphi) &= \sum_{n=0}^{\infty} \frac{E_{\alpha}^{(n)}\left(-\int_{\Lambda} \mathrm{d}\mu(x)\right)}{n!} \left(\int_{\Lambda} \mathrm{e}^{\mathrm{i}\varphi(x)} \mathrm{d}\mu(x)\right)^{n} \\ &= \sum_{n=0}^{\infty} \frac{E_{\alpha}^{(n)}\left(-\int_{\Lambda} \mathrm{d}\mu(x)\right)}{n!} \int_{\Lambda^n} \mathrm{e}^{\mathrm{i}(\varphi(x_1)+\varphi(x_2)+\cdots+\varphi(x_n))} \mathrm{d}\mu^{\otimes n} \\ &= E_{\alpha}\left(\int_{\Lambda} (\mathrm{e}^{\mathrm{i}\varphi(x)} - 1)\, \mathrm{d}\mu(x)\right) \end{aligned}$$

the last equality obtained by Taylor expansion of the Mittag-Leffler function. Similarly to the $\alpha = 1$ case, one sees that the probability measure $\pi^{\alpha}_{\mu}$ on $(\mathcal{D}'(M), \mathcal{C}_{\sigma}(\mathcal{D}'(M)))$ is actually supported on generalized functions of the form $\sum_{x\in\gamma} \delta_x$, $\gamma \in \Gamma$.

One sees from (A.10) that, instead of the uniform combinatorial weight $\frac{\exp(-\mu(\Lambda))}{n!}$ for $n$ particles of the Poisson case ($\alpha = 1$), one now has $\frac{E_{\alpha}^{(n)}(-\mu(\Lambda))}{n!}$, the rest being the same. Therefore the main difference in the fractional case ($\alpha \neq 1$) is that a different weight is given to each $n$ -particle space, although the support is the same. Different weights, multiplying the $n$-particle space measures, may be physically significant in that they have decays, for large volumes, smaller than the corresponding exponential factor in the Poisson measure.

It is not surprising that the support of the measure $\pi^{\alpha}_{\mu}$ coincides with the support of the Poisson measure ($\alpha = 1$). Using the spectral representation of the Mittag-Leffler function

$$E_{\alpha}(-z) = \int_0^{\infty} \mathrm{e}^{-\tau z} \mathrm{d}\nu_{\alpha}(\tau)$$

$\nu_{\alpha}$ being the probability measure in $\mathbb{R}_0^+$

$$\mathrm{d}\nu_{\alpha}(\tau) = \alpha^{-1} \tau^{-1-1/\alpha} f_{\alpha}\left(\tau^{-1/\alpha}\right) \mathrm{d}\tau$$

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and $f_\alpha$ the $\alpha$-stable probability density given by

$$\int_0^\infty \mathrm{e}^{-t\alpha} f_\alpha(\tau)\,\mathrm{d}\tau = \mathrm{e}^{-t^\alpha}, 0 < \alpha < 1$$

one may rewrite (A.9) as

$$C_\alpha(\varphi) = \int_0^\infty \exp\left(\tau \int (\mathrm{e}^{\mathrm{i}\varphi(x)} - 1)\,\mathrm{d}\mu(x)\right)\,\mathrm{d}\nu_\alpha(\tau)$$

the integrand being the characteristic function of the Poisson measure $\pi_{\tau\mu}$, $\tau > 0$. This shows that the characteristic functional (A.9) coincides with the characteristic functional of the measure $\int_0^\infty \pi_{\tau\mu}\,\mathrm{d}\nu_\alpha(\tau)$. By uniqueness, this implies the integral decomposition

$$\pi_\mu^\alpha = \int_0^\infty \pi_{\tau\mu}\,\mathrm{d}\nu_\alpha(\tau) \qquad \text{(A.11)}$$

meaning that $\pi_\mu^\alpha$ is an integral (or mixture) of Poisson measures $\pi_{\tau\mu}$, $\tau > 0$.

A fractional Poisson analysis may be developed along the lines of the inifinite-dimensional Poisson analysis [36].

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