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Reduction and approximation in guiding-center dynamics

Philippe Ghendrih[1], **Ricardo Lima**[2] **and R Vilela Mendes**[3,4,5]

[1] Association Euratom-CEA, DRFC/DSM/CEA Cadarache, 13108 St Paul lez Durance Cédex, France
[2] Centre de Physique Théorique, CNRS Luminy, case 907, F-13288 Marseille Cedex 9, France
[3] Centro de Fusão Nuclear—EURATOM/IST Association, Instituto Superior Técnico, Av Rovisco Pais 1, 1049-001 Lisboa, Portugal
[4] CMAF, Complexo Interdisciplinar, Universidade de Lisboa, Av Gama Pinto, 2 - 1649-003 Lisboa, Portugal

E-mail: philippe.ghendrih@cea.fr, lima@cpt.univ-mrs.fr and vilela@cii.fc.ul.pt

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Abstract
The guiding-center and gyrocenter formulations of plasmas in strong magnetic fields aim at the elimination of the angle associated with the Larmor rotation of charged particles around the magnetic field lines. At each finite level of a perturbative treatment these formulations are approximations to the true dynamics. Here we discuss the conditions under which guiding-center dynamics may become an exact operation in the framework of reduction of dynamical systems with symmetry.

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1. Introduction

In contrast with the many types of approximation schemes used to deal with complex problems, a *reduction* is an exact process. Reduction is a process by which, given a multidimensional dynamical system, one focuses on a small subset of variables, obtaining exact equations for the variables in this subset. Along the way, of course, one loses information about some of the dynamical details of the system. In particular, several distinct evolutions of the whole system may lead to the same dynamical trajectory when projected on the space of the reduced variables but, in any case, one is confident that the dynamics of the reduced variables is exact.

Reduction in mechanics had its origins in the classical works of Euler, Lagrange, Hamilton, Jacobi, Routh and Poincaré. Routh's elimination of cyclic variables and Jacobi's elimination of the node are among the first examples (for reviews of the historical aspects and the several

[5] Author to whom any correspondence should be addressed.

types of reduction we refer to [1–3]). In general, reduction applies when the system under study possesses some kind of symmetry. Then, the variables associated with the invariance directions are eliminated and one ends up with a dynamical description on a quotient space.

In the context of symplectic manifolds, a construction has now become standard [3–5] which, for reference and later use, we now summarize:

Let (P, ω) be a symplectic manifold where a Lie group G acts by symplectomorphisms $(\phi_g^* \omega = \omega)$. An equivariant moment map for the group action is a map $J : P \to g^*$ (g^* being the dual of the Lie algebra g of G) such that if $\widehat{J}$ denotes the dual map from g to the space of smooth functions on P, we have

$$d(\widehat{J}(\xi)) = i_{\xi_P}\omega \tag{1}$$

and $J(\phi_g(x)) = Ad^*_{g^{-1}}(J(x)), \forall x \in P, g \in G$. If the momentum map is not equivariant it can be converted into an equivariant one by central extension of the group [6]. Equation (1) means that each infinitesimal generator ξ_P of g has $\widehat{J}(\xi)$ as an Hamiltonian function.

Let μ be an element of g^* and G_μ its coadjoint isotropy subgroup. For any $G-$invariant Hamiltonian, $J^{-1}(\mu) \subset P$ is an invariant set for the dynamics and the reduced space $J^{-1}(\mu)/G_\mu = P_\mu$ is a symplectic manifold with the symplectic form Ω_μ determined by $\pi_\mu^* \Omega_\mu = i_\mu^* \Omega$. π_μ is the projection $J^{-1}(\mu) \to P_\mu$ and i_μ the inclusion $J^{-1}(\mu) \to P$.

In the symplectic reduction the emphasis is on the projection of the Poisson structure and Hamiltonian dynamics to a quotient space by the action of a symmetry group. In the Lagrangian approach, reduction follows a different approach. Reduced variables are identified and then one proceeds to carry the variational structure to a quotient space. In the process not only the completeness of the reduced variables set should be checked, but also whether the variational structure can be carried over to the quotient space. Here, in principle, a symmetry group need not to be known to begin with. One may start by conjecturing some tentative set of reduced variables (functions of the original variables). Then, using the Poisson brackets of the original variables, check whether this trial set is algebraically closed. If not, the Poisson brackets themselves will suggest new variables to close the set. Then, to be sure that the variational structure is carried over to the quotient space, it is necessary to check whether the (exact) dynamics of the reduced variables may be obtained from a Lagrangian written on these variables alone. If that is possible, an exact reduction has been performed.

Even when an exact reduction on some subset of variables is not possible, the reduction point of view provides a step-by-step approach to an approximation scheme. For example, it is possible that the trial set of reduced variables is algebraically closed with a consistent bracket structure, but that their dynamics cannot be entirely defined by an action principle containing only these variables. To proceed it must be necessary at this step to propose some kind of approximation, but at least the method makes it clear how much of the reduction process is exact and how much it implies an approximation.

The same methodology of approximation through reduction may be used in the group theoretical setting. Suppose that some dynamical process is known that is not exactly symmetric, but we are interested only in the dynamical features consistent with the symmetry. Then an *approximation-through-reduction* may be obtained by projecting the dynamics on the space of functions that possess the required symmetry.

In the context of plasma physics in strong magnetic fields, several different time scales rule the physical phenomena. One is the fast time scale of gyromotion of the ions around the magnetic field (of order $\frac{2\pi mc}{eB}$) and the others the longer bounce and drift time scales associated with the field gradients and curvature. An important simplification arises when the time scales can be treated separately. This might not be possible, for example in the study of fluctuation phenomena. However, for some stability and transport problems it is indeed a useful approximation. When the fast time scale and the field fluctuations are separated, or integrated over, one obtains the guiding-center and gyrocenter equations. The usual way this is done is either by simple averaging the single particle dynamics [7, 8] or, more accurately, by performing near-identity Lie-transformations to obtain an action two-form where the gyroangle dependence is to be asymptotically eliminated. In both cases the point of view is to address the gyrokinetics formulation as an approximation to the plasma dynamics. The Lie-transform perturbative approach has been extensively developed [9–12] leading, for example, to a third-order perturbative analysis of a plasma moving with a nonuniform fluid velocity [13, 14]. The Lie-transform approach provides successive approximations of practical interest to the reduced Vlasov and Maxwell equations. However to show that this procedure provides an exact reduction of the dynamics would require the proof of convergence of the perturbative series, which has not yet been done.

The main purpose of our program, of which this paper is a first step, is to find out how much of the gyrokinetics program may be framed as an exact reduction. In this paper we focus on the guiding-center reduction on an external electromagnetic field.

Exact reductions are usually based on the knowledge of a symmetry group of the dynamical systems. For general nonuniform field configurations is not obvious what symmetry group, if any, might be associated with guiding-center dynamics. Lacking this information, we look instead for exact invariants. The central result of the paper is the construction of an exact invariant as a Borel summable series. Then we show how, from the existence of the invariant, an exact reduction of the dynamics may be achieved.

2. Guiding-center dynamics as an exact reduction

In the extended phase-space $(\vec{x}, \vec{p}, t, h)$, the Hamiltonian of a particle moving in an electromagnetic field is [15]

$$H(\vec{x}, \vec{p}, t, h) = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 + e\Phi - h, \tag{2}$$

the nonvanishing elements of the Poisson tensor being

$$\{x^i, p^j\} = \delta^{ij} \qquad \{t, h\} = -1. \tag{3}$$

Changing coordinates to $(\vec{x}, \vec{v}, t, K)$ with

$$\vec{v} = \frac{1}{m}\left(\vec{p} - \frac{e}{c}\vec{A}\right) \qquad K = h - e\Phi \tag{4}$$

leads to

$$H(\vec{x}, \vec{v}, t, k) = \tfrac{1}{2}m\vec{v}^2 - K \tag{5}$$

and

$$\begin{aligned} \sigma^{i v^j} &= \{x^i, v^j\} = \frac{1}{m}\delta^{ij} & \sigma^{tK} &= \{t, K\} = -1 \\ \sigma^{v^i v^j} &= \{v^i, v^j\} = \frac{e}{m^2 c}B^{ij} & \sigma^{v^i K} &= \{v^i, K\} = -\frac{e}{m}E^i \end{aligned} \tag{6}$$

$B^{ij} = \epsilon^{ijk}B_k$.

It is convenient to decompose the velocity into magnetic field adapted components,

$$v_\parallel = v_i\widehat{b}^i \qquad \vec{v}_\perp = \vec{v} - \widehat{b}(\vec{v}\cdot\widehat{b}) \tag{7}$$

$\widehat{b} = \frac{\vec{B}}{|\vec{B}|}$, for which the equations of motion are obtained from (5) and (6) by $\frac{dF}{dt} = \{F, H\}$

$$\begin{aligned} \frac{d}{dt} v_{\|} &= \frac{e}{m} E_{\|} + \vec{v}_{\perp} \cdot \frac{d}{dt} \widehat{b} \\ \frac{d}{dt} \vec{v}_{\perp} &= \frac{e}{m} \vec{E}_{\perp} + \frac{e}{mc} (\vec{v}_{\perp} \times \vec{B}) - v_i \frac{d}{dt} (\widehat{b}^i \widehat{b}) \end{aligned} \tag{8}$$

or, if the fields have no explicit time dependence

$$\begin{aligned} \frac{d}{dt} v_{\|} &= \frac{e}{m} E_{\|} + \vec{v}_{\perp} \cdot (\vec{v} \cdot \nabla) \widehat{b} \\ \frac{d}{dt} \vec{v}_{\perp} &= \frac{e}{m} \vec{E}_{\perp} + \frac{e}{mc} (\vec{v}_{\perp} \times \vec{B}) - v_i (\vec{v} \cdot \nabla) (\widehat{b}^i \widehat{b}) \end{aligned} \tag{9}$$

For bounded electromagnetic fields with bounded derivatives, the right-hand side of the system of equations (9) is locally lipschitzian. This ensures existence and uniqueness of the solution.

To prove the existence of an exact reduction of a dynamical system one has to identify a symmetry group or, equivalently, the existence of one or more invariants. Our method depends on the construction of a formal invariant. Existence of such invariants, to all orders in perturbation theory, for charged particle motion in a strong magnetic field was first pointed out by Kruskal [16]. Here we attempt the explicit construction of an exact invariant. The technique hinges on transforming the equation for the transverse velocity to the form

$$\frac{d}{dt} (\vec{v}_{\perp} - \vec{u}_{\perp}) = g(\vec{x}, \vec{v}, t)(\vec{v}_{\perp} - \vec{u}_{\perp}) \times \vec{B} - \hat{b}\Gamma(\vec{x}, \vec{v}, t) - \alpha(\vec{x}, \vec{v}, t)(\vec{v}_{\perp} - \vec{u}_{\perp}). \tag{10}$$

The main result is summarized in the following:

Proposition. *In the domain of existence of bounded solutions of the system (9) and for bounded and sufficiently smooth electromagnetic fields, there are functions* $\Gamma(\vec{x}, \vec{v}, t)$, $\alpha(\vec{x}, \vec{v}, t)$, $g(\vec{x}, \vec{v}, t)$ *and a perpendicular vector function* $\vec{u}_{\perp}(\vec{x}, \vec{v}, t)$ *such that the equation (10) holds. For fields that have no explicit time dependence and sufficiently large magnetic field we also have* $\vec{u}_{\perp} = \vec{u}_{\perp}(\vec{x}, \vec{v})$, $\alpha = \alpha(\vec{x}, \vec{v})$, $g = g(\vec{x}, \vec{v})$ *and* $\Gamma = \Gamma(\vec{x}, \vec{v})$.

Proof. If we do not require the functions $\vec{u}_{\perp}$, α, g and Γ to have no explicit time dependence, the result is fairly trivial. It suffices to rewrite (10) as

$$\begin{aligned} &\frac{d}{dt} \vec{u}_{\perp} - g(\vec{x}, \vec{v}, t) \vec{u}_{\perp} \times \vec{B} + \alpha(\vec{x}, \vec{v}, t) \vec{u}_{\perp} \\ &\qquad = \left(\frac{d}{dt} + \alpha(\vec{x}, \vec{v}, t) + g(\vec{x}, \vec{v}, t) \vec{B} \times \right) \vec{v}_{\perp} + \hat{b}\Gamma(\vec{x}, \vec{v}, t) \end{aligned} \tag{11}$$

and then, the assumed boundedness of the fields and the solutions of (9) imply the existence of a solution $\vec{u}_{\perp}(\vec{x}, \vec{v}, t)$ for each choice of $\Gamma(\vec{x}, \vec{v}, t)$, $g(\vec{x}, \vec{v}, t)$ and $\alpha(\vec{x}, \vec{v}, t)$.

Less trivial is to show the existence of solutions for $\vec{u}_{\perp}$,α, g and Γ which have no explicit time dependence. Because we are going to construct the quantities $\vec{u}_{\perp}$, α, g and Γ as a formal series, it is important to control the magnitude of the operators action in the space of velocities, in particular to guarantee that we do not generate terms that grow with $|B|$, the magnetic field intensity. Let us assume the fields to have no explicit time- dependence. Then, in the second equation in (9) we bring the term $\frac{e}{mc}(\vec{v}_{\perp} \times \vec{B})$ to the left-hand side

$$\left(\frac{d}{dt} + \frac{e}{mc} \vec{B} \times \right) \vec{v}_{\perp} = \frac{e}{m} \vec{E}_{\perp} - v^i (\vec{v} \cdot \nabla)(\hat{b}^i \hat{b}) \tag{12}$$

and rewrite the operator $\left(\frac{\mathrm{d}}{\mathrm{d}t} + \frac{e}{mc}\vec{B}\times\right)$ as a derivation on the variables $\left(t, v_\perp^{(1)}, v_\perp^{(2)}\right)$

$$D = \frac{\mathrm{d}}{\mathrm{d}t} + \frac{e|B|}{mc}\left(v_\perp^{(1)}\frac{\partial}{\partial v_\perp^{(2)}} - v_\perp^{(2)}\frac{\partial}{\partial v_\perp^{(1)}}\right) \tag{13}$$

$v_\perp^{(1)}$ and $v_\perp^{(1)}$ being the components of $\vec{v}_\perp$ in an arbitrary perpendicular coordinate system. Then equations (9) become

$$Dv_\parallel = \frac{e}{m}E_\parallel + \vec{v}_\perp\cdot(\vec{v}\cdot\nabla)\widehat{b} \qquad D\vec{v}_\perp = \frac{e}{m}\vec{E}_\perp - v_i(\vec{v}\cdot\nabla)(\widehat{b}^i\widehat{b}) \tag{14}$$

We see that the right-hand side no longer involves explicit dependence on the magnetic field intensity, that is, the action of the D operator does not introduce frequencies comparable to the Larmor frequency. It is the choice of the D operator for the operator expansions that, for large B, implements the separation of time scales.

We now apply the identity

$$\vec{a}_\perp = -\frac{1}{|\vec{B}|^2}(\vec{a}_\perp\times\vec{B})\times\vec{B} \tag{15}$$

that holds for perpendicular fields, to the first and the perpendicular part of the second term on the right-hand side of the second equation in (14), to obtain

$$\begin{aligned} D\left(\vec{v}_\perp - c\frac{\vec{E}_\perp\times\vec{B}}{|B|^2} + \frac{mcv_\parallel}{e|B|^2}(\vec{v}\cdot\nabla)\widehat{b}\times\vec{B}\right) &= \frac{e}{mc}\left(-c\frac{\vec{E}_\perp\times\vec{B}}{|B|^2} + \frac{mcv_\parallel}{e|B|^2}(\vec{v}\cdot\nabla)\widehat{b}\times\vec{B}\right)\times\vec{B} \\ &\quad - D\left\{c\frac{\vec{E}_\perp\times\vec{B}}{|B|^2} - \frac{mcv_\parallel}{e|B|^2}(\vec{v}\cdot\nabla)\widehat{b}\times\vec{B}\right\} - \widehat{b}v_i(\vec{v}\cdot\nabla)\widehat{b}^i \end{aligned} \tag{16}$$

Separating the perpendicular and parallel components of the term $D\{\cdots\}$ in the right-hand side of equation (16), using the identity (15) for the perpendicular component and iterating the process one obtains

$$D(\vec{v}_\perp - \vec{u}_\perp) = -\frac{e}{mc}\vec{u}_\perp\times\vec{B} - \widehat{b}\cdot\{v_i(\vec{v}\cdot\nabla)\widehat{b}^i + D\vec{u}_\perp\}$$

and then, projecting $\left(\frac{\mathrm{d}}{\mathrm{d}t} - D\right)\vec{u}_\perp$ into perpendicular and parallel components and defining

$$\vec{u}_\perp = \left\{1 - \frac{mc}{e|B|^2}\vec{B}\times D\right\}^{-1}\left(c\frac{\vec{E}_\perp\times\vec{B}}{|B|^2} - \frac{mcv_\parallel}{e|B|^2}(\vec{v}\cdot\nabla)\widehat{b}\times\vec{B}\right) \tag{17}$$

$$\Gamma = \widehat{b}\cdot D\vec{u}_\perp + \widehat{b}\cdot\left(\frac{\mathrm{d}}{\mathrm{d}t} - D\right)\vec{u}_\perp + v_i(v\cdot\nabla)\widehat{b}^i \tag{18}$$

$$\alpha = \frac{\vec{v}_\perp - \vec{u}_\perp}{|\vec{v}_\perp - \vec{u}_\perp|^2}\bullet\left(\frac{\mathrm{d}}{\mathrm{d}t} - D\right)\vec{u}_\perp \tag{19}$$

$$\gamma = -\left(\frac{(\vec{v}_\perp - \vec{u}_\perp)}{|\vec{v}_\perp - \vec{u}_\perp|^2}\times\frac{\vec{B}}{|B|^2}\right)\bullet\left(\frac{\mathrm{d}}{\mathrm{d}t} - D\right)\vec{u}_\perp \tag{20}$$

one finally obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}(\vec{v}_\perp - \vec{u}_\perp) = \left(\frac{e}{mc} + \gamma\right)(\vec{v}_\perp - \vec{u}_\perp)\times\vec{B} - \widehat{b}\Gamma - \alpha(\vec{v}_\perp - \vec{u}_\perp), \tag{21}$$

which has the form of equation (10), as claimed.

When D is applied to the velocities it is understood that it is a replacement by the right-hand side of equations (14) and $\left(\frac{\mathrm{d}}{\mathrm{d}t} - D\right)$ is simply a derivation in the $\left(v_\perp^{(1)}, v_\perp^{(2)}\right)$ space. Therefore, for fields without explicit time dependence and if a proper meaning is given to the series in (17), one proves the existence of time-independent solutions $\vec{u}_\perp = \vec{u}_\perp(\vec{x}, \vec{v})$, $\Gamma = \Gamma(\vec{x}, \vec{v})$, $g(\vec{x}, \vec{v})$ and $\alpha(\vec{x}, \vec{v})$.

We now discuss the convergence of the formal series in (17). The operator $\underset{\sim}{O} = \frac{mc}{e|B|^2}\vec{B}\times D$ having the real line as a spectrum, $1 - \frac{mc}{e|B|^2}\vec{B} \times D$ cannot have an inverse defined in the whole of L^2. Instead we consider the action of the operator $\underset{\sim}{O}$ at successively higher orders in the space of velocities which, together with their derivatives, are assumed to be bounded by a constant L. Because of the choice of the operator D, the terms do not grow with B but, because of the nonlinear nature of the (14) action, there is a proliferation of terms and, at most, we obtain a bound

$$|\vec{u}_\perp| \leqslant \sum_n n! \left(\frac{L}{|B|}\right)^n \tag{22}$$

which does not ensure convergence of the series. However, multiplying and dividing by $n!$ and exchanging the order of sum and integral one obtains

$$|\vec{u}_\perp| \leqslant \int_d \sum_n \left(\frac{Lz}{|B|}\right)^n \mathrm{e}^{-z}\,\mathrm{d}z$$

where we have used the integral representation

$$n! = \int_d \mathrm{d}z\, z^n\, \mathrm{e}^{-z}$$

d denoting the integration along the line $(1+\varepsilon)\eta$ (η real $\in [0, \infty)$) in the complex plane. Then

$$|\vec{u}_\perp| \leqslant C \int_d \sum_n \left(1 - \frac{Lz}{|B|}\right)^{-1} \mathrm{e}^{-z}\,\mathrm{d}z$$

meaning that $|\vec{u}_\perp|$ is bounded by a series that is Borel summable along a direction not containing the real half-line. Then the formal series in (17) is also expected to be Borel summable. In this case, by a theorem of Borel [17, 18], one solution $|\underset{\sim}{\vec{u}}_\perp|$ exists for which (17) is an asymptotic series. This completes the proof of the proposition. □

Remarks.

(1) An apparently simpler construction might be carried out using $\frac{\mathrm{d}}{\mathrm{d}t}$ instead of the derivation D in equation (13). Then one would obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\vec{v}_\perp - \vec{u}'_\perp) = \frac{e}{mc}(\vec{v}_\perp - \vec{u}'_\perp) \times \vec{B} - \widehat{b} \cdot \left\{ v_i(\vec{v} \cdot \nabla)\widehat{b}^i + \frac{\mathrm{d}}{\mathrm{d}t}\vec{u}'_\perp \right\} \tag{23}$$

with $\vec{u}'_\perp$ defined by a series identical to (17) with D replaced by $\frac{\mathrm{d}}{\mathrm{d}t}$. However, in this case (22) does not hold for $\vec{u}'_\perp$ and Borel summability cannot be proved. The physical reason for this is that the derivation D does not introduce frequencies comparable to the Larmor frequency, whereas $\frac{\mathrm{d}}{\mathrm{d}t}$ does.

(2) The nature of (17) as an asymptotic series has the implication that for practical calculations the number of terms to be kept depends both on the magnitude of the velocities and of the magnetic field. The general rule is that, for an asymptotic series $\sum_n a_n$, the best approximation is obtained by keeping terms up to the order that minimizes a_n.

From the existence of a vector function $\vec{u}_\perp$, satisfying (10), an exact invariant may now be constructed

Corollary. *Existence of the functions* $\vec{u}_\perp$, α, g *and* Γ, *that provide the rewriting of the perpendicular velocity equation in the form (10) implies that*

$$M = \frac{|\vec{v}_\perp - \vec{u}_\perp|^2}{F} \tag{24}$$

is a constant of motion provided

$$F = \exp\left(-2\int^t \alpha(\vec{x}, \vec{v}, \tau)\,\mathrm{d}\tau\right) \tag{25}$$

Furthermore, existence of M implies that an exact guiding-center reduction is possible leading to a four-dimensional reduced phase space.

For uniform fields or in lowest order M is proportional to the magnetic moment but in the general case it cannot be identified with it.

That (10) and (25) imply that $\frac{\mathrm{d}}{\mathrm{d}t}M = 0$ is easily checked by a straightforward calculation. Once the existence of the invariant is established, the exact reduction follows from the Marsden–Weinstein theory [4]. M itself is the (dual) moment map (the $\widehat{J}(\xi)$ in equation (1)) that by

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \{Q, M\} \tag{26}$$

generates the action of the symmetry group on phase-space functions $Q(\vec{x}, \vec{v})$.

For each value μ of M one obtains a symplectic reduced space. Existence of a set of Q-coordinates in the reduced space follows from the existence of solutions for the linear (in Q) equation

$$\{Q, M\} = 0, \tag{27}$$

that is,

$$\frac{\partial M}{\partial v^i}\sigma^{v^i j}\frac{\partial Q}{\partial x^j} + \frac{\partial M}{\partial x^i}\sigma^{i v^j}\frac{\partial Q}{\partial v^j} + \frac{\partial M}{\partial v^i}\sigma^{v^i v^j}\frac{\partial Q}{\partial v^j} + \frac{\partial M}{\partial t}\sigma^{tK}\frac{\partial Q}{\partial K} + \frac{\partial M}{\partial v^i}\sigma^{v^i K}\frac{\partial Q}{\partial K} = 0 \tag{28}$$

for fields without explicit time dependence. Equation (28) may be written as

$$\gamma^i \frac{\partial}{\partial z^i} Q = 0 \tag{29}$$

with (in extended phase space) $\gamma = (\vec{\gamma}_x, \vec{\gamma}_v, \gamma_K)$ and $z = (\vec{x}, \vec{v}, t)$ and

$$\begin{aligned}
(\vec{\gamma}_x)^i &= (\vec{v}_\perp - \vec{u}_\perp)^j \left(\delta^{ij} - \frac{\partial \vec{u}_\perp^{\,j}}{\partial v^i}\right)\\
(\vec{\gamma}_v)^n &= (\vec{v}_\perp - \vec{u}_\perp)^j \left\{\frac{e}{mc}B_{ni}\left(\delta^{ij} - \frac{\partial \vec{u}_\perp^{\,j}}{\partial v^i}\right) + \left(v_\parallel \frac{\partial}{\partial x^n}\hat{b}^j + \frac{\partial \vec{u}_\perp^{\,j}}{\partial x^n}\right)\right\}\\
\gamma_K &= m|\vec{v}_\perp - \vec{u}_\perp|\alpha(\vec{x}, \vec{v}) + eE^i(\vec{v}_\perp - \vec{u}_\perp)^n\left(\delta_{ni} - \frac{\partial \vec{u}_\perp^{\,n}}{\partial v^i}\right).
\end{aligned} \tag{30}$$

3. Remarks and conclusions

(1) Attempting to frame gyrokinetics as an exact reduction, our methodology somehow goes in the opposite direction of the usual approach. Instead of approaching asymptotic gyroangle independence by successive near-identity transformations, we attempt to start from the establishment of exact invariants. Then, of course, because the invariant is defined by an asymptotic series, practical calculations should use truncations of the exact $\vec{u}_\perp$ and the corresponding approximations for the set of variables $\{Q\}$ in the reduced space.

(2) The reduced space coordinates are the variables that enter into the gyrokinetics reduced Vlasov equation. An iterative scheme may be developed to generate solutions of the linear system (29).

Consider an arbitrary unit vector $\hat{a}$ transverse to $\hat{b}$. We may take $\hat{a} = \frac{l\times\hat{b}}{|l\times\hat{b}|}$, l being a fixed vector in space, noncollinear with $\hat{b}$. Then define

$$\theta_0 = \frac{mc}{e|B|}\cos^{-1}\left(\frac{\hat{a}\cdot(\vec{v}_\perp - \vec{u}_\perp)}{|\vec{v}_\perp - \vec{u}_\perp|}\right), \tag{31}$$

where θ_0 is the angle variable conjugate to M in the case of uniform fields. Indeed, in this case $\gamma^i \frac{\partial}{\partial z_i}\theta_0$ reduces to

$$(\vec{v}_\perp - \vec{u}_\perp)^j \left\{\frac{\partial}{\partial x^j} + \frac{e}{mc}B_{kj}\frac{\partial}{\partial v_k}\right\}\theta_0 = 1. \tag{32}$$

We now write (29) as

$$\gamma^i \frac{\partial}{\partial z^i}(Q^{(0)} + Q^{(1)} + \cdots) = 0 \tag{33}$$

or

$$\gamma^i \frac{\partial}{\partial z^i}(Q^{(1)} + \cdots) = -\gamma^i \frac{\partial}{\partial z^i}Q^{(0)}. \tag{34}$$

Putting

$$Q^{(1)} = -\theta_0\gamma^i \frac{\partial}{\partial z^i}Q^{(0)} \tag{35}$$

one cancels the lowest order (in the field derivatives) terms on the right-hand side of (34). Iterating the procedure one would obtain

$$Q = \left\{1 - \theta_0 \sum_{n=0}(-1)^n(\gamma^i\partial_i\theta_0 - 1)^n\gamma^j\partial_j\right\}Q^{(0)}. \tag{36}$$

For the lowest order we may choose

$$\begin{aligned}\vec{Q}^{(0)} &= \vec{x} + \tfrac{mc}{e|\vec{B}|^2}(\vec{v}_\perp - \vec{u}_\perp)\times\vec{B}\\ Q_v^{(0)} &= \widehat{b}\cdot\vec{v}.\end{aligned} \tag{37}$$

Convergence of the formal series (36) will depend on the fast convergence to zero of higher order space derivatives of the fields.

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