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Symmetries and stable periodic orbits for one-dimensional maps

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Global $\phi^{-1} \circ f \circ \phi = f$ and infinitesimal $\alpha \circ f = Df(\alpha)$ symmetries are considered for dynamical maps f. The general solution of $\alpha \circ f(x) = f'(x)\alpha(x)$; $x \in [a,b] \subset \mathbb{R}$ is constructed, and for S-unimodal maps, the existence of a nontrivial continuous solution is shown to be equivalent to the existence of a stable periodic orbit.

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I. SYMMETRIES IN DISCRETE DYNAMICAL SYSTEMS

Symmetries play an important role in the study of dynamical systems defined by flows of vector fields. For Hamiltonian systems that remain invariant under the action of a group, the moment map construction[1] unifies the derivation of constants of motion. These, in turn, allow the elimination of a number of variables and the construction of a reduced phase space for the system.[2] Also, in bifurcation problems, symmetry considerations are particularly useful in the derivation of the branching equations.[3]

Although various genericity results already exist for equivariant diffeomorphisms,[4,5] symmetries do not seem to have played as important a role in the study of discrete dynamical dynamics as they have for flows.

By a *discrete dynamical system*, we mean a pair (M,f) where M is a differentiable manifold and f a smooth map. Mostly we will think of M as an open set in Euclidean space R^n. A discrete dynamical system will have a *symmetry* if f is equivariant for the action of a diffeomorphism ϕ, i.e.,

$$\phi^{-1} \circ f \circ \phi = f. \tag{1.1}$$

Now let f be equivariant for the action of a local one-parameter group ϕ_t. Let $\alpha = d\phi_t/dt\,|_{t=0}$ be the corresponding Lie algebra element. Then, differentiation of (1.1) yields

$$\alpha \circ f = Df(\alpha) \tag{1.2}$$

$\left[\text{in local Euclidean coordinates } \alpha^i(f(x)) = \alpha^j(x)\dfrac{\partial}{\partial x^j} f^i(x)\right]$.

Notice that Eq. (1.2) may, in principle, have a solution even when there is no corresponding one-parameter group of diffeomorphisms. When (1.2) has a nontrivial continuous solution, the map f will be said to have an *infinitesimal symmetry*.

As in the "continuous-time" systems, one may, in some cases, construct constants of motion from the knowledge of the symmetries. The following result is due to J. T. Duarte.[6]

Lemma: Let f be a $R^n \to R^n$ map of constant Jacobian. If f has a symmetry ϕ, then det $D\phi$ is constant on the orbits of f.

Proof: From $\phi \circ f = f \circ \phi$, differentiating, and computing the determinant, one obtains

$$\det D\phi\,(f(x)) \det Df(x) = \det Df(\phi(x)) \det D\phi\,(x).$$

Because $\det Df =$ const, the result follows:

$$\det D\phi\,(f(x)) = \det D\phi\,(x). \qquad \blacksquare$$

Notice that the same result holds if, instead of $\phi^{-1} \circ f \circ \phi = f$, one has $\phi^{-1} \circ f \circ \phi = f \circ h$ with det Df = const and det $Dh = 1$.

The main problem I will be concerned with in this paper is the relation between the existence of symmetries and the dynamical properties of maps. Here, I will restrict this study to one-dimensional maps. For them, a relation is established between nontrivial symmetries and the existence of stable periodic orbits.

II. SYMMETRIES AND STABLE PERIODIC ORBITS IN ONE-DIMENSIONAL MAPS

Let us examine the question of the existence of infinitesimal symmetries. For this purpose, one seeks the general solution $\alpha(x)$ to the functional equation

$$\alpha \circ f(x) = f'(x)\alpha(x), \tag{2.1}$$

where f is a differentiable function defined in an interval $[a,b]$ with a finite number of critical points in that interval. The method used is inspired by a technique of Kuczma[7] (see, in particular, Theorem 1.3, Chap. I, p. 40). However, the particular form of Eq. (2.1) allows an important simplification in the construction of the solution, namely, one avoids entirely explicit reference to the family f_λ^{-1} of inverses of f. Furthermore, it is also possible to define the solution in all the interval, whereas Theorem 1.3 of Kuczma gives it only in a submodulus set E $(f(E) \subset E)$.

Given a critical point x_i^c, let E_i^c be the set of antecedents of x_i^c plus the critical point itself,

$$E_i^c = \{x \colon \exists k \geqslant 0,\ f^k(x) = x_i^c\}.$$

Each E_i^c may contain other critical points besides x_i^c. We call *ground-level critical point* one for which the set E_i^c contains no other critical points [i.e., $x \in E_i^c$ and $x \neq x_i^c \Rightarrow f'(x) \neq 0$].

Z-orbit is the equivalence class defined by $x \sim y$ iff $\exists\ m,n \colon f^m(x) = f^n(y)$, $m,n \geqslant 0$. This should not be confused with the notion of *orbit* of x, i.e., $\{y \colon \exists\ k \geqslant 0, f^k(x) = y\}$. The set $E = [a,b] - U_i E_i^c$ will play an important role: In $[a,b]$ we consider the classes C_k, $k \geqslant 0$, $x \in C_k$, $k \geqslant 1$ iff $\exists\ j \colon f^{j+k}(x) = f^j(x)$, j being the smallest integer for which the equality holds, $x \in C_0$ otherwise. $U_{K>1} C_k$ is the set of points that either belong to a periodic orbit or fall into one.

We now define the class of functions which will be used to construct the general solution to the functional equation.

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A real-valued function ϕ is said to belong to the class $\psi[A]$ when the following is true.

(a) ϕ is defined in a set $A \subset [a,b]$.

(b) $\phi(x) = 0$ if $x \in C_k$ $(k \neq 0)$ and $f^{(i+k)'}(x) \neq f^{i'}(x)$, i.e., the periodic orbit where x falls has a multiplier different from unit $(f^{k'} \neq 1)$.

(c) $\phi(x) = 0$ if $x \in E$ and $\exists\ m > 0, n \geqslant 0$, and $x_i^c : f^m(x_i^c) = f^n(x)$, i.e., $x \in E$ and is in the Z-orbit of a critical point.

(d) $\phi(x) = 0$ if $x \in U_i E_i^c$ and x is not a ground level critical point or one of its antecedents.

After these preliminaries we can state the following theorem.

Theorem 2.1: Let A be a set which has exactly one point from each Z-orbit contained in $E = [a,b] - U_i E_i^c$ plus all ground level critical points. Then, for every function $\alpha_0(x)$ belonging to the class $\psi[A]$, there exists exactly one function $\alpha(x)$ defined in the interval $[a,b]$ that satisfies the functional Eq. (2.1) and such that $\alpha(x) = \alpha_0(x)$ for $x \in A$. This function is given by the following.

(a) For $x \in E$,

$$\alpha(x) = \alpha_0[a(x)]\, f^{n_x'}[a(x)]/f^{m_x'}(x), \qquad (2.2a)$$

where $a(x)$ is the point in A which belongs to the Z-orbit of x, and n_x and m_x are chosen such that

$$f^{n_x}[a(x)] = f^{m_x}(x).$$

(b) If $x \in U_i E_i^c$ and x is not a ground-level critical point or one of its antecedents, then

$$\alpha(x) = 0. \qquad (2.2b)$$

(c) If x is an antecedent of a ground-level critical point x_i^c,

$$\alpha(x) = \alpha_0(x_i^c)/f^{m_x'}(x) \qquad (2.2c)$$

with $f^{m_x}(x) = x_i^c$.

Proof: The existence of the set A follows from the axiom of choice.

It is easy to check that Eqs. (2.2) are solutions to (2.1) under each of the stated conditions. For example, for (2.2a), because $a(f(x)) = a(x)$,

$$\alpha \circ f(x) = \alpha_0[a(x)]\, \frac{f^{n_x'}[a(x)]}{f^{(m_x-1)'}[f(x)]} = f'(x)\alpha(x).$$

The choice of $\alpha_0(x)$ in the class $\psi[A]$ guarantees that

$$(f^{(j+k)'}(x) - f^{j'}(x))\alpha(x) = 0$$

holds whenever $x \in C_k$ as required by (2.1). Finally, to prove unicity, let $z(x)$ be another function satisfying (2.1) and such that $z(x) = \alpha_0(x)$, $x \in A$. Then, from $f^{n_x}[a(x)] = f^{m_x}(x)$ and $z[a(x)] = \alpha_0[a(x)]$,

$$z(f^{n_x}[a(x)]) = f^{n_x'}[a(x)]z[a(x)] = z(f^{m_x}(x)) = f^{m_x'}(x)z(x),$$

which implies

$$z(x) = \alpha_0[a(x)]\, f^{n_x'}[a(x)]/f^{m_x'}(x) = \alpha(x). \qquad \blacksquare$$

Formulas (2.2), defining a solution to the functional equation (2.1), parametrized by an arbitrary function of class $\psi[A]$, are too general to be of immediate use in the characterization of the dynamics. This arises from the fact that no practical prescription is given to find the set A, and no restrictions are imposed on $\alpha_0(x)$ besides being of class $\psi[A]$. A useful restriction is continuity of the solution. This motivated our definition of an *infinitesimal symmetry* $\alpha(x)$ of f to be a continuous solution to the functional equation (2.1) in $[a,b]$.

For the next result, I will restrict myself to the class of symmetric S-unimodal maps. A one-dimensional map f of $[-1,1]$ into itself is symmetric S-unimodal if

(1) $f(0) = 1$,
(2) $f(x) = f(-x)$,
(3) f is C^3,
(4) The Schwartzian derivative $Sf(x) < 0$, $x \neq 0$,
(5) $f'(0) = 0$ and $f'(x) \neq 0$, $x \neq 0$,
(6) f is strictly increasing in $[-1,0)$.

For this class of maps we prove the following theorem.

Theorem 2.2: For f symmetric S-unimodal, there is a continuous nontrivial solution to the functional equation (2.1) if and only if f has a stable periodic orbit.

Proof: Symmetry of f implies $\alpha(x) = -\alpha(-x)$, and continuity at $x = 0$ leads to $\alpha(0) = 0$. The functional equation then implies $\alpha = 0$ for all the antecedents of the critical point, i.e., $\alpha(x) = 0$, $x \in E^c$. If f is S-unimodal and has no stable periodic orbit, then the set of antecedents of 0 is dense in $[-1,1]$.[8] By continuity, $\alpha = 0$ in the whole interval. Therefore the existence of a stable periodic orbit is a necessary condition for a nontrivial infinitesimal symmetry.

When there is a stable periodic orbit, one proves the existence of continuous solutions by showing that the set A of Theorem 2.1 may be chosen to contain one or two intervals, where $\alpha_0(x)$ is defined as a continuous function.

Let the stable periodic orbit have period n. Then there are n fixed points of f^n, where $|f^{n'}| \leqslant 1$. A choice of semi-open B-intervals is made which is better described by Fig. 1, where

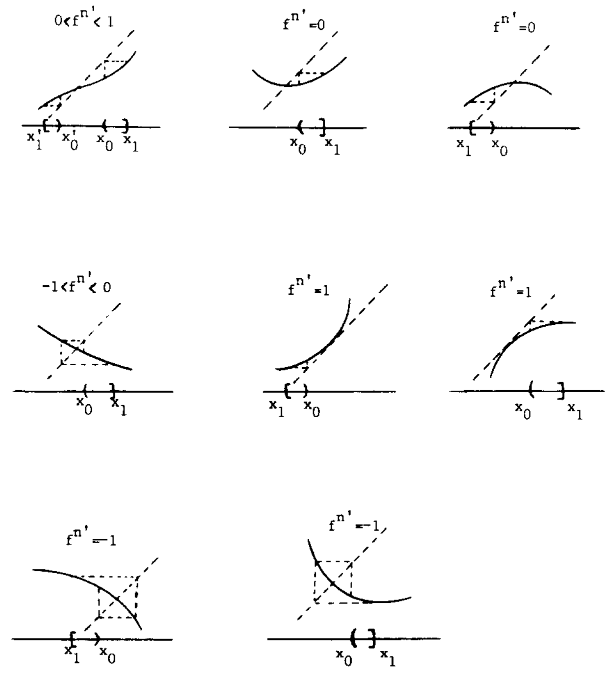

FIG. 1: B-intervals for the proof of Theorem 2.2.

the several possibilities for the behavior of f^n near the fixed points are illustrated.

It should be further specified that all B-intervals are chosen in a small enough neighborhood of one of the fixed points of f^n such that $f^k B \cap B = \varnothing, k = 1,...,n-1$. Also, by construction, $f^n B \cap B = \varnothing$.

Any two points $x,y \in B$, $x \neq y$ belong to distinct Z-orbits. Any point that converges to the stable periodic orbit is either in the Z-orbit of a point in B or in the Z-orbit of the periodic orbit.

Let A be the union of the B-interval(s) and a point ξ of the periodic orbit. For S-unimodal maps, it is known that the Lebesgue measure of those points which do not converge to the stable periodic orbit is zero. Therefore A contains one point from each Z-orbit of almost all points in the interval.

We now define a nonzero function $\alpha_0(x)$ in A such that

(1) $\alpha_0(x)$ is continuous in B,

(2) $\alpha_0(\xi) = 0$,

(3) $\lim_{x \to x_0} \alpha_0(x) = \alpha_0(x_1) = \lim_{x \to x_0'} \alpha_0(x) = \alpha_0(x_1') = 0$,

(4) $\alpha_0(x)$ is of class $\psi[A\ ]$,

(5) $\alpha_0(a(x_c)) = 0$, where $a(x_c) \in A$ is the point of A that is in the Z-orbit of the critical point.

Continuity of the solution $\alpha(x)$ now follows from the continuity of algebraic operations of continuous functions and Eqs. (2.2) of Theorem 2.1. Let y be a point that converges to the stable periodic orbit such that $a(y)$ [i.e., the point in A that belongs to the Z-orbit of y] falls in the interior of B. Then, from Eq. (2.2)

$$\alpha(y) = \alpha_0[a(y)]/f^{m_y'}(y), \tag{2.3}$$

with $a(y) = f^{m_y}(y)$.

Suppose now that for every $a \in B$, there is a point z_a in the neighborhood of y such that $a = f^{m_y}(z_a)$. In this case, by using (2.3) for all points $a \in B$, one constructs a continuous solution in an interval containing y. Furthermore, because of our condition (3) on $\alpha_0(x)$, the solution will vanish at the endpoints of the interval.

If, however, there are some points $a \in B$ which have no antecedent of order m_y in the neighborhood of y, it means that near y, there is an antecedent β of the critical point of order less than m_y. In this case, one uses (2.3) to construct the solution in an interval around y that has β as an endpoint. Again, the solution vanishes at the endpoints because they are either antecedents of the endpoints of B or of a critical point.

Now we pick another point that tends to the stable orbit where the solution is not yet defined and repeat the construction. Repeating the procedure until all points that tend to the stable orbit are exhausted, it follows from unicity and the conditions on $\alpha_0(x)$ that a continuous solution is defined in disjoint intervals vanishing at their endpoints. For the set of points outside the intervals which do not tend to the stable periodic orbit, the solution is defined to be zero. The vanishing of the solution at the endpoints of the intervals insures continuity in the whole interval $[-1,1]$. ∎

Using methods of iteration theory, the existence of continuous solutions to a functional equation was shown to be equivalent to the existence of a stable periodic orbit. This result provides an analytical handle on a dynamical problem because, in some cases, the existence of solutions to a functional equation can be decided purely by methods of analysis.

To close, I will add some remarks concerning directions of future research in this problem.

First, it would be interesting to know whether the relation between existence of symmetries and characteristic properties of the orbits can be extended to higher-dimension maps.

Once the relation between dynamical properties of maps and solutions of functional equations is established, such a relation becomes useful if, by analytical methods, one can decide about the existence or nonexistence of solutions with the required properties.

At least for one-dimensional maps, this could be achieved by writing the solution as a series in orthogonal functions $G(x)$ in $[-1,1]$:

$$\alpha(x) = \sum \alpha_n\ G_n(x). \tag{2.4}$$

By substitution in the functional equation (2.1), this one is converted into an (infinite) set of algebraic relations. If $\max_{[-1,1]}|G_n(x)|$ is finite, for $\alpha(x)$ to be continuous, it suffices that the solution $\alpha = \{\alpha_n\} \in l^1$.

For $f(x) = 1 - \mu x^2$, for example, if $\{G_n(x)\}$ is the set of Chebyshev polynomials, the proof of aperiodicity of the $\mu = 2$ point becomes a trivial calculation because the only solution is $\alpha_n = 0$.

With other orthogonal sets of appropriate weight, it is, in principle, possible to decide by purely analytical methods on the existence of stable periodic orbits for other μ values.

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