Journal of Mathematical Physics

Decomposition of vector fields and mixed dynamics

R. Vilela Mendes and J. Taborda Duarte

Citation: J. Math. Phys. **22**, 1420 (1981); doi: 10.1063/1.525063
View online: http://dx.doi.org/10.1063/1.525063
View Table of Contents: http://jmp.aip.org/resource/1/JMAPAQ/v22/i7
Published by the AIP Publishing LLC.

Decomposition of vector fields and mixed dynamics

R. Vilela Mendes and J. Taborda Duarte
CFMC-Instituto Nacional de Investigação Cientifica, Av. Gama Pinto, 2-1699 Lisboa Codex, Portugal

(Received 16 July 1980; accepted for publication 20 January 1981)

Some theorems are proved concerning the decomposition of vector fields into gradient and Hamiltonian components. A constructive method to carry out one of the decompositions is applied to some three- and four- dimensional dynamical models.

PACS numbers: 03.20. + i

1. DECOMPOSITION OF VECTOR FIELDS

A classical dynamical system is defined by the couple (M,X), M being a differentiable manifold and X a C^r vector field. The study of simple ways to describe general vector fields will lead therefore to a parametrization and classification of classical dynamical systems. A step in this direction was taken by Roels[1] who proved that in a two-dimensional symplectic manifold every vector field is locally the sum of a Hamiltonian and a gradient field. Our purpose in this paper is to obtain similar decompositions for N-dimensional manifolds. The proof of the main result uses the following local lemma.

Lemma : Let R^N (N even) be endowed with the canonical scalar product. Then on every compact neighborhood Ω there are $N-1$ nondegenerate 2-forms α_i with the properties :

(a) $d\alpha_i = 0$,

(b) $\underbrace{\alpha_i \wedge \cdots \wedge \alpha_i}_{N/2} = (N/2)!v$ (v volume form on R^N),

(c) $*\alpha_i = \frac{1}{(N/2-1)!} \underbrace{\alpha_i \wedge \cdots \wedge \alpha_i}_{N/2-1}$,

(d) $*\alpha_i \wedge \alpha_j = 0, \quad i \neq j$,

and such that given a C^∞ 2-form η, there are $N-1$ C^∞ functions a_i and a 2-form α satisfying locally:

(1) $\delta\alpha = \delta\eta$,

(2) $\alpha = \sum_{i=1}^{N-1} a_i \alpha_i$.

For the proof one uses an Euclidean coordinate system. In these coordinates a constructive recipe for a set of 2-forms α_i is

$$\begin{aligned} \alpha_1 &= dx^1 \wedge dx^2 + dx^{i_{13}} \wedge dx^{i_{14}} + \cdots + dx^{i_{1(N-1)}} \wedge dx^{i_{1N}}, \\ \alpha_2 &= dx^1 \wedge dx^3 + dx^{i_{23}} \wedge dx^{i_{24}} + \cdots + dx^{i_{2(N-1)}} \wedge dx^{i_{2N}}, \\ &\cdots \\ \alpha_{N-1} &= dx^1 \wedge dx^N + dx^{i_{(N-1)3}} \wedge dx^{i_{(N-1)4}} + \cdots \\ &\quad + dx^{i_{(N-1)(N-1)}} \wedge dx^{i_{(N-1)N}}, \end{aligned} \tag{1.1}$$

where in α_p the numbers $1, p+1, i_{p3}, i_{p4}, \cdots, i_{pN}$ are an even permutation of $1 \cdots N$, and no elementary 2-form $dx^i \wedge dx^j$ appears more than once in (1.1). For example, for $N = 4$ and 6, one has

$$\begin{aligned} \alpha_1 &= dx^1 \wedge dx^2 + dx^3 \wedge dx^4, \\ \alpha_2 &= dx^1 \wedge dx^3 + dx^4 \wedge dx^2, \\ \alpha_3 &= dx^1 \wedge dx^4 + dx^2 \wedge dx^3, \\ \alpha_1 &= dx^1 \wedge dx^2 + dx^3 \wedge dx^4 + dx^5 \wedge dx^6, \\ \alpha_2 &= dx^1 \wedge dx^3 + dx^2 \wedge dx^5 + dx^4 \wedge dx^6, \\ \alpha_3 &= dx^1 \wedge dx^4 + dx^2 \wedge dx^6 + dx^3 \wedge dx^5, \\ \alpha_4 &= dx^1 \wedge dx^5 + dx^2 \wedge dx^4 + dx^3 \wedge dx^6, \\ \alpha_5 &= dx^1 \wedge dx^6 + dx^2 \wedge dx^3 + dx^4 \wedge dx^5. \end{aligned} \tag{1.2}$$

It is straightforward to check that for general N the forms constructed according to (1.1) are nondegenerate and satisfy the conditions (a)–(d). Clearly for $N \geqslant 6$ one does not obtain a unique set.

To prove the lemma, one should now check that given a smooth 2-form η it is possible to find $N-1$ functions a_i such that

$$\delta \sum_{i=1}^{N-1} a_i \alpha_i = \delta\eta. \tag{1.3}$$

In Euclidean coordinates the codifferential of a 2-form β reads $\delta\beta = \partial_j \beta_{ij}\, dx^i$. Therefore, from the knowledge of the α_i forms (1.1), one writes (1.3) as a simple first-order partial differential system. To avoid the introduction of cumbersome index notation we will merely illustrate this for $N = 4$ and $N = 6$:

$$\begin{vmatrix} \partial_2 & \partial_3 & \partial_4 \\ -\partial_1 & -\partial_4 & \partial_3 \\ \partial_4 & -\partial_1 & -\partial_2 \\ -\partial_3 & \partial_2 & -\partial_1 \end{vmatrix} \begin{vmatrix} a_1 \\ a_2 \\ a_3 \end{vmatrix} = \sum_j \partial_j \eta_{ij},$$

$$\begin{vmatrix} \partial_2 & \partial_3 & \partial_4 & \partial_5 & \partial_6 \\ -\partial_1 & \partial_5 & \partial_6 & \partial_4 & \partial_3 \\ \partial_4 & -\partial_1 & \partial_5 & \partial_6 & -\partial_2 \\ -\partial_3 & \partial_6 & -\partial_1 & -\partial_2 & \partial_5 \\ \partial_6 & -\partial_2 & -\partial_3 & -\partial_1 & -\partial_4 \\ -\partial_5 & -\partial_4 & -\partial_2 & -\partial_3 & -\partial_1 \end{vmatrix} \begin{vmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{vmatrix} = \sum_j \partial_j \eta_{ij}.$$

The general rule for writing the matrix of partial derivatives in

$$\sum_{j=1}^{N-1} M(\partial)_{ij} a_j = \sum_{j=1}^{N} \partial_j \eta_{ij} \quad (i = 1 \cdots N) \tag{1.4}$$

is that, if the form $dx^i \wedge dx^j$ occurs in α_r, then the i, r and j, r matrix elements are ∂_j and $-\partial_i$.

By a smooth truncation of $\Sigma \partial_j \eta_{ij}$ outside the neighborhood Ω, one replaces the system (1.4) by

 0022-2488/81/071420-03$1.00

$$\sum_{j=1}^{N-1} M(\partial)_{ij} a_j = u_i \quad (i = 1 \cdots N), \tag{1.5}$$

where the u_i are $C_0^\infty(R^N)$ functions that coincide with $\Sigma \partial_j \eta_{ij}$ in Ω. The solutions of (1.5) will also coincide with solutions of (1.4) in Ω.

The system (1.5) has $N-1$ unknowns a_j and N equations. However, not all equations are independent because by construction

$$\sum_{i=1}^{N} \xi_i M(\xi)_{ij} = 0 \quad (j = 1 \cdots N-1).$$

Eliminating one of the rows in $M(\partial)_{ij}$, one is led to a system of $N-1$ equations with $N-1$ unknowns,

$$\sum_{j=1}^{N-1} \widetilde{M}(\partial)_{ij} a_j = u_i \quad (i = 1 \cdots N-1), \tag{1.6}$$

whose det $\widetilde{M}(\xi)$ is not identically zero.

The existence of a fundamental solution ($\widetilde{M}(\partial)E = \delta \mathbb{1}$) follows from the existence of a fundamental solution for the differential operator with constant coefficients det $\widetilde{M}(\partial)$.[2] From the fundamental solution, by convolution with the C_0^∞ functions u_i, one finally proves the existence of C^∞ solutions a_j to (1.4) in Ω.

For $N = 4$ the lemma is equivalent to the statement that there is a self-dual α such that $\delta\alpha = \delta\eta$. In this case one can apply the Hodge–de Rham theorem to write $\eta = d\beta + \delta\gamma$, and choosing $\alpha = d\beta + {}^*d\beta$, one proves the assertion in a coordinate free manner. Unfortunately, we could not find a similar proof for higher dimensions.

On the other hand, the coordinatewise proof of the lemma and in particular the system (1.4) provides a constructive method to obtain in practice the decomposition of vector fields whose existence is asserted in the following theorem.

Theorem 1: Given an N-dimensional (N even) C^∞-manifold we can find for every $x \in M$ a nbd Ω of x, a Riemannian metric $\tilde{g}$, and $N-1$ symplectic forms $\tilde{\alpha}_i$ on Ω such that every vector field X defined on the nbd can be decomposed into one gradient and $N-1$ Hamiltonian fields.

Let $\phi: \Omega \to R^N$ be a chart around x such that $\phi(x) = 0$. Defining $\tilde{g}$ as the pullback by ϕ of the Euclidean metric, and $\tilde{\alpha}_i$ as $\phi^*(\alpha_i)$, we observe that the 2-forms $\tilde{\alpha}_i$ have the same properties as the forms α_i in the lemma.

Let $\tilde{g}^b: \mathscr{X}(\Omega) \to \Omega^1(\Omega)$ be the isomorphism from the vector fields onto the 1-forms induced by $\tilde{g}$, and $\tilde{g}^\#$ its inverse. By the Hodge–de Rham theorem and Poincaré's lemma,

$$\tilde{g}^b(X) = dS + \delta\tilde{\eta}.$$

Hence $X = \tilde{g}^\#(dS) + \tilde{g}^\#(\delta\tilde{\eta})$, and $\tilde{g}^\#(dS)$ is a gradient vector field. For $\delta\tilde{\eta}$ we can write $\delta\tilde{\eta} = \delta(\Sigma_i^{N-1} b_i \tilde{\alpha}_i)$, where b_i are C functions defined on Ω, simply by carrying $\tilde{\eta}$ to R^N by the chart and applying the lemma. We have then

$$\tilde{g}^\#(\delta\tilde{\eta}) = \sum_i^{N-1} \tilde{g}^\#(\delta(b_i \tilde{\alpha}_i)).$$

It remains to prove that each $\tilde{g}^\#(\delta(b_i \tilde{\alpha}_i))$ is a Hamiltonian vector field for the symplectic form $\tilde{\alpha}_i$. This follows from a computation that uses the properties of the $\tilde{\alpha}_i$ and the equalities $i(Z)\alpha = \tilde{*}(\tilde{*}\alpha \wedge \tilde{g}^b(Z))$, and

$$\tilde{*} i(Z)\tilde{v} = -(-1)^N \tilde{g}_b(Z),$$

where $\tilde{*}$ and $\tilde{v}$ are the Hodge star and the volume form associated to $\tilde{g}$:

$$\begin{aligned} i(\tilde{g}^\# \delta(b_i \tilde{\alpha}_i))\tilde{\alpha}_i &= -i(\tilde{g}^\# \tilde{*}(db_i \wedge \tilde{*}\tilde{\alpha}_i))\tilde{\alpha}_i \\ &= -\tilde{*}(\tilde{*}\tilde{\alpha}_i \wedge \tilde{*}(db_i \wedge \tilde{*}\tilde{\alpha}_i)) \\ &= \frac{-1}{(N/2-1)!}\tilde{*}(\underbrace{\tilde{\alpha}_i \wedge \cdots \wedge \tilde{\alpha}_i}_{N/2-1} \wedge i(\tilde{g}^\# db_i)\tilde{\alpha}_i) \\ &= \frac{-1}{(N/2)!}\tilde{*} i(\tilde{g}^\# db_i)(\underbrace{\tilde{\alpha}_i \wedge \cdots \wedge \tilde{\alpha}_i}_{N/2}) = -\tilde{*} i(\tilde{g}^\# db_i)\tilde{v} \\ &= db_i \end{aligned}$$

The theorem states that, locally at least, one can erect a system of $N-1$ symplectic forms that together with the metric form a fixed framework enabling us to decompose any smooth motion into elementary gradient and Hamiltonian components. This is the situation that seems to be the most useful for the applications. However, there exists a different decomposition problem when for a given vector field one is allowed to choose either a metric or a symplectic form adapted to that particular vector field. The following results are almost trivial consequences of the "flow box" theorem.[3]

Theorem 2: Let X be a vector field on a Riemannian manifold M_g. Then for each $p \in M_g$ there is a neighborhood Ω of p and a symplectic form ω_X on Ω such that X is decomposed in, at most, one gradient and one Hamiltonian vector field.

Proof: Take $p \in M$. Either $X(p) \neq 0$ or $X(p) = 0$. If $X(p) \neq 0$ by the flow box theorem there is a neiborhood Ω and a local diffeomorphism $\phi: \Omega \to R^N$, $\phi(y) = (y_1, \ldots, y_N)$ such that $\phi_*(X) = \partial/\partial y_1$. Then $\phi_*(X)$ is Hamiltonian for the canonical symplectic form $\omega = \Sigma_{i=1}^{N/2} dy_{2i-1} \wedge dy_{2i}$ in R^N. Then X is Hamiltonian in Ω for $\phi^*\omega = \omega_X$.

If $X(p) = 0$, take X_g any gradient vector field (for the metric g) such that $X_g(p) \neq 0$. Then $Y = X + X_g$ does not vanish at p and we can apply the above argument so that Y is Hamiltonian, i.e., $X = X_g - Y$ as stated.

Theorem 2′: Let X be a vector field on a symplectic manifold M_ω. Then for each $p \in M$, there is a neighborhood and a Riemannian metric g_X on Ω such that X is decomposed in, at most, one gradient and one Hamiltonian vector field.

Proof: Take $p \in M$. Either $X(p) \neq 0$ or $X(p) = 0$. If $X(p) \neq 0$ by the flow box theorem we can find a nbd of p and a metric g_X on Ω such that X be gradient. If $X(p) = 0$, we choose a Hamiltonian vector field X_ω such that $X_\omega(p) \neq 0$. Then we take $Y = X + X_\omega$ and apply again the same argument.

Although simpler than those of Theorem 1, these decompositions are probably of little practical value because to find the flow box coordinate system is equivalent to finding the orbits. Hence, to write such a decomposition should not be much simpler than to find an exact solution of the equations of motion.

The results in this paper imply that general classical motions are mixed Hamiltonian or mixed Hamiltonian-gradient systems. Besides the obvious parametrization usefulness of such decompositions they may also provide new ways of studying classical systems, for example by developing a perturbative theory of gradient deformations of Hamiltonian systems. Preliminary results in this direction indicate that at least it is then simple to establish necessary conditions for the existence of constants of motion in dissipative systems.[7]

2. EXAMPLES

The proofs of Theorem 1 and the lemma provide a constructive method to obtain the corresponding decomposition once a vector field X is given. In Euclidean coordinates the function S of the gradient part is obtained as a solution of the Poisson equation

$$\Delta S = \operatorname{div} X,$$

and with the choice of the symplectic forms (1.1) the Hamiltonian functions are obtained by solving the differential system (1.4). Using this method on finds:

= for the van der Pol oscillator,

$$\dot{x} = y = \frac{\partial S}{\partial x} + \frac{\partial H}{\partial y}, \quad S = \alpha\left(\frac{x^2}{2} - \frac{x^4}{12}\right),$$

$$\dot{y} = \alpha(1 - x^2)y - x = \frac{\partial S}{\partial y} - \frac{\partial H}{\partial x},$$

$$H = \frac{y^2}{2} - \alpha\left(x - \frac{x^3}{3}\right)y + \frac{x^2}{2}.$$

= for Rossler's model for "hyperchaos,"[4]

$$\dot{x} = -y - z = \frac{\partial S}{\partial z} + \frac{\partial H}{\partial y} + \frac{\partial H'}{\partial z},$$

$$\dot{y} = x + \frac{y}{4} + w = \frac{\partial S}{\partial y} - \frac{\partial H}{\partial x} - \frac{\partial H'}{\partial w},$$

$$\dot{z} = 3 + xz = \frac{\partial S}{\partial z} + \frac{\partial H}{\partial w} - \frac{\partial H'}{\partial x},$$

$$\dot{w} = -\frac{z}{2} + 0.05w = \frac{\partial S}{\partial w} - \frac{\partial H}{\partial z} + \frac{\partial H'}{\partial y},$$

with

$$S = \tfrac{1}{6}x^3 + 0.3w^2/2 \quad H = -\tfrac{1}{2}(x^2 + y^2) + \tfrac{1}{4}z^2 + 3w,$$

$$H' = -\tfrac{1}{2}x^2 z - \tfrac{1}{2}z^2 - \tfrac{1}{2}w^2 - \tfrac{1}{4}yw.$$

Systems of odd dimensionality N may always be imbedded into a $(N + 1)$-dimensional manifold, the same methods become applicable and one obtains:

= for the Lorenz model[5]

$$\dot{x} = -\sigma x + \sigma y = \frac{\partial S}{\partial x} + \frac{\partial H}{\partial y},$$

$$\dot{y} = -xz + rx - y = \frac{\partial S}{\partial y} - \frac{\partial H}{\partial x},$$

$$\dot{z} = xy - bz = \frac{\partial S}{\partial z},$$

with

$$S = \tfrac{1}{2}\sigma x^2 - \tfrac{1}{2}y^2 - \tfrac{1}{2}bz^2 + xyz,$$

$$H = \tfrac{1}{2}y^2(\sigma - z) + x^2(z - \tfrac{1}{2}r).$$

= for the Gause–Lotka–Volterra equations (3 species)[6]

$$\dot{x} = x(1 - x - \alpha y - \beta z) = \frac{\partial S}{\partial x} + \frac{\partial H}{\partial y} + \frac{\partial H'}{\partial z},$$

$$\dot{y} = y(1 - \beta x - y - \alpha z) = \frac{\partial S}{\partial y} - \frac{\partial H}{\partial x},$$

$$\dot{z} = z(1 - \alpha x - \beta y - z) = \frac{\partial S}{\partial z} - \frac{\partial H'}{\partial x},$$

with

$$S = \tfrac{1}{2}(x^2 + y^2 + z^2) - \tfrac{1}{6}(2 + \alpha + \beta)(x^3 + y^3 + z^3),$$

$$H = -\tfrac{1}{2}(\alpha + \beta)xy^2 + y(\tfrac{1}{2}\beta x^2 + \alpha xz),$$

$$H' = -\tfrac{1}{2}(\alpha + \beta)xz^2 + z(\tfrac{1}{2}\alpha x^2 + \beta xy).$$

AKNOWLEDGMENTS

The authors are grateful to Professor Claude Bruter for bringing to their attention the potential value of generalizing Roels' Theorem.

[1] J. Roels, C. R. Acad. Sci. Ser. A **278**, 29 (1974).
[2] L. Hörmander, *Linear Partial Differential Operators* (Springer, Berlin, 1969), Chap. III.
[3] F. Dumortier, "Singularities of vector fields," IMPA math. monogr. No. 32, Rio de Janeiro, 1978, p. 8.
[4] O. E. Rossler, Phys. Lett.A **71**, 155 (1979).
[5] E. N. Lorenz; J. Atmos. Sci. **20**, 130 (1963).
[6] R. M. May, W. J. Leonard, SIAM J. Appl. Math. **29**, 243 (1975).
[7] J. T. Duarte and R. V. Mendes, "Deformation of Hamiltonian dynamics and constants of motion in dissipative systems" preprint CFMC, E-15/80.