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Deformation of Hamiltonian dynamics and constants of motion in dissipative systems

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A necessary condition for the existence of arcs of vector fields with constants of motion is found. The result is applied to arcs obtained by deformation of Hamiltonian dynamics and illustrated in the Van der Pol and Lorenz models.

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I. PRELIMINARIES

For the vast majority of nonlinear dynamical systems it is usually extremely difficult to extract analytical information from the defining equations. A possile exception arises when the system can be shown to belong to some family and one of the members of the family is either exactly solvable or has nice properties. Then, in some cases, one can use the simpler system to obtain information on the properties of the family. This is the situation in the perturbation theory around the linear approximation, where the validity of extrapolation from the linear to the perturbed system hinges on the smallness of some physical parameter.

Another example arises when in $\dot{x} = X(x)$ the vector field can be decomposed into components, each having well-studied properties; for example, $X = X^{(S)} + X^{(V)}$, $X^{(S)}$ being a gradient and $X^{(V)}$ a volume-conserving or even a Hamiltonian field. In this case one could, for example, define a family (arc of vector fields) $X_\epsilon = 2\epsilon X^{(S)} + 2(1-\epsilon)X^{(V)}$ which for $\epsilon = \frac{1}{2}$ coincides with the original system and for $\epsilon = 0$ and $\epsilon = 1$ is purely (V) on purely (S). Properties of the original system can therefore be obtained from the deformation of its components. In this case, one cannot rely on the smallness of the deformation parameter ϵ and the validity of extrapolations must depend on the differentiability properties (in ϵ) of the arc.

The main purpose in the research that led to this paper was to identify families of dynamics which would supply nonperturbative information about its members once the analytical behavior of one of them is known. In this sense one might say that the main result is the notion of "arc of vector fields with constants of motion" (Definition 2). Once one realizes that the ingredients in the definition are what one needs to carry nonperturbative information along the family, the remaining results are a matter of computation using the differentiability properties of composite maps.

In Sec. 2 a necessary condition is obtained for the existence of the arcs (Theorem 1), which is then particularized for arcs obtained by deformation of Hamiltonian dynamics (Theorem 2).

As far as applications are concerned, we suggest that our results might be useful to help understand the transition between conservative and dissipative regimes.

When dissipation is added to a conservative system, there is typically a reduction of the phase space with the system tending at $t \to \infty$ to an attracting subset of lower dimension. Looking at dissipative systems as belonging to some family obtained by deformation of a conservative dynamics, one might hope to find which subsets in the conservative phase space do not change qualitatively when dissipation is turned on. In other words, although conservative dynamics is structurally unstable, it may happen that when restricted to some subsets of the phase space it remains stable for dissipative deformations. Finding the deformation stable subsets would supply analytical information on the nature and approximate location of the attractors.

Although the purpose of this paper is not to make a full exploration of the mathematical results of Sec. II, we have included as an illustration in Sec. III a study of the van der Pol and Lorenz models which were chosen for their simplicity and availability of numerical studies.

To conclude this section a quick review is made of some mathematical results needed in the sequel. All the material can be found with proofs in Ref. 1.

Let E and F be Banach spaces and U an open set in E. We denote by $C_b^k(U,F)$ the set of C^k maps $f: U \to F$ with the first k derivatives $D^k f: U \to L_s^k(E,F)$ bounded in U, that is, $\exists$ $k(f) \in \mathbb{R}^+$ such that $\sup_{x\in U} \|D^i f(x)\| < k(f)$ for all $0 \leqslant i \leqslant k$.

For $F = E$ one also uses the notation $\mathscr{X}_b^k(U)$ instead of $C_b^k(U,E)$. In this case one thinks of $f \in \mathscr{X}_b^k(U)$ as a C^k-vector field.

For $f \in C_b^k(U;F)$ define the norm $\|f\|_k = \Sigma_{i=0}^k \sup_{x\in U} \|D^i f(x)\|$. We will use the following:

Theorem: (1) $C_b^k(U,F)$ is a Banach space. (2) (i) Let M be a compact interval in R and U an open set in F; then $C^k(M,U) = \{ f \in C^k(M,F) \mid f(M) \subset U \}$ is open in $C^k(M,F)$.

(ii) The map comp: $C_b^{r+s}(U,E) \times C^r(M;U) \to C^r(M;E)$ is a C^s map, its derivative being given by

$$D\,\mathrm{comp}(f,g)(\gamma,h)(x) = Df(g(x))(h(x)) + \gamma(g(x)),$$

$$\left.\begin{matrix} f \in C_b^{r+s}(U,E), & \gamma \in C_b^{r+s}(U,E), \\ g \in C^r(M;U), & h \in C^r(M;U) \end{matrix}\right\}, \quad x \in M.$$

II. ARCS OF VECTOR FIELDS AND CONSTANTS OF MOTION

Definition 1: Let (M,X) be a differentiable dynamical system. *A constant of motion* of (M,X) is any differentiable function $\Phi: M \to R$ such that for some solution γ of X we have $\Phi \circ \gamma = \mathrm{const}$.

This notion is a generalization of the concept of first

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integral. Many systems which have no nontrivial first integrals have constants of motion. Below we will define a class of dynamical systems which is of special interest to us.

Definition 2: Let M be a differentiable manifold. A family $\epsilon\mapsto X_\epsilon$, defined by associating with each $\epsilon\in I = [-a,a]$ a vector field over M, is called an *arc of vector fields with constants of motion* if the following conditions are satisfied:

(i) Each X_ϵ has a constant of motion ϕ_ϵ over a periodic solution γ_ϵ.

(ii) The constant of motion ϕ_0 of X_0 is a first integral in a neighborhood of γ_0.

(iii) The maps, $\epsilon\mapsto X_\epsilon$, $\epsilon\mapsto\gamma_\epsilon$, and $\epsilon\mapsto\phi_\epsilon$, resp. $I\to\mathscr{X}^1_b(U)$, $I\to C^1_b(\mathbb{R};U)$, and $I\to C^1_b(U;\mathbb{R})$, are C^1-differentiable, U being an open set in M.

Then we prove the following:

Theorem 1: Let $\epsilon\mapsto X_\epsilon$ be an arc of vector fields with constants of motion defined on an open set U of a Banach space E. Then there is an (X_0-dependent) nontrivial 2-form β on U such that

$$\int_{\gamma_0} i\left(\frac{d}{d\epsilon}X_{\epsilon|_{\epsilon=0}}\right)\beta = 0. \tag{2.1}$$

Remark: Eq. (2.1) can also be written as

$$\int_0^{T_0} i\left(\frac{d}{d\epsilon}X_{\epsilon|_{\epsilon=0}}\right)(d\phi_0)\gamma_0(t)\,dt = 0. \tag{2.2}$$

Proof: The steps used in the proof are:

(1) Take the ϵ derivative at $\epsilon = 0$.

(2) Take the t derivative.

(3) Introduce the assumption that ϕ_0 is a first integral.

Let $T(\epsilon)$ be the period of the periodic solutions γ_ϵ, $T = \sup_{\epsilon\in I} T(\epsilon)$ and denote $\widetilde{M} = [0,T]$.

First step: Consider the diagram

$$I \xrightarrow{A} C^2_b(U,\mathbb{R})\times C^1(\widetilde{M},U) \xrightarrow{\text{comp}} C^1(\widetilde{M},\mathbb{R})$$

defined by

$$\epsilon\mapsto(\phi_\epsilon,\gamma_\epsilon)\mapsto\phi_\epsilon\circ\gamma_\epsilon.$$

Therefore

$$\frac{d}{d\epsilon}(\phi_\epsilon\circ\gamma_\epsilon)_{|_{\epsilon=0}} = \frac{d}{d\epsilon}(\text{comp}\circ A)(\epsilon)_{|_{\epsilon=0}}.$$

Applying the chain rule and the theorem quoted in Sec. I, we conclude that $(d/d\epsilon)(\phi_\epsilon\circ\gamma_\epsilon)_{|_{\epsilon=0}}$ is a $C^1(\widetilde{M},R)$ map that associates with each $t\in\widetilde{M}$ the real number

$$D\phi_0(\gamma_0(t))\left(\frac{d}{d\epsilon}\gamma_{\epsilon|_{\epsilon=0}}(t)\right) + \frac{d}{d\epsilon}\phi_{\epsilon|_{\epsilon=0}}(\gamma_0(t))$$

or, denoting by primes the ϵ derivatives,

$$\frac{d}{d\epsilon}(\phi_\epsilon\circ\gamma_\epsilon)_{|_{\epsilon=0}} = D\phi_0(\gamma_0(t))(\gamma_0'(t)) + \phi_0'(\gamma_0(t)) = K_0', \tag{2.3}$$

where the last equality follows from the definition of constant of motion $(\phi_\epsilon\circ\gamma_\epsilon)(t) = K_\epsilon$.

Second step: As $K\in C^2(I\times\widetilde{M})$, it follows that $(d/dt)K'_{\epsilon|_{\epsilon=0}}$ is identically zero.

Taking the t derivative of Eq. (2.3),

$$D\phi_0'(\gamma_0(t))[X_0(\gamma_0(t))] + D\phi_0(\gamma_0(t))(\dot{\gamma}_0'(t)) + D^2\phi_0(\gamma_0(t))(\dot{\gamma}_0(t),\gamma_0'(t)) = 0$$

where the dot denotes the t derivative.

From $\dot{\gamma}_0 = X_0\circ\gamma_0$, as γ_0 is a solution to X_0, we compute $\dot{\gamma}_0'(t)$:

$$\dot{\gamma}_0'(t) = DX_0(\gamma_0(t))(\gamma_0'(t)) + X_0'(\gamma_0(t))$$

and obtain

$$D\phi_0'(\gamma_0(t))(X_0(\gamma_0(t))) + D\phi_0(\gamma_0(t))(DX_0(\gamma_0(t))(\gamma_0'(t))) + D\phi_0(\gamma_0(t))(X_0'(\gamma_0(t))) + D^2\phi_0(\gamma_0(t))(\dot{\gamma}_0(t),\gamma_0'(t)) = 0. \tag{2.4}$$

Third step: Because ϕ_0 is a first integral of X_0 in an open nbd V of γ_0, we have $\forall\, x\in V\; D\phi_0(x)(X_0(x)) = 0$; hence, deriving the map $x\mapsto D\phi_0(x)(X_0(x))$, we obtain for $x\in V$ and $y\in E$:

$$D\phi_0(x)(DX_0(x)(y)) + D^2\phi_0(x)(X_0(x),y) = 0.$$

In particular, for $x = \gamma_0(t)$ and $y = \gamma_0'(t)$,

$$D\phi_0(\gamma_0(t))(DX_0(\gamma_0(t))(\gamma_0'(t))) + D^2\phi_0(\gamma_0(t))(X_0(\gamma_0(t)),\gamma_0'(t)) = 0 \quad \forall\; t\in\widetilde{M}. \tag{2.5}$$

Equation (2.4) reduces then to

$$D\phi_0'(\gamma_0(t))(X_0(\gamma_0(t)) + D\phi_0(\gamma_0(t))(X_0'(\gamma_0(t))) = 0 \quad \forall\; t\in\widetilde{M}. \tag{2.6}$$

Denoting by i the interior multiplication of a differential form by a vector field:

$$i(X_0)(d\phi_0')(\gamma_0(t)) + i(X_0')(d\phi_0)(\gamma_0(t)) = 0. \tag{2.7}$$

Let T_0 be the period of γ_0. As $[0,T_0]\subset[0,T]$, we restrict ourselves to the interval $[0,T_0]$ and integrate

$$\int_0^{T_0} i(X_0)(d\phi_0')(\gamma_0(t))\,dt + \int_0^{T_0} i(X_0')(d\phi_0)(\gamma_0(t))\,dt = 0;$$

γ_0 being a solution of X_0, this becomes

$$\int_{\gamma_0} d\phi_0' + \int_0^{T_0} i(X_0')(d\phi_0)(\gamma_0(t))\,dt = 0,$$

and by Stokes' theorem

$$\int_0^{T_0} i(X_0')(d\phi_0)(\gamma_0(t))\,dt = 0. \tag{2.2}$$

Because ϕ_0 is a first integral of X_0,

$$i(X_0)(d\phi_0) = 0,$$

in a neighborhood V of γ_0, we can find a differential 2-form β such that

$$d\phi_0 = i(X_0)\beta \quad \text{on} \quad V,$$

and from Eq. (2.2) one obtains

$$\int_0^{T_0} [i(X_0')i(X_0)\beta](\gamma_0(t))\,dt = 0,$$

i.e.,

$$\int_{\gamma_0} i(X_0')\beta = 0. \tag{2.1}$$

In the theorem we have just proved X_0 may be any vector field with a first integral. Of special interest for the applications is the case where X_0 is an Hamiltonian field.

From Ref. 2 we recall the main result in that paper:

"If M is a diff manifold $X\in\mathscr{X}(M)$ and $x\in M$, then it is possible to find, on an nbd Ω of x, a Riemannian metric g and

$N-1$ sympletic forms ω_i such that in Ω, $X = X_s + \Sigma_{i=1}^{N-1} X_{H_i}$, where X_s is gradient w.r.t. g and the X_{H_i} Hamiltonian w.r.t. ω_i.

We can then obtain the following:

Theorem 2: Let X be a vector field on $U \subset \mathbb{R}^N$ and $\{X_S, X_{H_1},...,X_{H_{N-1}}\}$ its gradient and Hamiltonian components. If the family $\epsilon \mapsto X_\epsilon$ defined by $X_\epsilon = X_{H_i} + \epsilon(X_S + \Sigma_{k\neq i} X_{H_k})$ is an arc of vector fields with constants of motion, then there is a solution γ_0 of X_{H_i} such that

$$\int_0^{T_0} \left[(\nabla H_i \cdot \nabla S) + \sum_{k\neq i} \omega_k(\nabla H_i, \nabla H_k)\right] \gamma_0(t)) \, dt = 0. \tag{2.8}$$

Proof:

$$\frac{d}{d\epsilon} X_{\epsilon|_{\epsilon=0}} = X_S + \sum_{k\neq i} X_{H_k};$$

hence

$$\int_0^{T_0} i(X_0')(dH_i)(\gamma_0(t))\, dt = 0.$$

Let us make explicit the integrand function

$$\begin{aligned} i(X_0')(dH_i) &= i\left(X_S + \sum_{k\neq i} X_{H_k}\right)(dH_i) \\ &= i(X_S)(dH_i) + \sum_{k\neq i} i(X_{H_k})(dH_i) \\ &= (dH_i)(X_S) + \sum_{k\neq i} i(X_{H_k})(dH_i), \end{aligned}$$

and the result follows from the equalities

$$i(X_{H_k})(dH_i) = \omega_k(\nabla H_i, \nabla H_k), \quad dH_i(X_s) = \nabla H_i \cdot \nabla S.$$

III. APPLICATIONS

Combining the decomposition results[2] and the theorems in the previous section, we define the following strategy for searching for constants of motion (and attractors) in dynamical systems:

(a) Decompose the system into gradient and Hamiltonian components $\{X = X_S + \Sigma_{i=1}^{N-1} X_{H_i}\}$.

(b) Identify the constants of motion of each one of the Hamiltonian components X_{H_i}. These will be the Hamiltonian H_i itself plus a certain set $\{\phi_k\}$.

(c) Look for the closed orbits of the Hamiltonian components that satisfy the conditions of the theorem, Eq. (2.2), i.e., in coordinatewise notation

$$\int_0^{T_0} \left\{(\nabla \phi_k \cdot \nabla S) + \sum_{i=1}^{N-1} \omega_i(\nabla \phi, \nabla H_i)\right\} dt = 0. \tag{2.2'}$$

In general, for an N-dimensional system, the set of orbits that satisfy (2.2′) for each ϕ_k might span an $(N-1)$-dimensional subspace Ω_k. The nonempty intersections of subsets of the Ω_k, i.e., the set of orbits that satisfy (2.2′) simultaneously for certain subsets of $\{\phi_k\}$, would then supply information about the topological dimension and approximate location of the attractors for the arcs of vector fields associated to X_{H_i}.

(d) At this point, and before one gets the impression that a sure recipe has been obtained to find analytical approximations to the constants of motion of any dynamical system, one should remember that Eq. (2.2′) is only a *necessary* not a sufficient condition for the existence of an arc (in the sense of the Definition 2). By applying (2.2′), all one obtains are analytical approximations to the constants of motion of the arcs of vector fields associated to the components X_{H_i} of the system. Left open is the question of whether the system actually belongs to an arc of vector fields of its components, i.e., whether it satisfies the necesssary differentiability conditions in the deformation parameter.

Therefore, one should complement this study by other methods, for example, using this analysis to complement and interpret numerical studies.

(e) If the closed orbits of the Hamiltonian components do not cover the whole phase space, one might try other Hamiltonians to explore the remaining regions. A natural choice is to use blown-up versions of the X_{H_i} for $H = \lambda^2 H_i(x/\lambda)$.

Whereas in (c) the arc of vector fields to be used is $X_\epsilon = X_{H_i} + \epsilon(X_S + \Sigma_{k\neq i} X_{H_k})$, in the case of an Hamiltonian H that is not a component, the arc is

$$X_\epsilon = (1-\epsilon)X_H + \epsilon\left(X_S + \sum_i X_{H_i}\right) = (1-\epsilon)X_H + \epsilon X.$$

The result has the same form as Eq. (2.2′).

These techniques will now be illustrated in the Van der Pol and Lorenz models.

For the Van der Pol oscillator

$$\dot{x} = y = \frac{\partial S}{\partial x} + \frac{\partial H}{\partial y}, \quad S = \alpha\left(\frac{x^2}{2} - \frac{x^4}{12}\right),$$

$$\dot{y} = \alpha(1-x^2)y - x = \frac{\partial S}{\partial y} - \frac{\partial H}{\partial x}, \quad H = \frac{x^2+y^2}{2} - \alpha\left(x - \frac{x^3}{3}\right)y.$$

For values of the Hamiltonian H greater than

$$H > \tfrac{1}{2}\left[-\tfrac{1}{3}\alpha^2(2+\sqrt{1+3/\alpha^2})(1-\sqrt{1+3/\alpha^2})^2 + 2 + \sqrt{1+3/\alpha^2}\right],$$

the Hamiltonian orbits are not closed. In anticipation of the fact that one may need to explore wider regions of phase space than those covered by the closed orbits of H, we use a blown-up function

$$H_\lambda = \lambda^2 H(x/\lambda, y/\lambda) = \frac{x^2+y^2}{2} - \alpha\left(x - \frac{x^3}{3\lambda^2}\right)y.$$

In this two dimensional case, for the orbit $2H_\lambda = K$, Eq. (2.8) reduces to

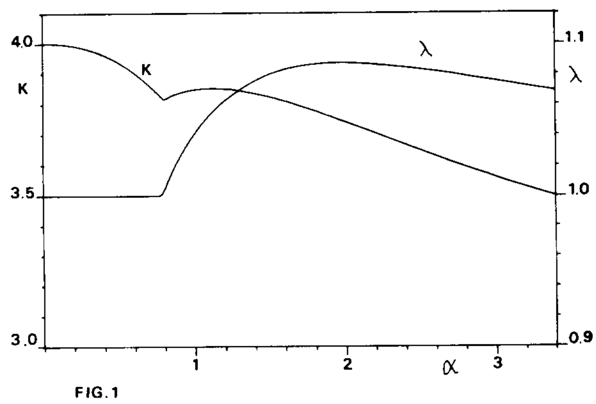


FIG. 1. Constant K for the Hamiltonian approximation to the limit cycle and blow-up parameter λ for the Van der Pol oscillator.

$$0=\int_A \Delta S=\alpha\int_A\int dxdy\,(1-x^2)$$
$$=4\alpha\int_0^{x_r} dx\,(1-x^2)\sqrt{\alpha^2x^2(1-x^2/3\lambda^2)^2-x^2+K}, \tag{3.1}$$

where the first equality follows from Stokes' theorem and x_r is the value for which the square root in the integrand vanishes.

The values of K and λ that satisfy Eq. (3.1) are plotted in the Fig. 1. One sees that for $\alpha > 0.75$ a certain amount of blowup is needed. Whenever $\lambda > 1$ is required, we have used the criterium of minimum blowup, i.e., we have chosen the smaller λ for which there is a closed orbit of $2H_\lambda$ satisfying Eq. (3.1). This means that for $\lambda > 1$ we use the separatrix between open and closed Hamiltonian orbits as an approximation to the limit cycle.

In the Figs. 2(a–c) we compare the exact limit cycle (dotted curve) obtained by numerical integration with our analytical Hamiltonian approximation

$$2H_\lambda = x^2+y^2-\alpha(x-x^3/3\lambda^2)y=K.$$

One sees that even for fairly large degrees of nonlinearity the overall shape and location of the limit cycle is reasonably approximated.

Our second example is the Lorenz model.[3] Extensive work on analytical approximations to the exact solutions of this model, in the limit of high Rayleigh number, has been done by Shimizu.[4] We will use the same change of variables as this author

$$x=\frac{X}{\sqrt{2\sigma(r-1)}}\quad m=\frac{Z}{r-1}-x^2,\quad \tau=t,$$

and write the Lorenz model $[\dot X=-\sigma X+\sigma Y$, $\dot Y=-Y+Xr-XZ, \dot Z=-bZ+XY]$ as

$$\dot x = p,$$
$$\dot p = -(\sigma+1)p-\sigma(r-1)x(x^2-1+m),$$
$$\dot m = -bm+(2\sigma-b)x^2.$$

Using these coordinates, our decomposition bear some formal resemblance to Shimizu's perturbation theory around the high Rayleigh number limit. This allowed some control and check on our results, which, however, are not restricted to any particular Rayleigh number range.

Although odd-dimensional, the Lorenz model can be considered as imbedded in a space with one extra dimension and the same decomposition techniques applied. We will use

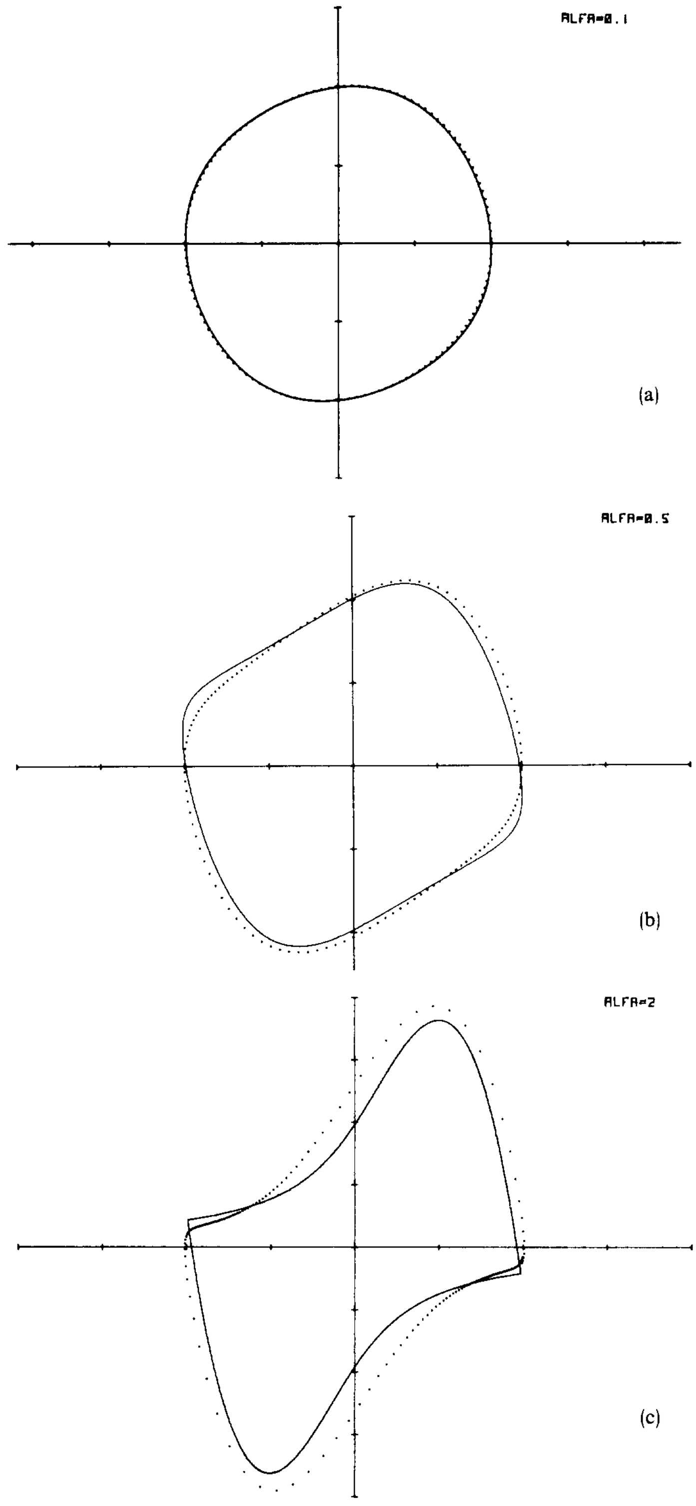
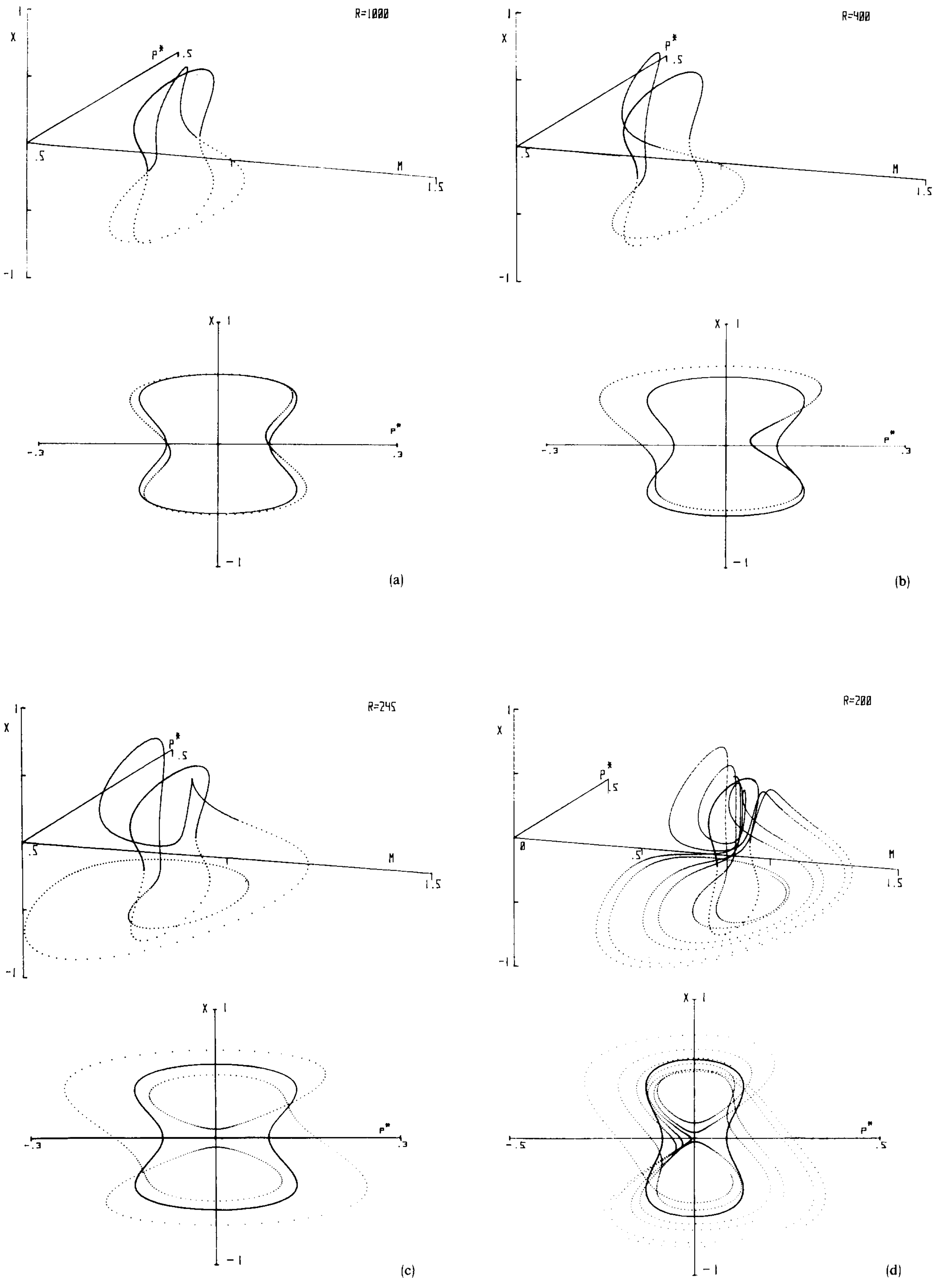


FIG. 2. Comparison of the limit cycle and the Hamiltonian orbit for three values of the nonlinearity parameter α in the Van der Pol.



FIGS. 3(a)–3(d). Numerical solution and the Hamiltonian orbit for several Rayleigh values in the Lorenz model ($\sigma = 16$, $b = 4$).

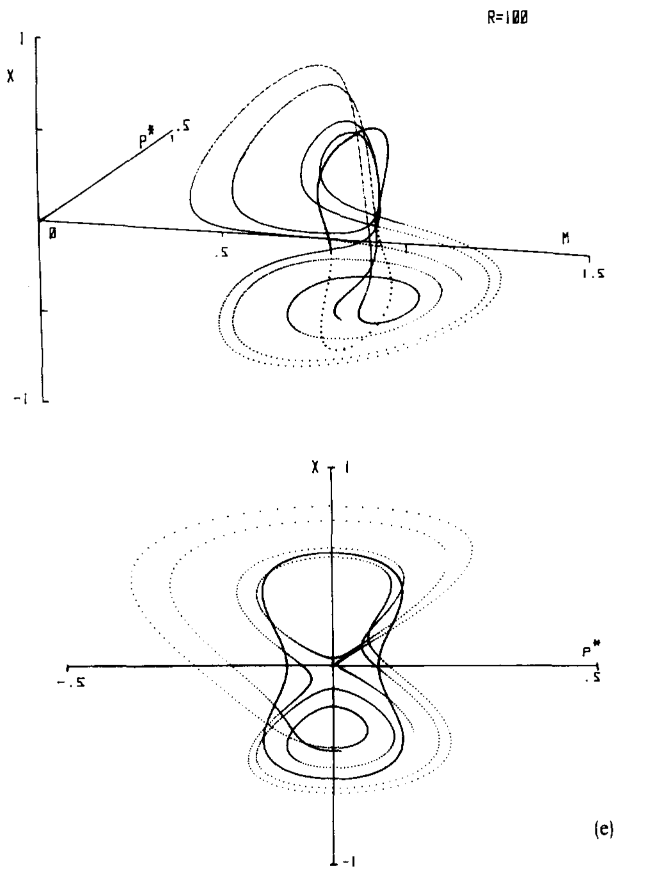


FIG. 3(e). Numerical solution and the Hamiltonian orbit for several Rayleigh values in the Lorentz model ($\sigma = 16$, $b = 4$).

the following decomposition:

$$\dot{x} = \frac{\partial S}{\partial x} + \frac{\partial H^1}{\partial p} + \frac{\partial H^2}{\partial m},$$
$$\dot{p} = \frac{\partial S}{\partial p} - \frac{\partial H^1}{\partial x}, \qquad (3.2)$$
$$\dot{m} = \frac{\partial S}{\partial m} - \frac{\partial H^2}{\partial x},$$

with

$$S = -\tfrac{1}{2}(\sigma+1)p^2 - \tfrac{1}{2}bm^2,$$
$$H^1 = \tfrac{1}{2}p^2 + \sigma(r-1)(\tfrac{1}{4}x^4 + \tfrac{1}{2}x^2(m-1)),$$
$$H^2 = -\tfrac{1}{3}(2\sigma - b)x^3.$$

The symplectic forms associated to the decomposition (3.2) are $\omega_1 = dx \wedge dp + dm \wedge dw$ and $\omega_2 = dx \wedge dm + dw \wedge dp$, w being the extra coordinate in the four-dimensional imbedding.

The Hamiltonian H^1 has closed orbits. Choosing H^1 to define the zero point in the arc, it follows from (3.2) that at $\epsilon = 0$ there are two constants of motion, namely

$$\phi_1 = H^1, \quad \phi_2 = m.$$

Application of the theorem leads to

$$0 = \int \{\nabla S\cdot\nabla H^1 + \omega_2(\nabla H^1, \nabla H^2)\}\, dt$$
$$= \int \{-(\sigma+1)p^2 + \tfrac{1}{2}\sigma(r-1)x^2[(2\sigma - b)x^2 - bm]\}\, dt. \qquad (3.3a)$$

$$0 = \int [\nabla S\cdot\nabla m + \omega_1(\nabla m, \nabla H^1) + \omega_2(\nabla m, \nabla H^2)]\, dt$$
$$= \int [-bm + (2\sigma - b)x^2]\, dt. \qquad (3.3b)$$

On the H^1 Hamiltonian orbits

$$p^2 + \sigma(r-1)[\tfrac{1}{2}x^4 + x^2(m-1)] = h,$$

where h is a constant. Making the replacement

$$dt = dx/2\sqrt{h - \sigma(r-1)[x^4/2 + x^2(m-1)]}$$

and the appropriate change of variables in Eqs. (3.3), they become

$$-(\sigma+1)hI_1 + \sigma(r-1)[(\sigma+1)(m-1) - \tfrac{1}{2}bm]I_2$$
$$+ \tfrac{1}{2}\sigma(r-1)(3\sigma - b + 1)I_4 = 0, \qquad (3.4a)$$
$$-bmI_1 + (2\sigma - b)I_2 = 0, \qquad (3.4b)$$

where

$$I_1 = \frac{2}{\sqrt{\alpha^2+\beta^2}} K\left(\frac{\alpha^2}{\alpha^2+\beta^2}\right), \qquad (3.5a)$$
$$I_2 = -\beta^2 I_1 + 2\sqrt{\alpha^2+\beta^2}\, E\left(\frac{\alpha^2}{\alpha^2+\beta^2}\right), \qquad (3.5b)$$
$$I_4 = \frac{2h}{3\sigma(r-1)} I_1 + \tfrac{4}{3}(1-m)I_2, \qquad (3.5c)$$
$$\alpha^2 = (1-m) + \sqrt{(1-m)^2 + 2h/\sigma(r-1)} \qquad (3.5d)$$
$$\beta^2 = -(1-m) + \sqrt{(1-m)^2 + 2h/\sigma(r-1)}. \qquad (3.5e)$$

K and E are the complete elliptic integrals of first and second kind.[5]

Using Eqs. (3.5), Eqs. (3.4a)–(3.4b) are converted into the following equivalent set of equations:

$$\frac{2h}{\sigma(r-1)} = \frac{bm[(6\sigma - 2 - b)(1-m) - 3b]}{(2\sigma - b)(b+2)}, \qquad (3.6a)$$
$$\frac{bm + (2\sigma - b)\beta^2}{\alpha^2+\beta^2} K\left(\frac{\alpha^2}{\alpha^2+\beta^2}\right) = (2\sigma - b)E\left(\frac{\alpha^2}{\alpha^2+\beta^2}\right). \qquad (3.6b)$$

Replacing (3.6a) into (3.5d) and (3.5e), one concludes that Eq. (3.6b) determines a solution for m independently of the Rayleigh number r. Once a solution for m is obtained numerically from Eq. (3.6b) for a given pair (σ, b), a solution for h is always obtained from (3.6a) for any r.

The existence of a *one-dimensional* Hamiltonian approximation to the constant of motion does *not* depend on the Rayleigh number. Referring back to our comments about topological dimension of attractors in (c) this sheds some doubt on the full turbulent nature of the r regions found in between the ranges of parameters for which limit cycles were found. It suggests that a (perhaps dense) set of periodic orbits might also exist in the turbulent regions. These remarks however, are only speculative, because of the limitations mentioned in (d).

For the popular values $\sigma = 16$ and $b = 4$, the numerical solution of Eq. (3.6b) leads to

$$m = 0.8586,$$
$$h = 0.11896(r-1). \qquad (3.7)$$

In Figs. 3(a)–3(e) the Hamiltonian orbit

$$\frac{p^2}{\sigma(r-1)} + \left[\frac{x^4}{2} + x^2(m-1)\right] = \frac{h}{\sigma(r-1)}, \quad m = \text{const.}$$

for the values given in (3.7) is compared with the exact solution obtained by numerical integration for several Rayleigh number values. For the numerical integration, an integration step $\Delta t = 0.001$ is used in a Runge–Kutta algorithm, the solution being followed after 10^4 steps to remove the transient behavior. Plotted are points at 0.002 intervals and the initial conditions are chosen near the Hamiltonian orbit, namely $m(0) = 0.859$, $x(0) = 0$, $p(0) = \sqrt{h}$. In the figures $p^* = p/\sqrt{\sigma(r-1)}$.

From the inspection of the results, one notices that the Hamiltonian orbit gives a good estimate of the size and average position of the limit cycles. The perspective view and the projection in the p^*–x plane (where the H^1 dynamics takes place) allow an interpretation of these cycles as a distortion in the m direction of the Hamiltonian orbit obtained from the theorem.

Also in the "turbulent" regions ($r = 100, 200$) the exact solution winds around the Hamiltonian orbit, the projection in the p^*–x plane revealing the analytical orbit as an organizing center for the dynamics.

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