An infinite-dimensional calculus for generalized connections on hypercubic lattices

R. Vilela Mendes[a)]
IPFN - EURATOM/IST Association, Instituto Superior Técnico, Av. Rovisco Pais 1, 1049-001 Lisboa, Portugal, and CMAF, Complexo Interdisciplinar, Universidade de Lisboa, Av. Gama Pinto, 2 - 1649-003, Lisboa, Portugal

(Received 11 May 2010; accepted 30 April 2011; published online 27 May 2011)

A space for gauge theories is defined, using projective limits as subsets of Cartesian products of homomorphisms from a lattice on the structure group. In this space, non-interacting and interacting measures are defined as well as functions and operators. From projective limits of test functions and distributions on products of compact groups, a projective gauge triplet is obtained, which provides a framework for the infinite-dimensional calculus in gauge theories. The gauge measure behavior on non-generic strata is also obtained. © *2011 American Institute of Physics*. [doi:10.1063/1.3592919]

I. PRELIMINARIES

A. Gauge theory: Basic definitions

In its usual formulation a classical gauge theory consists of four basic objects:

(i) A principal fiber bundle $P(M, G)$ with structure group G and projection $\pi : P \to M$, the base space M being an oriented Riemannian manifold.

(ii) An affine space $\mathcal{C}$ of connections ω on P, modeled by a vector space $\mathcal{A}$ of 1-forms on M with values on the Lie algebra LG of G.

(iii) The space of differentiable sections of P, called the *gauge group* $\mathcal{G}$.

(iv) A $\mathcal{G}-$invariant functional (the Lagrangian) $\mathcal{L} : \mathcal{A} \to \mathbb{R}$.

The statement (iv) presumes the existence of a reference measure in the configuration space $\mathcal{A}/\mathcal{G}$, the exponential of the Lagrangian being a Radon-Nykodim derivative with respect to this reference measure. Because that might not always be possible to achieve, it is better to replace (iv) by:

(iv′) A well-defined measure in the configuration space $\mathcal{A}/\mathcal{G}$.

Choosing a reference connection, the affine space of connections on P may be modeled by a vector space of LG-valued 1-forms ($C^\infty(\Lambda^1 \otimes LG)$). Likewise, the curvature F is identified with an element of $C^\infty(\Lambda^2 \otimes LG)$. In a coordinate system one writes,

$$A = A^a_\mu dx^\mu t_a, \qquad x \in M, \qquad t_a \in LG,$$

with the action of $\gamma = \{g(x)\} \in \mathcal{G}$ on $A \in \mathcal{A}$ given by

$$\gamma : A_\mu(x) \to (gA_\mu)(x) = g(x)A_\mu(x)g^{-1}(x) - (\partial g)(x) \cdot g^{-1}(x). \tag{1}$$

In this paper G will always be considered to be a compact group.

The action of $\mathcal{G}$ on $\mathcal{A}$ leads to a stratification of $\mathcal{A}$ corresponding to the classes of equivalent *orbits* $\{gA; g \in \mathcal{G}\}$. Let S_A denote the *isotropy (or stabilizer) group* of $A \in \mathcal{A}$,

$$S_A = \{\gamma \in \mathcal{G} : \gamma A = A\}. \tag{2}$$

a)Electronic mail: vilela@cii.fc.ul.pt.

The *stratum* $\Sigma(A)$ of A is the set of connections having isotropy groups $\mathcal{G}-$conjugated to that of A,

$$\Sigma(A) = \{B \in \mathcal{A} : \exists\gamma \in \mathcal{G} : S_B = \gamma S_A \gamma^{-1}\}. \tag{3}$$

The *configuration space of the gauge theory* is the quotient space $\mathcal{A}/\mathcal{G}$ and, therefore, a stratum is the set of points in $\mathcal{A}/\mathcal{G}$ that correspond to orbits with conjugated isotropy groups.

B. Generalized connections and the Ashtekar-Lewandowski measure

Whenever a Lagrangian is defined the calculation of physical quantities in the path integral formulation,

$$\langle\phi\rangle = \int_{\mathcal{A}/\mathcal{G}} \phi(\xi) e^{-\mathcal{L}(\xi)} d\mu(\xi) \tag{4}$$

requires a measure in $\mathcal{A}/\mathcal{G}$ and no such measure is found for Sobolev connections. Therefore, it turned out to be more convenient to work in a space of *generalized connections* $\overline{\mathcal{A}}$, defining parallel transports on piecewise smooth paths as simple homomorphisms from the paths on M to the group G, without a smoothness assumption.[1] The same applies to the generalized gauge group $\overline{\mathcal{G}}$. Then, there is, in $\overline{\mathcal{A}}/\overline{\mathcal{G}}$, an induced Haar measure, the Ashtekar-Lewandowski measure.[2,3] Sobolev connections are a dense zero measure subset of the generalized connections.[4]

A generalized connection is simply a homomorphism from the groupoid of paths $\mathcal{P}$ to the structure group. Different choices for the groupoid of paths have been proposed. Piecewise analyticity was first proposed,[5] later extended to the smooth category[6] and to a more general setting covering both cases.[7,8] This led to the notions of graphs,[2] webs, and hyphs.[9] Here, being mostly concerned with the Yang-Mills theory (and not with gravity where diffeomorphism invariance is important), the piecewise analytic case will be considered.

The space of generalized connections is characterized by using the representation theory of C^* algebras. The relevant algebra is the algebra $\mathcal{HA}$ of functions on $\mathcal{A}/\mathcal{G}$ obtained by taking finite linear combinations of finite products of traces of holonomies (Wilson loop functions $W_\alpha(A)$) around closed loops α. The completion $\overline{\mathcal{HA}}$ of $\mathcal{HA}$ in the sup norm is a C^* algebra. $\overline{\mathcal{HA}}$ is an abelian C^* algebra and its Gel'fand spectrum is, by definition, the space of generalized connections $\overline{\mathcal{A}/\mathcal{G}}$. An equivalence class of holonomically equivalent loops is called a *hoop* . By composition, hoops generate a hoop group $\mathcal{HG}$. Every point $\overline{A} \in \overline{\mathcal{A}/\mathcal{G}}$ gives rise to a homomorphism $\breve{H}$ from the hoop group $\mathcal{HG}$ to the structure group G and every homomorphism defines a point in $\overline{\mathcal{A}/\mathcal{G}}$ and, from $\breve{H}_{\overline{A}}(\beta) = Tr H(\beta, \overline{A})$, where $H(\beta, \overline{A})$ denotes the holonomy, it follows that the correspondence has the trivial ambiguity that $\breve{H}$ and $g^{-1}\breve{H}g$ define the same point in $\overline{\mathcal{A}/\mathcal{G}}$.[5]

An important notion is the notion of independent hoops. Denote by $\breve{\alpha}$ the hoop for which α is a representative loop. In a set of independent hoops $\{\breve{\beta}_1, \breve{\beta}_2, \cdots, \breve{\beta}_n\}$ every representative loop β_i must contain an open interval, that is, traversed exactly once and no finite segment of which is shared by any other loop in a different hoop. Furthermore, given a set of hoops $\{\breve{\alpha}_1, \breve{\alpha}_2, \cdots, \breve{\alpha}_k\}$, it is always possible to find a set of independent hoops such that the hoop subgroup generated by the $\breve{\alpha}_i$'s is contained in the hoop subgroup generated by the $\breve{\beta}_i$'s and for every $(g_1, g_2, \cdots, g_n) \in G^n$, there is a connection $A \in \mathcal{A}$, such that $H(\beta_i, A) = g_i$, $i = 1, \cdots, n$.[5] $H(\beta_i, A)$ denotes the holonomy $(W_\beta(A) = Tr H(\beta_i, A))$.

The notion of independent hoops provides a simple definition of cylindrical functions. Given a set of independent loops $\beta_1, \beta_2, \cdots, \beta_n$, consider the hoop subgroup S^* that they generate and define an equivalence relation in $\overline{\mathcal{A}/\mathcal{G}}$ by $\overline{A}_1 \sim \overline{A}_2$ iff $\breve{H}_{\overline{A}_1}(\breve{\gamma}) = g^{-1}\breve{H}_{\overline{A}_2}(\breve{\gamma})g$ for some $g \in G$ and all $\breve{\gamma} \in S^*$. Denoting by $\pi(S^*)$ the projection on the quotient space $(\overline{\mathcal{A}/\mathcal{G}})/\backsim$, cylindrical functions are the pull-backs under $\pi(S^*)$ of the functions f on $(\overline{\mathcal{A}/\mathcal{G}})/\sim$. $(\overline{\mathcal{A}/\mathcal{G}})/\sim$ is isomorphic to G^n/Ad, the algebra of the cylindrical functions is a C^* algebra and its completion in the sup norm is isomorphic to the algebra $\overline{\mathcal{HA}}$.

A natural integration measure for the cylindrical functions is the Haar measure on G, which, being invariant under G, projects down naturally to G^n/Ad. It satisfies the required compatibility condition in the sense that if f is a cylindrical function on $\overline{\mathcal{A}/\mathcal{G}}$ with respect to two different finitely generated hoop subgroups S_i^*, $i = 1, 2$, then

$$\int_{G^{n_1}/Ad} f_1 d\mu_1 = \int_{G^{n_2}/Ad} f_2 d\mu_2.$$

The measure on $\overline{\mathcal{A}/\mathcal{G}}$, whose restriction to cylindrical functions is the Haar measure on G^n/Ad, is the Ashtekar-Lewandowski measure.

C. Projective limits and physical interpretation

Let $\{I, \succ\}$ be a set endowed with an order relation $\succ$ and suppose that with each element $i \in I$, a set X_i is associated and for each pair (i, j), in which $j \succ i$, there is a mapping $\pi_{ij} : X_j \to X_i$ such that π_{ii} is the identity and $\pi_{ki}\pi_{ij} = \pi_{kj}$. Then a set X is called the projective limit $X = \varprojlim X_i$ of the family $\{X_i\}$ of sets if the following conditions are satisfied:

(a) there is a family of mappings $\pi_i : X \to X_i$ such that for any pair (i, j), in which $j \succ i$, $\pi_{ij}\pi_j = \pi_i$.

(b) for any family of mappings $\alpha_i : Y \to X_i$, from an arbitrary set Y, for which the equalities $\pi_{ij}\alpha_j = \alpha_i$ hold for $j \succ i$, there exists a unique mapping $\alpha : Y \to X$ such that $\alpha_i = \pi_i\alpha$ for every $i \in I$.

An explicit construction of the projective limit, which is particularly suited to the physical interpretation, is the following. Consider the direct product $\prod_{i\in I} X_i$ and select in it the subset X which satisfies the consistency condition $j \succ i \Longrightarrow \pi_{ij}\overset{\frown}{X}_j = \overset{\frown}{X}_i$ with $\overset{\frown}{X}_i \subset X_i$ and $\overset{\frown}{X}_j \subset X_j$. This subset is the projective limit of the family $\{X_i\}$ of sets.

Notice that this construction emphasizes the fact that the projective limit is not the limit of the sequence X_i. Instead it is a particular subset of the direct product. If, for example, the physical meaning of the index set $\{I, \succ\}$ is a refinement to successive smaller scales, the projective limit, once the consistency condition is fulfilled, contains a description of all the scales and not only the small scale limit.

II. THE GAUGE PROJECTIVE SPACE, KINEMATICAL MEASURE, FUNCTIONS, AND OPERATORS

In the past, the Ashtekhar-Lewandowski measure has been constructed in very general settings[1–3] using projective limits of floating lattices and weak smoothness conditions. Here, staying closer to the usual physical setting of lattice gauge theory, one uses fixed hypercubic lattices with piecewise analytic parametrization. Projective techniques on hypercubic lattices have also recently been found to provide a framework for the construction of operators and semiclassical states in the "kinematical" space of loop quantum gravity.[10]

Consider a sequence of hypercubic lattices in $\mathbb{R}^4$ of edge $\frac{a}{2^k}$, $k = 0, 1, 2, \cdots$ constructed in such a way that the lattice of edge $\frac{a}{2^k}$ is a refinement of the $\frac{a}{2^{k-1}}$ lattice (all vertices of the $\frac{a}{2^{k-1}}$ lattice are also vertices in the $\frac{a}{2^k}$ lattice). Finite volume hypercubes Γ in this lattice are a directed set $\{\Gamma, \succ\}$ under the inclusion relation $\succ$. $\Gamma \succ \Gamma'$ meaning that all edges and vertices in Γ' are contained in Γ, the inclusion relation satisfies

$$\begin{aligned}
&\Gamma \succ \Gamma, \\
&\Gamma \succ \Gamma' \text{ and } \Gamma' \succ \Gamma \Longrightarrow \Gamma = \Gamma', \\
&\Gamma \succ \Gamma' \text{ and } \Gamma' \succ \Gamma'' \Longrightarrow \Gamma \succ \Gamma''.
\end{aligned} \tag{5}$$

For convenience, one considers that the lattice refinement from size $\frac{a}{2^{k-1}}$ to $\frac{a}{2^k}$ is made one plaquette at a time so that all intermediate configurations are present in the directed set. This directed set will cover both, successively higher volumes and finer lattices. Let p_0 be a point that does not belong to any lattice of the directed family. One assumes an analytic parametrization of each edge, to each edge l associates a p_0-based loop and for each generalized connection A considers the holonomy $h_l(A)$.

For definiteness, each edge is considered to be oriented along the coordinates in positive direction and the set of edges of the lattice Γ is denoted $E(\Gamma)$. The set $\mathcal{A}_\Gamma$ of generalized connections for the lattice hypercube Γ is the set of homomorphisms $\mathcal{A}_\Gamma = Hom(E(\Gamma), G) \sim G^{\#E(\Gamma)}$, obtained by associating to each edge the holonomy $h_l(\cdot)$ on the associated p_0 -based loop. The set of gauge-independent generalized connections $\mathcal{A}_\Gamma/Ad$ is obtained by factoring the adjoint representation at p_0, $\mathcal{A}_\Gamma/Ad \sim G^{\#E(\Gamma)}/Ad$. However because, for gauge independent functions, integration in $\mathcal{A}_\Gamma$ coincides with integration in $\mathcal{A}_\Gamma/Ad$, for simplicity from now on one uses only $\mathcal{A}_\Gamma$. The space of generalized connections that one considers here is the projective limit $\mathcal{A} = \underleftarrow{\lim}\mathcal{A}_\Gamma$ of the family,

$$\{\mathcal{A}_\Gamma, \pi_{\Gamma\Gamma'} : \Gamma' \succ \Gamma\}, \tag{6}$$

where $\pi_{\Gamma\Gamma'}$ and π_Γ denote the surjective projections $\mathcal{A}_{\Gamma'} \longrightarrow \mathcal{A}_\Gamma$ and $\mathcal{A} = \underleftarrow{\lim}\mathcal{A}_\Gamma$.

Recall that the projective limit of the family $\{\mathcal{A}_\Gamma, \pi_{\Gamma\Gamma'}\}$ is the subset $\mathcal{A}$ of the Cartesian product $\prod_\Gamma \mathcal{A}_\Gamma$ that satisfies the consistency condition

$$\mathcal{A} = \left\{ a \in \prod_\Gamma \mathcal{A}_\Gamma : \Gamma' \succ \Gamma \Longrightarrow \pi_{\Gamma\Gamma'}\mathcal{A}_{\Gamma'} = \mathcal{A}_\Gamma \right\}.$$

The projective topology in $\mathcal{A}$ is the coarsest topology for which each π_Γ mapping is continuous.

For a compact group G, each $\mathcal{A}_\Gamma$ is a compact *Hausdorff space.* Then $\mathcal{A}$ is also a compact Hausdorff space. In each $\mathcal{A}_\Gamma$, one has a natural (Haar) normalized product measure $\nu_\Gamma = \mu_H^{\#E(\Gamma)}$, μ_H being the normalized Haar measure in G. Then, according to a theorem of Prokhorov, as generalized by Kisynski,[11,12] if

$$\nu_{\Gamma'}\left(\pi_{\Gamma\Gamma'}^{-1}(B)\right) = \nu_\Gamma(B), \tag{7}$$

for every $\Gamma' \succ \Gamma$ and every Borel set B in $\mathcal{A}_\Gamma$, there is a unique measure ν in $\mathcal{A}$ such that $\nu(\pi_\Gamma^{-1}(B)) = \nu_\Gamma(B)$, for every Γ. Furthermore, this measure is tight, that is, for every $\varepsilon > 0$ there is a compact subset K of $\mathcal{A}$ such that $\nu_\Gamma(\mathcal{A}_\Gamma - \pi_\Gamma(K)) < \varepsilon$. The measure ν, so constructed, is a version of the Ashtekar-Lewandowski measure.

The consistency condition (7) is easy to check in the present context. It suffices to consider $\Gamma' = \Gamma_k$ as the refinement of $\Gamma = \Gamma_{k-1}$, when the edge size goes from $\frac{a}{2^{k-1}}$ to $\frac{a}{2^k}$. Then, if g_i are group elements associated to the finer lattice (size $\frac{a}{2^k}$)

$$\begin{aligned}\nu_{k-1}(g_1 \times g_2 \times g_3 \times g_4 \in B) &= \nu_{k-1}\left(g_i \times g_j \times g_k \times \left(g_i \times g_j \times g_k\right)^{-1} B : g_i, g_j, g_k \in G\right),\\ &= \mu_H(G)^3 \mu_H\left(\left(g_i \times g_j \times g_k\right)^{-1} B\right) = \nu_k(B),\end{aligned} \tag{8}$$

and the consistency condition (7) follows from the normalization and invariance of the μ_H measure. The second equality in (8) reflects the factorized nature of the product measure. $\mathcal{A} = \underleftarrow{\lim}\mathcal{A}_\Gamma$ will be called *the gauge space* and ν will be *the kinematical measure*.

In the following one also needs to define functions and operators in the projective family. The correspondence $\mathcal{A}_\Gamma \sim G^{\#E(\Gamma)}$ means that functions on the projective family are constructed from equivalent classes of functions in $G^{\#E(\Gamma)}$.

G being a compact connected Lie group with Lie algebra LG, one chooses an $AdG-$invariant inner product $(\cdot,\cdot)$ on LG. For each $\xi \in LG$ define ∂_ξ by

$$(\partial_\xi f)(g) := \left.\frac{d}{dt} f\left(e^{t\xi} g\right)\right|_{t=0}; \qquad g \in G. \tag{9}$$

Choosing an orthonormal basis in LG, $\{\xi_1 \cdots \xi_n\}$, write $\partial_i := \partial_{\xi_i}$. With these operators one has a notion of $C^n(G)$ functions and, with the Haar measure $d\mu_H$, of $L^2(G, d\mu_H)$ space as well.

The Laplacian operator is

$$\Delta_G := \sum_{i=1}^{n} \partial_i^2, \tag{10}$$

which does not depend on the choice of the basis and is symmetric with respect to the $L^2(G, d\mu_H)$ inner product.

For any finite $\#E(\Gamma)$, the extension of these notions to $\mathcal{A}_\Gamma \sim G^{\#E(\Gamma)}$ is straightforward. To carry the notion of C^n function in (finite) product spaces to the projective family, introduce in the union

$$\bigcup_\Gamma C^n\left(\mathcal{A}_\Gamma\right),$$

the equivalence relation

$$f_{\Gamma_1} \sim f_{\Gamma_2} \qquad \text{if} \qquad \pi^*_{\Gamma_1\Gamma_3} f_{\Gamma_1} = \pi^*_{\Gamma_2\Gamma_3} f_{\Gamma_2}, \tag{11}$$

for any $\Gamma_3 \succ \Gamma_1, \Gamma_2$. $\pi^*_{\Gamma\Gamma'}$ is the pull-back map from the space of functions on Γ to the space of functions on $\Gamma^{'}$.

The set of C^n cylindrical functions associated to the projective family $\{\mathcal{A}_\Gamma, \pi_{\Gamma\Gamma'}\}$ is then

$$\bigcup_\Gamma C^n\left(\mathcal{A}_\Gamma\right) / \sim . \tag{12}$$

On the other hand, for families of operators $\{O_\Gamma, D(O_\Gamma)\}_{\Gamma\in S}$ with domains $D(O_\Gamma)$ defined on a subset of labels S, one requires the following consistency conditions:

$$\pi^*_{\Gamma\Gamma'} D\left(O_\Gamma\right) \subset D\left(O_{\Gamma'}\right), \tag{13}$$

$$O_{\Gamma'}\pi^*_{\Gamma\Gamma'} = \pi^*_{\Gamma\Gamma'} O_\Gamma, \tag{14}$$

for every $\Gamma^{'} \succ \Gamma$ such that $\Gamma, \Gamma^{'} \in S$.

III. THE GAUGE PROJECTIVE TRIPLET

Here one considers the same directed set $\{\Gamma, \succ\}$, spaces $\mathcal{A}_\Gamma = Hom(E(\Gamma), G) \sim G^{\#E(\Gamma)}$, and the projective limit $\mathcal{A} = \varprojlim \mathcal{A}_\Gamma$ as in Sec. II. To formulate an infinite-dimensional calculus in $\mathcal{A}$ one starts by defining test functions and distributions in $G^{\#E(\Gamma)}$, and then considers the corresponding projective limits.

Several families of norms may be used to construct test functions and distribution spaces in $G^{\#E(\Gamma)}$. Here the heat kernel norm will be used. For a compact connected group G, one uses the same construction of Hida spaces as in Ref. 13 to which the reader is referred for details and proofs. In G, there is a heat kernel $p_t(g)$ defined as the fundamental solution of

$$\partial_t p = \frac{1}{2}\Delta p, \tag{15}$$

Δ being the operator defined in (10). The heat kernel measure is $d\mu_t = p_t d\mu_H$ and it is easy to check that

$$L^2(G, d\mu_t) = L^2(G, d\mu_H). \tag{16}$$

By analogy with the Gaussian case a collection $\mathcal{H}_t\left(G\right)$ of spaces is defined as domains of

$$\mathcal{H}_t\left(G\right) = \mathcal{D}\left(p_{1/t}^{-1/2} \circ e^{-\frac{1}{2}\left(1-\frac{1}{t}\right)\triangle} \circ p_1\right), \tag{17}$$

for $t \geq 1$, which are Hilbert spaces with norm

$$\|f\|_t^2 = \int_G \left|\left(p_{1/t}^{-1/2} \circ e^{-\frac{1}{2}\left(1-\frac{1}{t}\right)\triangle} \circ p_1\right) f\right|^2 d\mu_H. \tag{18}$$

Equivalently, if the eigenvalues of the Laplacian are φ_n

$$\triangle\varphi_n = \lambda_n \varphi_n, \tag{19}$$

then

$$f \in \mathcal{H}_t(G) \Longleftrightarrow f = \sum_{n=1}^{\infty} c_n \frac{\varphi_n}{p_1}, \tag{20}$$

with

$$\sum_{n=1}^{\infty} |c_n|^2 \, e^{-\left(1-\frac{1}{t}\right)\lambda_n} < \infty. \tag{21}$$

Because $\triangle$ has negative spectrum, $\mathcal{H}_{t'}(G) \subset \mathcal{H}_t(G)$, if $t' > t$

From the family $\{\mathcal{H}_t(G), t \geq 1\}$ of Hilbert spaces one defines the *test function space on* G as

$$\mathcal{H}(G) = \bigcap_{t\geq 1} \mathcal{H}_t(G) = \bigcap_{n\in\mathbb{N}} \mathcal{H}_n(G). \tag{22}$$

$\mathcal{H}(G)$ is equipped with the projective limit topology of the spaces $\mathcal{H}_t(G)$, which coincides with the metric topology defined by the metric

$$d(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\| f - g\|_n}{1 + \| f - g\|_n}. \tag{23}$$

$\mathcal{H}(G)$ is dense in each $\mathcal{H}_t(G)$ and is a nuclear space of analytic functions on G.[13]

Because $(\mathcal{H}(G), d)$ is a countably Hilbert space it follows[14] that the topological dual $\mathcal{H}^*(G)$ of $\mathcal{H}(G)$ is given by

$$\mathcal{H}^*(G) = \bigcup_{n=1}^{\infty} \mathcal{H}_n^*(G), \tag{24}$$

$\mathcal{H}_n^*(G)$ being the dual space of $\mathcal{H}_n(G)$. That is, each continuous linear functional on $\mathcal{H}(G)$ must already be continuous for some norm $\|\bullet\|_n$. The nuclearity of $\mathcal{H}(G)$ also implies that $\mathcal{H}^*(G)$ carries many probability measures defined by characteristic functions and the Bochner-Minlos theorem. $\mathcal{H}^*(G)$ is the *space of distributions* on G. By a canonical embedding one has the chain (triplet)

$$\mathcal{H}(G) \subset L^2(G, d\mu_t) \subset \mathcal{H}^*(G). \tag{25}$$

For each finite hypercube Γ, taking direct products of $\#E(\Gamma)$ copies of the spaces, the generalization of this triplet construction to $G^{\#E(\Gamma)}$is straightforward

$$\mathcal{H}\left(G^{\#E(\Gamma)}\right) = \bigcap_{t\geq 1} \mathcal{H}_t\left(G^{\#E(\Gamma)}\right) = \bigcap_{n\in\mathbb{N}} \mathcal{H}_n\left(G^{\#E(\Gamma)}\right), \tag{26}$$

$$\mathcal{H}^*\left(G^{\#E(\Gamma)}\right) = \bigcup_{n=1}^{\infty} \mathcal{H}_n^*\left(G^{\#E(\Gamma)}\right), \tag{27}$$

$$\mathcal{H}\left(G^{\#E(\Gamma)}\right) \subset L^2\left(G^{\#E(\Gamma)}, \prod_{E(\Gamma)} d\mu_t\right) \subset \mathcal{H}^*\left(G^{\#E(\Gamma)}\right). \tag{28}$$

This provides, for each finite hypercube Γ, a space of test functions $\mathcal{H}(G^{\#E(\Gamma)})$ and distributions $\mathcal{H}^*(G^{\#E(\Gamma)})$ on $\mathcal{A}_\Gamma = Hom(E(\Gamma), G) \sim G^{\#E(\Gamma)}$.

With the directed set $\{\Gamma, \succ\}$ of finite volume hypercubes, one has the surjective projections $\mathcal{H}(\mathcal{A}_{\Gamma'} \sim G^{\#E(\Gamma')}) \xrightarrow{\pi^S_{\Gamma\Gamma'}} \mathcal{H}\left(\mathcal{A}_\Gamma \sim G^{\#E(\Gamma)}\right)$ and

$\mathcal{H}^*\left(\mathcal{A}_{\Gamma'} \sim G^{\#E(\Gamma')}\right) \xrightarrow{\pi^{S^*}_{\Gamma\Gamma'}} \mathcal{H}^*\left(\mathcal{A}_{\Gamma} \sim G^{\#E(\Gamma)}\right)$ for $\Gamma' \succ \Gamma$, the maps $\pi^{S}_{\Gamma\Gamma'}$ and $\pi^{S^*}_{\Gamma\Gamma'}$ meaning the restriction of functions and distributions on $G^{\#E(\Gamma')}$ to the elements of $G^{\#E(\Gamma)}$.

One now considers, in the Cartesian products $\prod_{\Gamma}\mathcal{H}(\mathcal{A}_\Gamma)$ and $\prod_{\Gamma}\mathcal{H}^*(\mathcal{A}_\Gamma)$ the subsets

$$\mathcal{H}(\mathcal{A}) = \left\{ s(a) \in \prod_{\Gamma} \mathcal{H}(\mathcal{A}_\Gamma) : \Gamma' \succ \Gamma \Longrightarrow \pi^{S}_{\Gamma\Gamma'} \mathcal{H}\left(\mathcal{A}_{\Gamma'}\right) = \mathcal{H}(\mathcal{A}_\Gamma) \right\}, \tag{29}$$

and

$$\mathcal{H}^*(\mathcal{A}) = \left\{ s\left(a^*\right) \in \prod_{\Gamma} \mathcal{H}^*(\mathcal{A}_\Gamma) : \Gamma' \succ \Gamma \Longrightarrow \pi^{S^*}_{\Gamma\Gamma'} \mathcal{H}^*\left(\mathcal{A}_{\Gamma'}\right) = \mathcal{H}^*(\mathcal{A}_\Gamma) \right\}, \tag{30}$$

which define spaces of test functions and distributions on $\mathcal{A}$. It is this projective triplet

$$\mathcal{H}(\mathcal{A}) \subset L^2(\mathcal{A}, d\nu) \subset \mathcal{H}^*(\mathcal{A}) \tag{31}$$

that provides the framework for an infinite-dimensional calculus in the gauge theory. A particularly useful tool for this purpose is the $S-$ transform, which for $\mathcal{H}^*(G)$ is

$$(S\Phi)(x) = \Phi(e_x), \tag{32}$$

$\Phi \in \mathcal{H}^*(G)$, $x \in G$ and

$$e_x(y) = \frac{p_1\left(x^{-1}y\right)}{p_1(y)}. \tag{33}$$

The $S-$transform is an injective map from $\mathcal{H}^*(G)$ onto $\mathcal{U}(G_{\mathbb{C}})$, the space of holomorphic functions of second order exponential growth on $G_{\mathbb{C}}$ (the complexification of G),

$$\mathcal{U}(G_{\mathbb{C}}) = \left\{ f \in Hol(G_{\mathbb{C}}) \,|\, \exists k, c > 0 : |f(z)| \leq c e^{k|z|^2}, \forall z \in G_{\mathbb{C}} \right\}. \tag{34}$$

The extension of this transform to $G^{\#E(\Gamma)}$ is straightforward and through the Cartesian product construction allows to deal with distributions in $\mathcal{H}^*(\mathcal{A})$ as functions in $\mathcal{H}(\mathcal{A})$.

Notice that all spaces in the gauge projective triplet (31) are subsets of a Cartesian product, not just the corresponding small distance limit. Therefore the triplet, here proposed, is the basic framework for a gauge theory calculus at all length scales.

The basic variables in the lattice are the p_0-based loop holonomies associated to the edges. The kinematical ν measure in $L^2(\mathcal{A}, d\nu)$ is factorizable in these variables and this fact plays an important role in checking the consistency condition (7). However, the measures constructed in Secs. IV and V are not factorizable in these variables and it is in this sense that they are called "interacting measures."

IV. CONVOLUTION SEMIGROUPS AND INTERACTION MEASURES

In (8) the consistency condition (7) is easy to check because of the factorized nature of the kinematical measure ν. However, interaction measures have to be constructed from entities involving more than one of the edge-based holonomies. The basic element will be

$$U_\square(A_\Gamma) = h_1 h_2 h_3^{-1} h_4^{-1}, \tag{35}$$

h_1 to h_4 being the holonomies associated to the loops based on the links of a plaquette. Then $U_\square(A_\Gamma)$ is the holonomy along the plaquette which, according to the orientation conventions used here, is obtained by the product of two p_0-based loop holonomies and two inverse p_0-based loop holonomies.

To construct an interaction measure, one first considers, on the finite-dimensional spaces $\mathcal{A}_\Gamma \sim G^{\#E(\Gamma)}$, measures that are absolutely continuous with respect to the Haar measure

$$d\mu_{\mathcal{A}_\Gamma} = 1/Zp(\mathcal{A}_\Gamma)(d\mu_H)^{\#E(\Gamma)}, \tag{36}$$

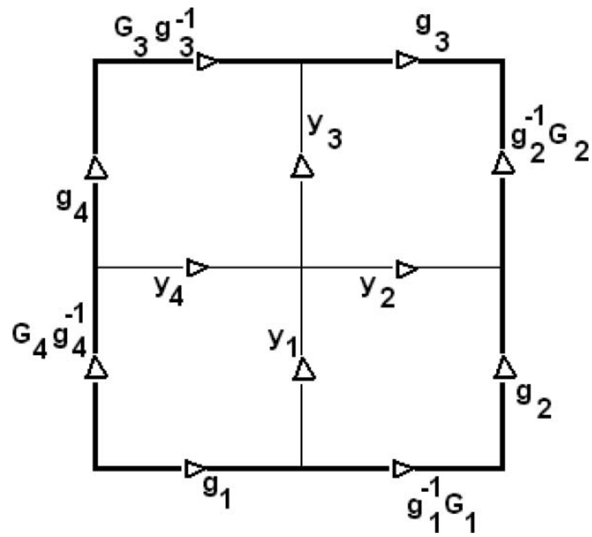

FIG. 1. Subdivision of one plaquette.

where $p(\mathcal{A}_\Gamma)$ is a continuous function in $\mathcal{A}_\Gamma$ and Z is a normalizing constant. In particular, make the simplifying assumptions:

- that $p(\mathcal{A}_\Gamma)$ is a product of plaquette functions

$$p(\mathcal{A}_\Gamma) = p\left(U_{\square_1}\right) p\left(U_{\square_2}\right) \cdots p\left(U_{\square_n}\right), \tag{37}$$

the product running over the n plaquettes contained in Γ and

- that $p(\cdot)$ is a central function, $p(xy) = p(yx)$ or, equivalently $p\left(y^{-1}xy\right) = p(x)$ with $x, y \in G$.

To be able to construct an interaction measure on the projective limit one has to check the consistency condition (7). In the directed set $\{\Gamma, \succ\}$ consider two elements Γ and Γ' which differ only in subdivision of a single plaquette (see Fig. 1), all the others being the same.

If there is a choice of densities $p(\cdot)$ that fulfill the consistency condition in this case, then it can be satisfied for whole directed set. For this case the consistency condition is simply

$$\int p'\left(g_1^{-1}G_1g_2y_2^{-1}y_1^{-1}\right) p'\left(y_2g_2^{-1}G_2g_3^{-1}y_3^{-1}\right) p'\left(y_4y_3g_3G_3^{-1}g_4^{-1}\right) p'\left(g_1y_1y_4^{-1}g_4G_4^{-1}\right)$$
$$\prod_{i=1}^{4} d\mu_H(g_i)\, d\mu_H(y_i)$$
$$= p\left(G_1G_2G_3^{-1}G_4^{-1}\right), \tag{38}$$

because integration over all the other plaquettes is the same in Γ and Γ'. p' and p denote the densities for plaquettes of size $\frac{a}{2^k}$ and $\frac{a}{2^{k-1}}$, respectively.

Using centrality of p', redefining

$$g_1y_1 = X_1, \qquad g_2y_2^{-1} = X_2, \qquad y_3g_3 = X_3^{-1}, \qquad y_4^{-1}g_4 = X_4^{-1}, \tag{39}$$

and using invariance of the normalized Haar measure, one may integrate over y_1, y_2, y_3, y_4 obtaining for the left-hand side of (38)

$$\int p'\left(X_1^{-1}G_1X_2\right) p'\left(X_2^{-1}G_2X_3\right) p'\left(X_3^{-1}G_3^{-1}X_4\right) p'\left(X_4^{-1}G_4^{-1}X_1\right) \prod_{i=1}^{4} d\mu_H(X_i).$$

Finally, if there is a sequence of central functions p', p'', p satisfying

$$\int p'(G_i X)\, p'\left(X^{-1} G_j\right) d\mu_H(X) \sim p''\left(G_i G_j\right),$$
$$\int p''(G_i X)\, p''\left(X^{-1} G_j\right) d\mu_H(X) \sim p\left(G_i G_j\right), \tag{40}$$

the consistency condition (38) would be satisfied, the proportionality constants being absorbed by the normalization constant Z'. p', p'', and p are the functions associated to the square plaquette with links of size $\frac{a}{2^k}$, the rectangular plaquette with links of size $\frac{a}{2^k}$ and $\frac{a}{2^{k-1}}$ and, finally, the square plaquette with links of size $\frac{a}{2^{k-1}}$. The sequence $\left(p', p'', p\right)$ corresponds to the subdivision of one plaquette. If such a sequence exists for all k, because all elements in the directed set $\{\Gamma, \succ\}$ may be reached by one-plaquette subdivisions, one obtains the following general result.

Theorem 1: *An interaction measure on the projective limit* $\mathcal{A} = \underleftarrow{\lim} \mathcal{A}_\Gamma$ *exists if a sequence of functions is found satisfying (40) for plaquette subdivisions of all sizes.*

Notice that:

- Eq. (40) is not necessarily a convolution semigroup property because p', p'', and p might be different functions and (40) is a proportionality relation, not an equality.

- The interaction measure may not be absolutely continuous with respect to the kinematical measure ν, constructed in Sec. II, because not all functions in the sequence (mostly in the small scale limit) might be continuous functions.

Although (40) is not exactly a convolution semigroup property, functions satisfying this condition may be obtained out of convolution semigroup kernels. Three cases will be separately analyzed, namely $G = U(1)\,, SU(2)\,, SU(3)$.

A. $G = U(1)$

An important convolution semigroup in $U(1)$,

$$U(1) = \left\{e^{i2\pi\theta}; \theta \in [0, 1)\,, \right\} \tag{41}$$

is the heat kernel semigroup,

$$K_1\left(e^{i2\pi\theta}, \beta\right) = \sum_{n\in\mathbb{Z}} \exp\left\{i2\pi n\theta - (2\pi n)^2 \beta\right\}, \tag{42}$$

which, by convolution with any initial condition $u_0(\theta)$, provides a solution to the $U(1)-$heat equation,

$$\left(\frac{\partial}{\partial\beta} - \frac{\partial^2}{\partial\theta^2}\right) u\left(e^{i2\pi\theta}, \beta\right) = 0. \tag{43}$$

Let us now use the $U(1)-$heat kernel in (42) with

$$p\left(e^{i2\pi\theta}\right) = K_1\left(e^{i2\pi\theta}, \beta\right), \tag{44}$$

to check the condition (40). One obtains

$$\int K_1\left(e^{i2\pi(\theta_1+\alpha)}, \beta'\right) K_1\left(e^{i2\pi(\theta_2-\alpha)}, \beta'\right) d\alpha = K_1\left(e^{i2\pi(\theta_1+\theta_2)}, 2\beta'\right). \tag{45}$$

Iterating this relation, one concludes that the measure consistency condition is satisfied with the choice (44) if

$$\beta' = \frac{\beta}{4}, \tag{46}$$

that is, each time one plaquette is subdivided the "time" label β in the densities associated to that plaquette should be divided by 4. Therefore, using the heat kernel as the density of the measure, a consistent measure is constructed in the projective limit.

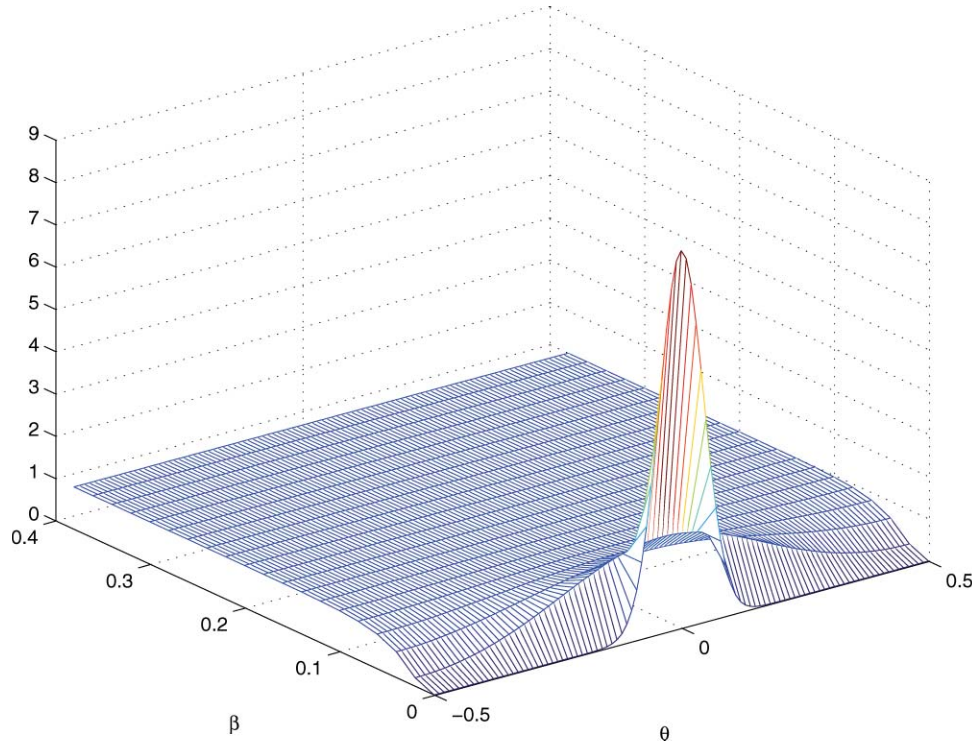

FIG. 2. (Color online) The $U(1)$ −heat kernel density.

An important consideration in establishing measures for the gauge spaces is to check whether, in the small scales, these measures correspond (or not) to the measures used by physicists for the same phenomena. In the $U(1)$ case this is easily seen by rewriting the heat kernel using the Jacobi identity

$$K_1\left(e^{i2\pi\theta},\beta\right)=\frac{1}{\sqrt{4\pi\beta}}\sum_{n\in\mathbb{Z}}\exp\left\{-\frac{(\theta+n)^2}{4\beta}\right\}. \tag{47}$$

Then, at very small scales β becomes extremely small. Therefore, in the sum of (47) only the $n=0$ and the $\theta\approx 0$ neighborhood contribute and, for the plaquette, one obtains a density proportional to

$$\exp\left\{-\frac{(\theta_1+\theta_2-\theta_3-\theta_4)^2}{4\beta}\right\}, \tag{48}$$

which indeed corresponds to the usual (small scale) $U(1)-$ measure.

In the lattice used to define the $\{\Gamma,\succ\}$ directed set when, in the lattice size $\frac{a}{2^k}$, $k\to\infty$ then $\beta\to 0$. In this limit the density (47) is no longer continuous, therefore, the interaction measure is not absolutely continuous with respect to the kinematical ν measure. Nevertheless the interaction measure has a generalized density in $\mathcal{H}^*(\mathcal{A})$, the distribution space of the projective gauge triplet described before. Summarizing:

Theorem 2: *Using the density (47) for each link-based loop, with $\beta_k=c\frac{1}{4^k}$ (c is an arbitrary constant) for each $\Gamma_k$in the directed set, a consistent measure is shown to exist in the projective limit$\mathcal{A}=\underleftarrow{\lim}\mathcal{A}_{\Gamma_k}$ of the $U(1)$ gauge theory. Furthermore, the measure coincides at small scales with the Gaussian measure (48). The measure in the projective limit is not absolutely continuous with respect to the kinematical measure, but it has a generalized density in $\mathcal{H}^*(\mathcal{A})$.*

In Fig. 2, one plots the density (47) for $\theta\in[0,1]$ and $\beta\in[0.001,0.4]$. One sees that for small β (small scales) the measure concentrates around $\theta=0$.

B. Non-abelian compact groups

Now that the $U(1)$ case is understood, a simple argument shows that a similar construction is possible for general compact groups. In a compact Lie group the heat kernel is

$$K(g,\beta) = \sum_{\lambda\in\Lambda^+} d_\lambda e^{-c(\lambda)\beta}\chi_\lambda(g), \tag{49}$$

with $g \in G$ and $\beta \in \mathbb{R}^+$. Λ^+ is the set of highest weights, d_λ and $\chi_\lambda(\cdot)$ are the dimension and the character of the $\lambda-$representation, and $c(\lambda)$ is the spectrum of the Laplacian (10)

$$(\Delta_G \chi_\lambda)(g) = c(\lambda)\chi_\lambda(g). \tag{50}$$

Using, as before, the heat kernel for the construction of the interaction measure, the condition (40) becomes

$$\begin{aligned}&\int d\mu_H(x) K(g_1x,\beta) K\left(x^{-1}g_2,\beta\right)\\ &= \sum_{\lambda,\lambda'\in\Lambda^+} d_\lambda d_{\lambda'} e^{-c(\lambda)\beta} e^{-c(\lambda')\beta} \int d\mu_H(x)\chi_\lambda(g_1x)\chi_{\lambda'}\left(x^{-1}g_2\right),\\ &= K(g_1g_2, 2\beta),\end{aligned} \tag{51}$$

the last equality following from Schur's orthogonality relations. Therefore, whenever the heat kernel is chosen as the density for the loops, the situation is quite similar to the $U(1)$ case, namely, on each subdivision of a plaquette

$$\beta \to \beta' = \frac{\beta}{4}. \tag{52}$$

Hence,

Theorem 3: *Using the heat kernel (49) for the density of each plaquette, with* $\beta_k = c\frac{1}{4^k}$ *(c is an arbitrary constant) for each* Γ_k *in the directed set, a consistent measure is shown to exist in the projective limit* $\mathcal{A} = \underleftarrow{\lim}\mathcal{A}_{\Gamma_k}$ *of the gauge theory with compact structure group G.*

Notice that in the verification of the condition (40) by (51) what is important is the characters orthogonality relation. Therefore, a different set of $c(\lambda)$'s might be used. This would lead to a different measure. The choice of which measure to choose would depend on physical considerations in the small scale limit.

The $SU(2)$ and $SU(3)$ cases will now be analyzed in detail.

1. *SU*(2)

Here

$$\begin{aligned}\Lambda^+ &= \{\lambda : 2\lambda \in \mathbb{N}, \lambda \geq 0\},\\ c(\lambda) &= \frac{1}{2}\lambda(\lambda+1),\\ d_\lambda &= 2\lambda+1,\end{aligned} \tag{53}$$

$$K_2(g,\beta) = \sum_{\lambda\in\Lambda^+}(2\lambda+1)e^{-\frac{1}{2}\lambda(\lambda+1)\beta}\frac{\sin\{(2\lambda+1)\pi x(g)\}}{\sin\{\pi x(g)\}}, \tag{54}$$

where $\pi x(g)$ is the angle coordinate of g in a maximal torus. It may be obtained from the 2×2 matrix representation of g by

$$\pi x(g) = \cos^{-1}\left(\frac{Tr(g)}{2}\right). \tag{55}$$

$K_2(g,\beta)$ may be rewritten as

$$K_2(g,\beta) = 2(2\pi)^{3/2}\frac{e^{\beta/8}}{\beta^{3/2}}\sum_{n\in\mathbb{Z}}\frac{x(g)+2n}{\sin\{\pi x(g)\}}\exp\left\{-2\pi^2\frac{(x(g)+2n)^2}{\beta}\right\}. \tag{56}$$

One also sees that for small β (small scales) the heat kernel density is dominated by the $n=0$ term in the sum above and by field configurations near $x(g)=0$. As in the $U(1)$ case, at $\beta=0$, the "densities" are no longer continuous functions and the measure in the projective limit space is not absolutely continuous with respect to the kinematical measure. There is however a generalized density in $\mathcal{H}^*(\mathcal{A})$.

The structure of the measure at small scales may now be compared with the naive continuum limit of lattice theory, as discussed, for example, in Ref. 15. There,

$$U = e^{icaA_\mu^b\tau_b}, \tag{57}$$

where c is a coupling constant, a is the lattice spacing, A_μ^b is the continuum gauge field, and τ_b is a Lie algebra basis. Then, for a plaquette in the 1, 2 plane

$$U_\square(A) = e^{icaA_1^b\left(x_\mu-\frac{1}{2}a\delta_{\mu2}\right)\tau_b}e^{icaA_2^b\tau_b\left(x_\mu+\frac{1}{2}a\delta_{\mu1}\right)}e^{-icaA_1^b\left(x_\mu+\frac{1}{2}a\delta_{\mu2}\right)\tau_b}e^{-icaA_2^b\left(x_\mu-\frac{1}{2}a\delta_{\mu1}\right)\tau_b}. \tag{58}$$

which for small a leads to

$$Tr(U_\square(A)) \simeq 2 - \frac{c^2a^4}{4}F_{12}^bF_{12}^b. \tag{59}$$

On the other hand, from (55), one knows that

$$TrU_\square = 2\cos(\pi x(U_\square)),$$

or $TrU_\square \simeq 2\left(1-\frac{1}{2}\pi^2x^2(U_\square)\right)$ for small x. Comparing with (59) one concludes that

$$x^2(U_\square) = \frac{c^2a^4}{4\pi^2}F_{12}^bF_{12}^b.$$

Therefore, for small β, the measure that uses (56) as its density coincides with the usual physical continuum measure.

In Fig. 3, one plots the density (56) for different values of $\theta=\pi x$ and β. As in the $U(1)$ case, one sees that for small β (small scales) the measure concentrates around $\theta=0$.

2. SU(3)

The irreducible representations of $SU(3)$ are labeled by two positive integers (p,q) ((1, 1) is the one-dimensional representation, $(2,1)=\underset{\smile}{3}$, $(1,2)=\underset{\smile}{3}^*$, etc.). The eigenvalues of the Laplacian and dimensions of the representations are

$$c(p,q) = \frac{1}{3}\left(p^2+q^2+pq\right)-1,$$
$$d_{p,q} = \frac{1}{2}pq(p+q). \qquad p,q=1,2,\cdots. \tag{60}$$

With $(\theta_1(g),\theta_2(g))$ denoting the angle coordinates of g in the maximal torus $diag\{\exp(i\theta_1),\exp(i\theta_2),\exp(-i(\theta_1+\theta_2))\}$, obtained from

$$\cos\theta_1(g)+\cos\theta_2(g)+\cos(\theta_1-\theta_2)(g) = ReTr(g),$$
$$\sin\theta_1(g)+\sin\theta_2(g)-\sin(\theta_1-\theta_2)(g) = ImTr(g), \tag{61}$$

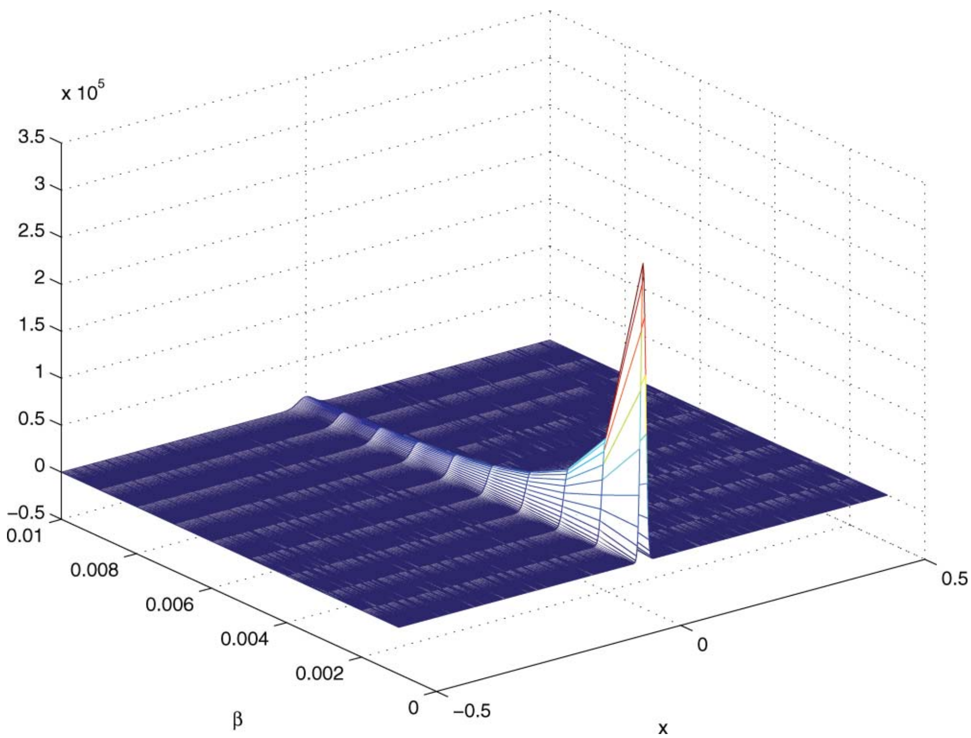

FIG. 3. (Color online) The $SU(2)$ − heat kernel density.

the $SU(3)$ heat kernel is[16]

$$K_3(g,\beta) = \frac{1}{s(\theta_1,\theta_2)} \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \{\theta_1-\theta_2+2\pi(l-m)\}\{\theta_1+2\theta_2+2\pi(l+2m)\} \times \{2\theta_1+\theta_2+2\pi(2l+m)\} \exp\left\{-\frac{1}{2\beta}\left[(\theta_1+2\pi l)^2+(\theta_2+2\pi m)^2+(\theta_1+2\pi l)(\theta_2+2\pi m)\right]\right\} \tag{62}$$

with

$$s(\theta_1,\theta_2) = 8\sin\frac{1}{2}(\theta_1-\theta_2)\sin\frac{1}{2}(2\theta_1+\theta_2)\sin\frac{1}{2}(\theta_1+2\theta_2). \tag{63}$$

That the choice $p(g) = K_3(g,\beta)$ and $p'(g) = K_3\left(g,\frac{\beta}{4}\right)$, for each plaquette subdivision, satisfies the consistency condition (40) follows from the general result (Theorem 3).

Now one checks the consistency of the result with the physical small scale (small β) limit. Inspection of (62) leads to the conclusion that for $\beta = 0$, the sums become dominated by the $l = m = 0$ term and the neighborhood ($\theta_1 \approx 0, \theta_2 \approx 0$). Then

$$K_3(g,\beta) \underset{\beta\to 0}{\longrightarrow} \exp\left\{-\frac{1}{2\beta}\left(\theta_1^2+\theta_2^2+\theta_1\theta_2\right)\right\}.$$

From (61), one also sees that for small θ_1, θ_2

$$3-\theta_1^2-\theta_2^2-\theta_1\theta_2 \simeq Tr(g),$$

which, equating with (59), leads to

$$\theta_1^2+\theta_2^2+\theta_1\theta_2 \simeq 1-\frac{c^2a^4}{4}F_{12}^bF_{12}^b.$$

That is, up to an irrelevant constant factor (to be absorbed in the measure normalization), one also obtains in the $SU(3)$ case the usual physical measure in the $\beta \to 0$ limit. Therefore, both for $SU(2)$ and $SU(3)$, one may take the measures so constructed as a definition of the Yang-Mills measure.

For small β, the measure is always concentrated in the neighborhood ($\theta_1 \approx 0, \theta_2 \approx 0$).

V. GAUGE MEASURE AND THE STRATA

Let S_A be the stabilizer (isotropy group) of a generalized connection $A \in \mathcal{A}$,

$$S_A = \{\gamma \in \mathcal{G} : \gamma A = A\}. \tag{64}$$

The action of the gauge group $\mathcal{G}$ on $\mathcal{A}$ leads to a stratification of $\mathcal{A}$ corresponding to the classes of equivalent *orbits* $\{gA; g \in \mathcal{G}\}$. The *stratum* $\Sigma(A)$ of A is the set of connections having isotropy groups $\mathcal{G}$−conjugated to that of A,

$$\Sigma(A) = \left\{B \in \mathcal{A} : \exists\gamma \in \mathcal{G} : S_B = \gamma S_A \gamma^{-1}\right\}. \tag{65}$$

The *configuration space of the gauge theory* is the quotient space $\mathcal{A}/\mathcal{G}$ and, therefore, a stratum is the set of points in $\mathcal{A}/\mathcal{G}$ that correspond to orbits with conjugated isotropy groups.

When G is a compact group the stratification is topologically regular. The map that, to each orbit, assigns the conjugacy class of its isotropy group is called the *type*. The set of strata carries a partial ordering of types, $\Sigma_\tau \subseteq \Sigma_{\tau'}$ with $\tau \le \tau'$, if there are representatives S_τ and $S_{\tau'}$ of the isotropy groups such that $S_\tau \supseteq S_{\tau'}$. The maximal element in the ordering of types is the class of the center $Z(G)$ of G and the minimal one is the class of G itself. Furthermore, $\cup_{t \ge \tau} \Sigma_t$ is open and Σ_τ is open in the relative topology in $\cup_{t \le \tau} \Sigma_t$.

Because the isotropy group of a connection is isomorphic to the centralizer of its holonomy group, the strata are in one-to-one correspondence with the Howe subgroups of G, that is, the subgroups that are centralizers of some subset in G. Given an holonomy group H_τ associated to a connection A of type τ, the stratum of A is classified by the conjugacy class of the isotropy group S_τ, that is, the centralizer of H_τ,

$$S_\tau = Z(H_\tau). \tag{66}$$

An important role is also played by the centralizer of the centralizer,

$$H'_\tau = Z(Z(H_\tau)), \tag{67}$$

that contains H_τ itself. If H'_τ is a proper subgroup of G, the connection A reduces locally to the subbundle $P_\tau = \left(M, H'_\tau\right)$. Global reduction depends on the topology of M, but it is always possible if P is a trivial bundle. H'_τ is the structure group of the *maximal subbundle* associated to type τ. Therefore, the types of strata are also in correspondence with types of reductions of the connections to subbundles. If S_τ is the center of G the connection is called *irreducible* and all others are called *reducible*. The stratum of the irreducible connections is called the *generic stratum*.

Now, for $G = SU(2)$ and $SU(3)$ one describes the strata and how they stand in relation to the measures defined before. In $G = SU(2)$, the isotropy groups (equivalently, the centralizers of the holonomy) and the structure groups of the maximal subbundles are

	S_A	H'_A
1	$\mathbb{Z}_2$	$SU(2)$
2	$U(1)$	$U(1)$
3	$SU(2)$	$\mathbb{Z}_2$

(68)

There are three strata. Stratum 1 is the generic stratum. The other two are reducible strata.

A $SU(2)$ transformation may be parametrized by

$$\exp\left(-i\frac{\theta}{2}\vec{n}\cdot\vec{\sigma}\right) = \mathbf{1}\cos\frac{\theta}{2} + i\left(\vec{n}\cdot\vec{\sigma}\right)\sin\frac{\theta}{2}. \tag{69}$$

Geometrically, the $SU(2)$ group may be pictured as a sphere of radius 4π, with all the points at radial distance 2π identified to $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and the points at radius 4π identified with the center of the sphere. The reducible stratum 3 corresponds to a $\mathbb{Z}_2-$bundle, that is, to homomorphisms of the loops to a two point space ($\theta = 0, \theta = 2\pi$). Each reducible stratum of type 2 is a $U(1)-$bundle corresponding to homomorphisms of the loops to the $SU(2)$ transformations along one radius (fixed $\vec{n}$, variable θ). Because adjoint transformations transform any radius into any other, all $U(1)-$bundles are equivalent and represent the same gauge configurations. Finally, the (generic) stratum 1 corresponds to homomorphisms to arbitrary $SU(2)$ transformations.

From (55), one sees that the intensity of the measure (54) only depends on the first term $\mathbf{1}\cos\frac{\theta}{2}$ in the parameterization (69). Therefore, all strata approach the small $x(g)$ region where the measure is peaked and, therefore, they all are expected to be relevant in the physical behavior of the gauge theory.

For $G = SU(3)$, the isotropy groups and the structure groups of the maximal subbundles are

	S_A	H'_A
1	$\mathbb{Z}_3$	$SU(3)$
2	$U(1)$	$U(2)$
3	$U(1)\times U(1)$	$U(1)\times U(1)$
4	$U(2)$	$U(1)$
5	$SU(3)$	$\mathbb{Z}_3$

(70)

There are five strata. Stratum 1 is the generic stratum. All others are reducible strata. To find out their relevance in the framework of the measure (62) one uses the following parametrization[17] for an arbitrary $SU(3)$ transformation,

$$U = u_0 + i u_k \lambda_k, \tag{71}$$

where $u_k = \alpha a_k + \beta d_{ijk} a_i a_j$ for an octet vector a_k, with u_0, α, and β being functions of the invariants

$$\begin{aligned} I_2(a) &= a_i a_i, \\ I_3(a) &= d_{ijk} a_i a_j a_k, \end{aligned} \tag{72}$$

which can be built from the vector a_k. Then

$$\begin{aligned} u_0 &= \tfrac{1}{3}\textstyle\sum_n e^{i\varphi_n}, \\ \alpha &= -\textstyle\sum_n \varphi_n e^{i\varphi_n}\left(3\varphi_n^2 - I_2\right)^{-1}, \\ \beta &= -\textstyle\sum_n e^{i\varphi_n}\left(3\varphi_n^2 - I_2\right)^{-1}, \end{aligned} \tag{73}$$

with

$$\varphi_n = 2\left(\frac{I_2}{3}\right)^{\frac{1}{2}} \cos\frac{1}{3}\left(2\pi n + \cos^{-1}\left(\sqrt{3} I_3 I_2^{-\frac{3}{2}}\right)\right); \quad n = 1, 2, 3. \tag{74}$$

As before, one sees from (61) that the intensity of the measure depends only on $Re u_0$ and $Im u_0$, whereas the choice of the subbundle for each stratum depends on the choice of the u_k coefficients. Therefore, there is, for all strata, a range of parameters that approaches the region where the measure is peaked.

VI. REMARKS AND CONCLUSIONS

(1) The construction of the kinematical (Haar) measure on hypercubic lattices is, as stated before, a particular case of the general projective techniques in Refs. 1, 2, and 3. What is essentially new is the construction of the gauge projective triplet, essential for the infinite-dimensional calculus and, of course, the interaction measures.

(2) The Cartesian product point of view, in the construction of the projective limit $\mathcal{A} = \varprojlim \mathcal{A}_\Gamma$ and of the triplet $\mathcal{H}(\mathcal{A}) \subset L^2(\mathcal{A}, d\nu) \subset \mathcal{H}^*(\mathcal{A})$, means that a consistent framework is obtained for the description of gauge theories at all length scales. In this setting the lattice structure, underlying the Cartesian product, is not an approximation scheme but a framework to characterize the theory at the several length scales.

(3) In Ref. 18, Ashtekar *et al.* use the exponential of the Wilson action to construct the Yang-Mills measure in two dimensions. This measure does not lead naturally to a proof of the consistency condition needed for the projective construction. Of course, their construction leads, in two dimensions, to a consistent construction of the continuum limit but, lacking the projective consistency, cannot be a framework to characterize the theory at all length scales.

(4) For each $\mathcal{A}_\Gamma = Hom(E(\Gamma), G) \sim G^{\#E(\Gamma)}$, the interaction measure $\frac{1}{Z} p(\mathcal{A}_\Gamma)(d\mu_H)^{\#E(\Gamma)}$, constructed in Sec. IV is absolutely continuous with respect the kinematical measure. However, when $\beta \to 0$ the "density" ceases to be a continuous function. Therefore, for the full projective limit, the interaction measure is not absolutely continuous with respect to the kinematical measure. That such a result was to be expected, follows also from the analysis of Fleischhack *et al.*[19,20] who, starting from very general conditions concluded that in gauge theories there is a "breakdown of the action method." However, in the gauge triplet framework, one may consider that a generalized density exists in $\mathcal{H}^*(\mathcal{A})$.

(5) From a mathematical physics point of view, a theory is completely determined whenever its measure is specified. In that sense, the construction in this paper, insuring a consistent measure on the projective limit is a rigorous specification of *a* Yang-Mills theory. Traces of holonomies, for example, merely play the role of "moments" of the measure, carrying, therefore, the same (or less) information than the measure itself.

The fact that the small β component of the interaction measures coincides with the usual small distance representations of the abelian and the Yang-Mills measure, mean that they will inherit the same qualitative physical properties. In particular, the absence of a mass gap in the abelian case and, for the non-abelian case, the same qualitative properties as obtained, for example, from asymptotic dynamics,[21] namely the fact that either there is spontaneous violation of the symmetry or all asymptotic states are color singlets.

For example, an area law at small scales ($\beta \to 0$) for 2D Wilson loops follows from a simple calculation. Consider the $SU(2)$ case and a loop containing N plaquettes. The trace of the product of the N group elements is

$$2\Pi_{i=1}^{N} \cos\theta(g_i) + R,$$

where the $\theta(g_i)'s$ are the maximal torus angles and R refers to the terms that depend on the other group angles. Under integration, only the first term will contribute because the heat kernel density only depends on the maximal torus angles and in 2D one may change coordinates from link variables to isolate the plaquette maximal torus coordinate. The replacement of the kinematical measure on the based loops associated to the links by the Haar measure on plaquette loops, holds for integration of gauge invariant functions and two dimensions because of the independence[22] of plaquette loops. Such simple calculation does not extend to higher dimensions.

Then, from (56) it follows that, for small β the heat kernel density being dominated by the $n = 0$ term, one obtains for each plaquette an integral

$$\frac{1}{Z}\int d\theta \cos\theta \frac{\theta}{\sin\theta} e^{-2\frac{\theta^2}{\beta}} \underset{\beta\to 0}{\simeq} e^{-\frac{\beta}{8}},$$

where $Z = \int d\theta \frac{\theta}{\sin\theta} e^{-2\frac{\theta^2}{\beta}}$. With N plaquettes one obtains $e^{-\frac{N\beta}{8}}$, and this is an area law. Notice that when the lattice edge $a \to \frac{a}{2}$, $\beta \to \frac{\beta}{4}$, and $N \to 4N$, therefore one obtains the right scaling. One notices that, for two dimensions in the continuum limit, this result coincides with the simple loop result of.[18]

(6) As to the role of non-generic strata in gauge theories, the fact that there is, for all strata, a range of parameters that approaches the region where the measure is peaked, emphasizes the

importance of these strata for the structure of low-lying excitations. A similar result[23] had been obtained using a ground-state approximation[24,25] for the non-abelian ground-state.

[1] A. Ashtekar and C. J. Isham, Class. Quantum Grav. **9**, 1433 (1992).
[2] A. Ashtekar and J. Lewandowski, J. Geom. Phys. **17**, 191 (1995).
[3] A. Ashtekar and J. Lewandowski, J. Math. Phys. **36**, 2170 (1995).
[4] D. Marolf and J. M. Mourão, Commun. Math. Phys. **170**, 583 (1995).
[5] A. Ashtekar and J. Lewandowski, "Representation theory of analytic holonomy C* algebras," in *Knots and Quantum Gravity*, edited by J. Baez (Oxford University Press, Oxford, 1994).
[6] J. Baez and S. Sawin, J. Funct. Anal. **150**, 1 (1997).
[7] C. Fleischhack, Commun. Math. Phys. **214**, 607 (2000).
[8] C. Fleischhack, e-print arXiv:math-ph/0107022.
[9] C. Fleischhack, J. Geom. Phys. **45**, 231 (2003).
[10] J. Aastrup and J. M. Grimstrup, e-print arXiv:0911.4141.
[11] J. Kisynski, Stud. Math. **30**, 141 (1968).
[12] K. Maurin, *General Eigenfunction Expansions and Unitary Representations of Topological Groups* (PWN-Polish Scientific Publication, Warszawa, Polland, 1968).
[13] T. Deck, Infinite Dimen Anal., Quantum Probab., Relat. Top. **3**, 337 (2000).
[14] I. M. Gelfand and G. E. Shilov, *Generalized Functions* (Academic, New York, 1968), Vol. 2.
[15] M. Creutz, *Quarks, Gluons, and Lattices* (Cambridge University Press, Cambridge, 1983).
[16] B. E. Baaquie, J. Phys. A **21**, 2651 (1988).
[17] A. J. Macfarlane, A. Sudbery, and P. H. Weisz, Commun. Math. Phys. **11**, 77 (1968).
[18] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourão, and T. Thieman, J. Math. Phys. **38**, 5453 (1997).
[19] C. Fleischhack,e-print arXiv:math-ph/0107022.
[20] C. Fleischhack, e-print arXiv:math-ph/0109030.
[21] R. Vilela Mendes, Phys. Rev. **D18**, 4726 (1978).
[22] C. Fleischhack, J. Math. Phys. **41**, 76 (2000).
[23] R. Vilela Mendes, J. Phys. A **37**, 11485 (2004).
[24] R. Vilela Mendes, Z. Phys. C: Part. Fields **54**, 273 (1992).
[25] R. Vilela Mendes in *Proceedings of the International Conference on Mathematical Analysis of Random Phenomena*, Hammamet, Tunisia, 12–17 September 2005 (World Scientific, Singapore, 2006), pp. 169–177.