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Complex hidden symmetries in real spacetime and their algebraic structures

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ABSTRACT

Considering real spacetime as a Lorentzian fiber in a complex manifold, there is a mismatch of the elementary linear representations of their symmetry groups, the real and complex Poincaré groups. No spinors are allowed as linear irreducible representations for the complex case, but when a spin^h structure is implemented on the associated principal bundles, one is naturally led to an algebraic structure similar to the one of the standard model. This last (dynamical) structure might therefore be inherited from the kinematical symmetries of a larger space.

1. Introduction

The coordinates, used in the space-time we live in, take values on the reals, with four dimensions and a Lorentzian metric. However in many instances, in the formulation of physical theories, it is convenient to consider the real field to be embedded into higher dimension division algebras. For example, in quantum field theory calculations it is convenient to study Wick-rotated Green's functions, and complex extensions shed light on the study of singularities and topological effects in general relativity [1] [2] [3] [4]. Also, in the framework of complex manifolds, twistor theory [5] renders null lines and null surfaces as the basic building blocks, with space-time points as derived entities, etc. A more extreme view, of course, would be to consider that our real-coordinates space-time is a lower dimensional subspace of an actual larger space, of which our senses are unaware of.

Whatever the point of view, a relevant question is: When real space-time is embedded into a higher dimensional division algebra structure, what properties of the higher structure are inherited or non-inherited by the real space-time? In particular this refers to the symmetry groups and their representations. Here, the following particular setting will be considered: In a complex manifold $\mathcal{M}$ with (local) metric $G=(1,-1,-1,-1)$ and symmetry group $L_{\mathbb{C}}=U(1,3)$

$$\Lambda^{\dagger} G \Lambda = G$$

$\left(\Lambda \in L_{\mathbb{C}}\right)$, real space-time is considered to be a $4-$dimensional subspace $\mathcal{S}$ with Lorentzian metric, that is, a Lorentzian fiber in the Grassmannian $Gr_4(\mathcal{M})$ of four-dimensional frames of $\mathcal{M}$. The symmetry group of $\mathcal{S}$ is the Lorentz group $L_{\mathbb{R}}=SO(1,3)$. Because here only local manifold effects are considered, one also includes as symmetries of $\mathcal{M}$ and $\mathcal{S}$, respectively, the complex and real translations. Therefore the (local) symmetry group of $\mathcal{M}$ is the 24$-$parameter complex Poincaré group $P_{\mathbb{C}}$ and the (local) symmetry group of $\mathcal{S}$ is the 10$-$parameter real Poincaré group $P_{\mathbb{R}}$.

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According to a well established tradition [6], the elementary physical states corresponding to these symmetry groups would be their unitary irreducible representations. When these were studied [7], a remarkable difference was found between the elementary representations of $P_{\mathbb{C}}$ and those of $P_{\mathbb{R}}$. Namely, half-integer spin states are compatible with $P_{\mathbb{R}}$ but not with $P_{\mathbb{C}}$ and, for integer spin states, a superselection rule emerges when discrete symmetries are included [7] [8]. Here the main concern will be the half-integer spin states. Their absence as elementary states of $P_{\mathbb{C}}$ may be physically interpreted as meaning that such states cannot be "rotated" away from each fiber to the other fibers of the Grassmannian $Gr_4(\mathcal{M})$ of four-dimensional frames in $\mathcal{M}$. As for symmetries, one might accept that the symmetry group of each Lorentzian fiber is simply the real Poincaré group $P_{\mathbb{R}}$ and that nothing is inherited from the larger complex group $P_{\mathbb{C}}$. Alternatively one might insist that $P_{\mathbb{C}}$ is actually also the symmetry group of the fiber, but that the symmetry is implemented in a subtler way. This second point of view leads to an analysis of when spin structures are or are not realized in coset manifolds. And to what additional quantum numbers appear when one insists on their implementation.

The coset manifolds that will be considered are those associated to the isotropy (little) groups of massive and massless states, $SU(3)/SO(3)$, $SO(3)/SO(2)$. The tools of algebraic topology provide a quick answer to the existence or non-existence of spin states, but the exact nature of the additional algebraic structures that appear, when spin structures are implemented, requires a detailed analysis of the group representations. A partial announcement of these results was included in a recent letter [9].

The commutation relations of the Lie algebras of $P_{\mathbb{C}}$ and $P_{\mathbb{R}}$ are listed in Appendix A, with $p_{\mathbb{C}} = \{K_\mu, H_\mu, M_{\mu\nu}, N_{\mu\nu}\}$ and $p_{\mathbb{R}} = \{K_\mu, M_{\mu\nu}\}$. A convenient 4−dimensional matrix representation of $M_{\mu\nu}$ and $N_{\mu\nu}$ is included in Appendix B.

2. The coset manifold SU(3)/SO(3)

The elementary unitary representations of $P_{\mathbb{C}}$ are classified by the eigenvalues of the first Casimir operator P^2

$$P^2 = K^\mu K_\mu + H^\mu H_\mu \tag{1}$$

and the representations of their isotropy (little) groups. For massive states, $P^2 > 0$, the little group is $U(3) \simeq U(1) \oplus SU(3)$ with Lie algebra (cf. Appendix B)

$$\mathcal{L}_{U(3)} = \{R_i,\ U_i,\ C_1, C_2, C_3\}\, ; i = 1,2,3, \tag{2}$$

the generators for $SU(3)$ and $U(1)$ being

$$\mathcal{L}_{SU(3)} = \{R_i,\ U_i,\ C_1 - C_2, C_1 + C_2 - 2C_3\}\, ; i = 1,2,3$$
$$\mathcal{L}_{U(1)} = \{C_1 + C_2 + C_3\}$$

For $P_{\mathbb{R}}$ the first Casimir operator is

$$p^2 = K^\mu K_\mu \tag{3}$$

The rotation subgroup in $SU(3)$ is generated by $\{R_i\}$, which are also the generators of the little group for massive states in the real fiber $\mathcal{S}$

$$\mathcal{L}_{SO(3)} = \{R_i\} \tag{4}$$

The little group $SO(3)$ in each real fiber has (double-valued) half-integer spin representations, whereas $SU(3)$ has no half-integer representations for the subgroup generated by $\{R_i\}$. This has a clear algebraic topology interpretation when the $SU(3)$ group is represented as a $SO(3)$ principal bundle $\mathcal{P}_{SO(3)}$. The base of this bundle is the coset manifold $SU(3)/SO(3)$ and existence of half-integer states in $SU(3)$ is synonymous with the problem of existence of spin structures in the coset manifold. A spin structure implies the existence of a double cover $\mathcal{P}_{Spin(3)}$ of $\mathcal{P}_{SO(3)}$ and a commutative diagram

$$\begin{array}{ccc} \mathcal{P}_{Spin(3)} & \rightarrow & \mathcal{P}_{SO(3)} \\ & \searrow\ \swarrow & \\ & SU(3)/SO(3) & \end{array}$$

A necessary and sufficient condition for the existence of a spin structure on the bundle is the vanishing of the second Stiefel-Whitney class [10]. Let

$$\mathcal{L}_{SU(3)} = \mathcal{H}^{(1)} \oplus \mathcal{K}^{(1)} = \{R_i\} \oplus \{U_i, C_1 - C_2, C_1 + C_2 - 2C_3\} \tag{5}$$

Because $[\mathcal{H}^{(1)}, \mathcal{K}^{(1)}] \subset \mathcal{K}^{(1)}$ and $[\mathcal{K}^{(1)}, \mathcal{K}^{(1)}] \subset \mathcal{H}^{(1)}$ the coset space $SU(3)/SO(3)$ is reductive and a symmetric space. $SU(3)/SO(3)$ is a 5−dimensional manifold known as the Wu manifold [11] [12] [13]. Its topological properties are known. Its Stiefel-Whitney class being nonzero, it is not a spin-manifold, nor a spin^c-manifold. It is however a spin^h-manifold [14] [15], meaning that to implement a spin structure over $SU(3)/SO(3)$ one has to add, to the $\mathcal{P}_{SO(3)}$ manifold, another manifold $\mathcal{P}'_{SU(2)}$ with a quaternion ($Sp(1) \simeq SU(2)$) structure group in the fibers and the same base space (a Whitney sum $\mathcal{P}_{SO(3)} \oplus \mathcal{P}'_{SU(2)}$). In this way one may have spinor states, consistent with both the $U(3)$ of the full isotropy group of massive states and the $SO(3)$ of the fibers. Hence, the physical consequences of insisting on a full $P_{\mathbb{C}}$− symmetry for spinor massive states would be:

- The emergence of additional quantum numbers associated to the $\mathcal{P}'_{SU(2)}$ auxiliary bundle.

- A degeneracy of the spinor states parametrized by the coset space $SU(3)/SO(3)$,[1] where a $SU(3)$ group operates transitively.

This is as far as one may go from the algebraic topology data. More precise information on the nature of the additional quantum numbers is obtained from comparing the representations of $SU(3)$ with those in $\mathcal{P}_{SO(3)} \oplus \mathcal{P}'_{SU(2)}$.

The following conclusions might probably be obtained from simple inspection of the matrices in the Appendix B, but become clearer by using a creation operator representation of the Lie algebra of $SU(3)$. Let $\left\{ a_i^\dagger, a_i; i = 1,2,3 \right\}$ be a set of bosonic creation and annihilation operators. Then the generators of the $SU(3)$ algebra have the representation

$$R_i = \epsilon_{ijk} a_j^\dagger a_k;\ U_i = i \left| \epsilon_{ijk} \right| a_j^\dagger a_k;\ C_1 - C_2 = i\sqrt{2} \left(a_1^\dagger a_1 - a_2^\dagger a_2 \right);$$
$$C_1 + C_2 - 2C_3 = i\sqrt{2} \left(a_1^\dagger a_1 + a_2^\dagger a_2 - 2a_3^\dagger a_3 \right) \tag{6}$$

A $SU(3)$ triplet is formed by 3 spin-one states

$$|1) = \frac{1}{\sqrt{2}} \left(a_1^\dagger + i a_2^\dagger \right) |0\rangle;\ |-1) = \frac{1}{\sqrt{2}} \left(a_1^\dagger - i a_2^\dagger \right) |0\rangle;\ |0) = a_3^\dagger |0\rangle \tag{7}$$

Now one has to match these states by composing an observable spin $\frac{1}{2}$ state in the real fiber with another spin $\frac{1}{2}$ state from the auxiliary $\mathcal{P}'_{SU(2)}$ bundle. Notice however that the three states in (7) not only transform under the R_i rotation generators, but also have nontrivial transformation properties under the other $SU(3)$ generators, in particular the U_i's. This must also be matched by the composite states.

Let $c_i^{\uparrow\dagger}$ and $c_j^{\downarrow\dagger}$ be the spin up and spin down creator operators of the observable state in the real fiber and $b_i^{\uparrow\dagger}$ and $b_j^{\downarrow\dagger}$ the corresponding operators of the auxiliary bundle. The i and j labels stand for the additional quantum numbers needed to match the $SU(3)$ transformation properties of the states. The following scheme satisfies the matching requirements

$$|1) \to b_i^{\uparrow\dagger} c_i^{\uparrow\dagger} |0\rangle;\ |-1) \to b_j^{\downarrow\dagger} c_j^{\downarrow\dagger} |0\rangle;\ |0) \to \left(b_i^{\uparrow\dagger} c_j^{\downarrow\dagger} + b_j^{\downarrow\dagger} c_i^{\uparrow\dagger} \right) |0\rangle \tag{8}$$

Notice that the “internal” quantum numbers i and j of the spinor states $c_i^{\uparrow\dagger}$ and $c_j^{\downarrow\dagger}$ may be different for different spin projections, showing the chiral nature of this quantum number. The spinor doublets in the real fiber are thus endowed with a doublet of “internal” quantum numbers, that is, a $SU(2)$ structure inherited from the $\mathcal{P}'_{SU(2)}$ bundle. On the other hand when two such spinors are coupled as in (8) they form a state that transforms under $SU(3)$. Therefore the spin^h mechanism not only generates an additional $SU(2)$ structure on the elementary states, but also a kind of “flavor $SU(3)$” on the coupled states. This $SU(3)$ structure, that at the complex space-time level would be an exact symmetry, may connect states with different $p^2 = K^\mu K_\mu$ values (the real mass square) because they are eigenvalues of P^2, not of p^2, and from the commutation relations in Appendix A one sees how the U_i operators permute the K_μ and H_μ operators. Summarizing, an exact complex symmetry may appear as a broken symmetry in the real fibers. An additional feature of the (8) structure is the existence of a spin zero state

$$\left(b_i^{\uparrow\dagger} c_j^{\downarrow\dagger} - b_j^{\downarrow\dagger} c_i^{\uparrow\dagger} \right) |0\rangle$$

With the additional $\mathcal{P}'_{SU(2)}$ bundle, spinor states are consistently constructed at each point of the coset space $SU(3)/SO(3)$. Therefore, a second consequence is a degeneracy of the spinor states parametrized by the points of the coset space. Recalling [7] that the little group of massive states for the complex Poincaré group $P_\mathbb{C}$ is actually $U(3)$ one should in fact consider the 6−dimensional manifold $\mathcal{M}^{(1)} = U(3)/SO(3)$. A detailed characterization of this space is important, particularly if the $U(3)$ symmetry, in its double role, is gauged. A coordinate system in $\mathcal{M}^{(1)}$ is obtained by choosing a point in each coset $(\mathbf{x})$ and when an element $g \in U(3)$ acts upon $(\mathbf{x})$

$$g(\mathbf{x}) = \left(\mathbf{x}' \right) h(\mathbf{x}, g) \tag{9}$$

with $h(\mathbf{x}, g) \in SO(3)$.

In the spinor degeneracy manifold $\mathcal{M}^{(1)} = U(3)/SO(3)$, which will be tentatively called the “color manifold”, a set of coordinates may be obtained in several ways:

1) Let $A = \left\{ a_{ij} \right\}$ be an arbitrary $U(3)$ matrix. Then $\Sigma_{j=1}^3 a_{ij} a_{ij}^* = 1$ but $\Sigma_{j=1}^3 a_{ij} a_{ij} = C_i$, a non trivial complex number. Acting on A on the right with a $SO(3)$ matrix, C_i remains invariant. Therefore the 3 complex numbers C_i $(i = 1,2,3)$ provide a set of 6 coordinates for $U(3)/SO(3)$.

2) Alternatively, starting from the identity of $U(3)$, $U(3)/SO(3)$ may be parametrized by choosing a representative element in each coset. Choose a symmetric set of coordinates in $\mathcal{M}^{(1)}$ by defining

$$Q_1 = U_1 + C_1; Q_2 = U_1 - C_1$$

[1] Actually $U(3)/SO(3)$.

$$Q_3 = U_2 + C_2; Q_4 = U_2 - C_2$$
$$Q_5 = U_3 + C_3; Q_6 = U_3 - C_3 \tag{10}$$

with the representative group element in each coset being

$$(\mathbf{x}) = e^{\left(x_1 Q_1 + x_2 Q_2\right)} e^{\left(x_3 Q_3 + x_4 Q_4\right)} e^{\left(x_5 Q_5 + x_6 Q_6\right)} \tag{11}$$

Using the matrices in Appendix B, $(\mathbf{x})$ is represented by the 3×3 matrix $L(\mathbf{x})$ with elements

$$\begin{aligned}
L_{11}(\mathbf{x}) &= e^{i\sqrt{2}(x_1 - x_2)} \cos\left(x_3 + x_4\right) \cos\left(x_5 + x_6\right) \\
L_{12}(\mathbf{x}) &= e^{i\sqrt{2}(x_1 - x_2)} \cos\left(x_3 + x_4\right) \sin\left(x_5 + x_6\right) \\
L_{13}(\mathbf{x}) &= i e^{i\sqrt{2}(x_1 - x_2)} e^{i\sqrt{2}(x_5 - x_6)} \sin\left(x_3 + x_4\right) \\
L_{21}(\mathbf{x}) &= -\sin\left(x_1 + x_2\right) \sin\left(x_3 + x_4\right) \cos(x_5 + x_6) \\
&\quad + i e^{i\sqrt{2}(x_3 - x_4)} \cos\left(x_1 + x_2\right) \sin\left(x_5 + x_6\right) \\
L_{22}(\mathbf{x}) &= -i \sin\left(x_1 + x_2\right) \sin\left(x_3 + x_4\right) \sin(x_5 + x_6) \\
&\quad + e^{i\sqrt{2}(x_3 - x_4)} \cos\left(x_1 + x_2\right) \cos\left(x_5 + x_6\right) \\
L_{23}(\mathbf{x}) &= i e^{i\sqrt{2}(x_5 - x_6)} \sin\left(x_1 + x_2\right) \cos\left(x_3 + x_4\right) \\
L_{31}(\mathbf{x}) &= i \cos\left(x_1 + x_2\right) \sin\left(x_3 + x_4\right) \cos(x_5 + x_6) \\
&\quad - e^{i\sqrt{2}(x_3 - x_4)} \sin\left(x_1 + x_2\right) \sin\left(x_5 + x_6\right) \\
L_{32}(\mathbf{x}) &= -i \sin\left(x_1 + x_2\right) \sin\left(x_3 + x_4\right) \sin(x_5 + x_6) \\
&\quad + e^{i\sqrt{2}(x_3 - x_4)} \cos\left(x_1 + x_2\right) \cos\left(x_5 + x_6\right) \\
L_{33}(\mathbf{x}) &= e^{i\sqrt{2}(x_5 - x_6)} \cos\left(x_1 + x_2\right) \cos\left(x_3 + x_4\right)
\end{aligned} \tag{12}$$

The set $\left\{Q_i; i = 1, \cdots, 6\right\}$ is an orthogonal set

$$Tr\left(Q_i Q_j\right) = -2\delta_{ij}$$

A (Riemannian) metric g may be established in $\mathcal{M}^{(1)}$ by transporting this quadratic form by the group elements $L(\mathbf{x})$

$$g_{\mathbf{x}}(X, Y) = Tr(L^{-1}(\mathbf{x})^* X L^{-1}(\mathbf{x})^* Y)$$

with $X, Y \in T\mathcal{M}^{(1)}$, the tangent space to $\mathcal{M}^{(1)}$. The metric is computed from (12) by

$$\begin{aligned}
g_{ij} &= Tr\left(L^{-1}(\mathbf{x})\left(\partial_{x_i} L(\mathbf{x})\right) L(\mathbf{x}) L^{-1}(\mathbf{x})\left(\partial_{x_j} L(\mathbf{x})\right) L(\mathbf{x})\right) \\
&= Tr\left(\partial_{x_i} L(\mathbf{x}) \partial_{x_j} L(\mathbf{x})\right)
\end{aligned} \tag{13}$$

From the Lévy connection associated to this metric a set of gauge fields is obtained in the "color manifold" $\mathcal{M}^{(1)}$. Notice however that because of the Whitney sum $\mathcal{P}_{SO(3)} \oplus \mathcal{P}'_{SU(2)}$ there is also a connection for the $SU(2)$ structure and, therefore, new gauge fields associated to the degrees of freedom associated to the $SU(2)$ structure.

The metric so obtained in the $\mathbf{x}$ coordinate representation has complex nondiagonal elements for arbitrary points of the $\mathcal{M}^{(1)}$ manifold. A diagonal metric may be obtained on a vielbein of Lie-algebra valued $1-$forms Γ_k by

$$\Omega(\mathbf{x}) = L^{-1}(\mathbf{x}) \partial_{x_i} L(\mathbf{x}) d x_i$$
$$\Gamma_k(\mathbf{x}) = Tr\left(Q_k \Omega(\mathbf{x})\right)$$

Γ_k^2 are then the diagonal elements of the metric, but the complexity is, of course, in the vielbein.

Summarizing: When $P_{\mathbb{C}}-$ symmetry is inherited by the real Lorentzian fibers, existence of spinor states implies a degeneracy of massive spinor states, parametrized by a $6-$dimensional manifold as well as, through a spin^h implementation on a Wu manifold, additional degrees of freedom ($SU(2)$ and $SU(3)$), which would correspond to an exact symmetry in the (complex) $P_{\mathbb{C}}$ group, but may appear as a broken symmetry for $P_{\mathbb{R}}$.

This structure is evocative of the structure of the particle physics standard model, but is different. In the standard model the color space is a linear representation of $SU(3)$ and the color states are linear elements of this representation. Here, in contrast, color and anti-color states would be particular directions in a curved manifold. More precisely, a frame of sections in a vector bundle. Also in the $SU(3) \times SU(2) \times U(1)$ group of the standard model, the $SU(2)$ structure is an independent structure, whereas here it appears as a result of the Whitney sum with $\mathcal{P}'_{SU(2)}$ and, when the representations for composite states are matched, it turns out to be related to the $SU(3)$ subgroup of the $U(1,3)$ group.

3. Massless states and U(2)/SO(2)

For the irreducible representations of $P_{\mathbb{C}}$, in the case $P^2 = K^\mu K_\mu + H^\mu H_\mu = 0$, $p^\mu \neq 0$, the algebra of the little group $G_2^{\mathbb{C}}$ for the momentum $\left(p^0, 0, 0, p^0\right)$ is a semi-direct sum of two subalgebras N^c and H^c with $[N^c, N^c] \subset N^c$; $[H^c, N^c] \subset N^c$; $[H^c, H^c] \subset H^c$. N^c, the algebra of the normal subgroup $\mathcal{N}^c$, is a 2-dimensional Heisenberg algebra

$$N^c = \left\{ l_1 = L_1 + R_2; m_1 = M_1 + U_2; l_2 = L_2 - R_1; m_2 = M_2 + U_1; m_3 = M_3 + \frac{1}{\sqrt{2}}\left(C_3 - C_0\right) \right\}$$

and H^c is the algebra of $U(2)$

$$H^c = \left\{ \frac{R_3}{2}; \frac{U_3}{2}; \frac{1}{2\sqrt{2}}\left(C_1 - C_2\right); \frac{1}{\sqrt{2}}\left(C_1 + C_2\right) \right\}$$

Notice the normalization of the generators, which is the one that is consistent with the usual $SU(2)$ spectrum of integer and half-integer values. The representations are obtained by the induced representation method [7]. As in the $P_{\mathbb{R}}$ case (the real Lorentzian fibers) there are two types of representations, the continuous-spin and the discrete-spin representations. Restricting our attention to this last case, it corresponds to the trivial representations of $\mathcal{N}^c$, that is, the generators l_i, m_i are mapped on the zero operator. Then, from $\mathcal{H}^c = U(2)$ one concludes that the states are labeled by the quantum numbers of $SU(2)$ and a $U(1)$ phase. Each spin projection in the $SU(2)$ multiplet may correspond to a different particle state in the real Lorentzian fiber, because only R_3 among the $SU(2)$ generators belongs to the real Lorentz group.

The coset space in this case is $U(2)/SO(2)$. $SO(3)/SO(2) \simeq \mathbb{C}P^1$ is a spin manifold, therefore, in principle, there should be no topological obstruction to the implementation of spinors in the real Lorentz fibers. A finer analysis, though, is obtained when the global representations of $SO(3)$ are compared with those of the $SO(2)$ fiber. The spectrum of a representation of the group $\mathcal{H}^c$ means that $\frac{R_3}{2}$ may take integer or half-integer values. But then R_3 will only take integer values. Therefore to have consistency, $R_3 = \frac{1}{2}$ states in the fiber can only exist if that spin is complemented by an additional spin $\frac{1}{2}$. The situation, then becomes identical to the massive case, only now the additional quantum number has to match the $SU(2)$ structure of $\mathcal{H}^c$ rather than the $SU(3)$ structure of the little group as in the massive case. Hence one also has the emergence of a "flavor-type" structure in the massless case, following from the insistence on the (hidden) implementation of $P_{\mathbb{C}}$ symmetry.

The $SO(3)/SO(2)$ coset space has two dimensions which may simply be assigned to the particle-antiparticle duality. No "color-type" degeneracy appears in the massless case.

4. Remarks

1) In the $U(1,3)$ context, massive and massless particles appeared associated to the coset manifolds $U(3)/SO(3)$ and $SO(3)/SO(2)$. Another relevant coset manifold is $SU(1,3)/SO(1,3)$ which relates Lorentz invariance with the generalized complex Lorentz invariance. The Dirac operator, acting in elementary or coupled spinors, has its domain in vector bundles with basis in this manifold and representations of $SO(1,3)$ as fibers. Because maximal compact subgroups are homotopical equivalent to the Lie groups that contain them, one expects [16] the same obstructions to a spin structure over $SU(1,3)/SO(1,3)$ as in the Wu manifold, $SU(3)/SO(3)$. A similar conclusion might be obtained from the representations of the total group $SU(1,3)$. The representations of $SU(1,3)$ may be decomposed as direct sums of representations of $SU(3) \otimes U(1)$. Then, considering the subgroup chain

$$SU(1,3) \supset SU(3) \supset SO(3)$$

one concludes that there are no half-integer spins associated to the (physical) $SO(3)$ on this chain. As in the Wu manifold, a spin^h construction is needed to have 4-component spinors in the fibers.

Several authors in the past, looking for spectrum generating algebras, have dismissed the $SU(1,3)$ algebra, as useless, for not having irreducible spinor representations. However, the fact that $SO(1,3)$ has spinor representations and $SU(1,3)$ does not, is the most interesting feature of the latter.

2) The basic idea in this paper was the exploration of higher space-time symmetries, which although not readily apparent in the analysis of linear representation of the groups, might still be present and manifested in subtler nonlinear ways. The inspiration was, of course, the embedding of real space-time in a wider complex domain. However there are other possibilities. An amusing connection arises from the observation of the action of the Nambu relativistic string [17]. As already pointed out by Mita [18], when the string is covariantly quantized in the orthonormal gauge, the invariance group of the Hamiltonian is exactly $SU(1,3)$. The possible relevance of this higher group might therefore have a dynamical origin, instead of being associated to the kinematical symmetries of a complex space-time. Conversely, one might instead say that the relevance of the string interpretation of some hadronic phenomena might have its origin on the complex embedding. The reasoning goes both ways.

3) Here, the inspiration was the embedding of space-time as a real fiber in a complex manifold. A relevant question is what would be the consequences of considering real space time as a real fiber of a space time parametrized by higher division algebras, quaternions or octonions [7]. The obstructions to spin and superselection rules in discrete symmetries are similar to the complex space. However much more quantum numbers are involved and more hidden symmetries might be at play.

4) In the construction of the representations associated to $SU(1,3)$, the translation part has been considered to be the one of the complex Poincaré group. That is, the present analysis is a local one, which does not consider gravitational curvature effects. Likewise possible small distance non-commutative space-time effects are not considered [19] [20]. What would be the effect of embedding higher hidden symmetries in those contexts is, of course, an interesting issue to be explored in the future.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Complex Poincaré group algebra

The generators of the algebra of the complex Poincaré group $P_{\mathbb{C}}$ are $\{K_\mu, H_\mu, M_{\mu\nu}, N_{\mu\nu}\}$, with K_μ and H_μ the generators of real and imaginary translations, $M_{\mu\nu} = -M_{\nu\mu}$ real rotations and boosts and $N_{\mu\nu} = N_{\nu\mu}$ imaginary transformations. The algebra of the real Poincaré group $P_{\mathbb{R}}$ is $\{K_\mu, M_{\mu\nu}\}$.

$$
\begin{aligned}
\left[M_{\mu\nu}, M_{\rho\sigma}\right] &= -M_{\mu\sigma} g_{\nu\rho} - M_{\nu\rho} g_{\mu\sigma} + M_{\nu\sigma} g_{\rho\mu} + M_{\mu\rho} g_{\nu\sigma} \\
\left[M_{\mu\nu}, N_{\rho\sigma}\right] &= -N_{\mu\sigma} g_{\nu\rho} + N_{\nu\rho} g_{\mu\sigma} + N_{\nu\sigma} g_{\rho\mu} - N_{\mu\rho} g_{\nu\sigma} \\
\left[N_{\mu\nu}, N_{\rho\sigma}\right] &= M_{\mu\sigma} g_{\nu\rho} + M_{\nu\rho} g_{\mu\sigma} + M_{\nu\sigma} g_{\rho\mu} + M_{\mu\rho} g_{\nu\sigma} \\
\left[M_{\mu\nu}, K_\rho\right] &= -g_{\nu\rho} K_\mu + g_{\mu\rho} K_\nu \\
\left[M_{\mu\nu}, H_\rho\right] &= -g_{\nu\rho} H_\mu + g_{\mu\rho} H_\nu \\
\left[N_{\mu\nu}, K_\rho\right] &= -g_{\nu\rho} H_\mu - g_{\mu\rho} H_\nu \\
\left[N_{\mu\nu}, H_\rho\right] &= g_{\nu\rho} K_\mu + g_{\mu\rho} K_\nu \\
\left[K_\mu, H_\rho\right] &= 0
\end{aligned}
$$

Appendix B. Matrix representation of the Lie algebra generators of U(1,3) and U(3)

For the characterization of the subalgebras of the Lie algebra of $U(1,3)$ it is useful to define the set $\{R_i, L_i, U_i, M_i, C_i, C_0\}$ of 4×4 matrix representations. For $U(3)$ the generators are the lower 3×3 blocks of $\{R_i, U_i, C_i\}$

$$
R_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}
R_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}
R_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

$$
L_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
L_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
L_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}
$$

$$
U_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}
U_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}
U_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

$$
M_1 = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
M_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
M_3 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}
$$

$$
C_1 = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
C_2 = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
C_3 = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}
$$

$$C_0 = \sqrt{2}\begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Their correspondence to the generators in Appendix A is as follows

$$R_i = \frac{1}{2}\epsilon_{ijk} M_{jk}; L_i = M_{0i}; M_i = N_{0i}; U_i = N_{jk}; C_{\mu+3} = -\frac{1}{\sqrt{2}} N_{\mu\mu} g_{\mu\mu}$$

Data availability

No data was used for the research described in the article.

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