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FRACTIONAL NETWORKS, THE NEW STRUCTURE

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ABSTRACT

Real world networks have, for a long time, been modelled by scale-free networks, which have many sparsely connected nodes and a few highly connected ones (the hubs). However, both in society and in biology, a new structure must be acknowledged, the fractional networks. These networks are characterized by the existence of very many long-range connections, display superdiffusion, Lévy flights and robustness properties different from the scale-free networks.

1. INTRODUCTION

The scale-free (SF) network [1] has been for some time the preferred paradigm for modeling real world networks in society, biology, etc. Characterized by asymptotic power-law degree distribution, these networks have many sparsely connected nodes and a few highly connected ones (the *hubs*). The hubs are the critical nodes to address (or protect) in a network, because they control the robustness of the network and the diffusion of information. This is a fact well known by politicians, advertisement agencies and hackers. The importance of hubs has been known for a long time, even at the time of the Inquisition [2]. Several mechanisms, preferential attachment or fitness for example, have been proposed to explain the formation of this network structure. Many networks have been reported to be scale-free although careful statistical analysis has questioned others [3].

A feature that has recently emerged in some social networks (see for example [4, 5, 6]) is the existence of very many long-range connections, rather than hubs. In a sense society imitates Nature, because also brain network phenomena, for example, have been shown to be dependent on many long-range connections [7, 8, 9, 10]. Also the human mobility network has long range connections of great relevance in epidemiology [11]. It is to be expected that the existence of a sufficient number of long range connections in a network would lead to new phenomena and have a strong effect on the propagation of information.

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Some authors have already studied dynamics on networks involving jumps over many links or cascades of many unit jumps, leading to anomalous diffusion (see for example [12, 13, 14, 15, 16]). What I want to emphasize here is that, rather than imposing a multijump dynamics on a regular network, anomalous diffusion and other phenomena emerge naturally as a structural property in networks with long range connections of a particular type. Hence these networks should by themselves be classified as a new network structure.

2. Long-Range Connections: Lévy Flights and Superdiffusion

2.1. The Laplacian and Random Walk Matrices

The Laplacian and the Random Walk matrices are the main tools in the study of dynamical properties in the network. Here, for completeness, one summarizes the basic definitions. The Laplacian matrix is

$$L = G - A \tag{1}$$

G being the degree matrix ($G_{ij}=\delta_{ij}\times$ number of connections of node i) and A the adjacency matrix ($A_{ij}=1$ if i and j are connected, $A_{ij}=0$ otherwise).

The Random-Walk matrix is

$$R = G^{-1}A \tag{2}$$

then, Pιι= 0, Pιφ= Αιφ/δεγρεε (ι) ($i \neq j$) if i and j are connected and Pιφ =0 otherwise.

For a node i connected to two other nodes i+1 and i−1 the action of the Laplacian matrix on a vector $(\ldots,\psi(i-1), \psi(i), \psi(i+1),\ldots)^T$ leads to $-\psi(i-1)+2\psi(i)-\psi(i+1)$, which is a discrete version of $-d^2$ (minus the second derivative). Let now $\psi(i)$ for each node i be the intensity of some function ψ across the network. It is reasonable to think that ψ diffuses from i to j proportional to $\psi(i) - \psi(j)$ if i and j are connected. Then,

$$\frac{d\psi(i)}{dt} = -k \sum_j A_{ij} \left(\psi(i) - \psi(j)\right) = -k\left(\psi(i) \sum_j A_{ij} - \sum_j A_{ij}\psi(j)\right) \tag{3}$$

which in matrix form is

$$\frac{d\psi}{dt} + kL\psi = 0 \tag{4}$$

a heat-like equation. Therefore the Laplacian matrix controls the diffusion of quantities in the network.

On the other hand, the R matrix controls the random motion of a walker on the network. The probability for a random walker to be at the node i at time t given that at time $t-1$ was at the node j is

$$p_i(t) = \sum_j \frac{A_{ij}}{degree(j)} p_j(t-1) \tag{5}$$

or, in matrix form

$$p(t)=G^{-1}Ap(t-1) \tag{6}$$

2.2. A Network with Power-Law Connection Probability

Let a network N be embedded into a Euclidean space network so that distances may be defined. In the actual network the distances might mean geographical distances, separation of communities, functional separation as in a brain network, etc.

In the network, with A_{ij}=0 or 1, let the probability of establishment of a link at distance d be proportional to a power of the distance

$$P_{ij} = cd_{ij}^{-\gamma}\ with\ \gamma \leq 3 \tag{7}$$

Several authors (see for example [17, 18]) have already considered the possibility of adding to an existing network links with a power law probability distribution to optimize navigation in the network subjected to a cost function. However, the power law exponents obtained for that purpose differ from those that characterize the different types of diffusion.

To find the nature of the diffusion in such a network, consider a block renormalized network N^* where each set of q nearby nodes of the original network N are mapped to a node of the N^* network[1]. Therefore in the N^* network the connections are

$$A_{ij}^* \cong cqd_{ij}^{-\gamma} \tag{8}$$

Then denoting by L^* and G^* the Laplacian and degree matrices of the N^* network

$$L^*\psi(i) = G_{ii}^*\psi(i) - cq\sum_{j\neq i} d_{ij}^{-\gamma}\psi(j) \tag{9}$$

To find what kind of diffusion the Laplacian matrix $L^*=G^*-A^*$ implies for the network N^*, one compares the distance dependence of the elements of the Laplacian matrix L^* along one of the coordinate axis of the embedding Euclidean space with a discrete one-dimensional

[1] With this "block-renormalization," the connection probability (7) leads to actual connection strengths in the renormalized network.

representation of a fractional derivative. The symmetrized Grünwald-Letnikov representation of the fractional derivative ($a<x<b$) (see for example [19])

$$D^{\beta}\psi(x) = \frac{1}{2} lim_{h\to 0} \frac{1}{h}\left\{\sum_{n=0}^{\left[\frac{x-a}{h}\right]}(-1)^n \binom{\beta}{n}\psi(x-nh) + \sum_{n=0}^{\left[\frac{b-x}{h}\right]}(-1)^n \binom{\beta}{n}\psi(x+nh)\right\} \tag{10}$$

with coefficients

$$\left|\binom{\beta}{n}\right| = \frac{\Gamma(\beta+1)|\sin(\pi\beta)|}{\pi}\frac{\Gamma(n-\beta)}{\Gamma(n+1)} \approx_{n\ large} \frac{\Gamma(\beta+1)|\sin(\pi\beta)|}{\pi} n^{-(\beta+1)} \tag{11}$$

and $sign\binom{\beta}{n} = (-1)^{n+1}$.

Comparing Eq.(10) with the expression (9) for $L^*\psi(i)$, the conclusion is that diffusion in the N^* network is fractional diffusion of exponent $\beta = \gamma - 1$. $\beta = 2$ would be normal diffusion, all $\beta < 2$ corresponding to superdiffusions.

The comparison of the power law dependence of the coefficients of the Laplacian matrix along one embedding coordinate with a one-dimensional fractional derivative is adequate to find the nature of the diffusion, but not sufficient to obtain the solution of the corresponding diffusion equations,

$$\frac{d\psi}{dt} = -k(-\Delta)^{\frac{\beta}{2}}\psi \tag{12}$$

Here the dimension of the embedding Euclidean space should be taken into account. For normal diffusion the fundamental solution factorizes, but this is not the case for exponents $\beta \neq 2$ as one sees from the fundamental solution, that is [20]

$$G(t,x) = \frac{(kt)^{-\frac{n}{\beta}}}{(2\pi)^n}\int d^n k exp\left(-ik\cdot\frac{x}{(kt)^{1/\beta}} - k^{\beta}\right)$$

n being the dimension of the embedding Euclidean space.

A similar result is obtained by analyzing the structure of the random walks controlled by $G^{-1}A$ (Eq.6) the conclusion being that, whereas for normal diffusion the jumps are of one step, for $\gamma < 3$ arbitrarily large jumps occur with a power law (Lévy flights).

CONCLUSION

1. The first general conclusion is that in these networks both mobility and diffusion of information may occur at a very fast rate. They may be considered as a new structure distinct from SF networks, a Fractional Network (FR). These networks are

characterized by the power law connection exponent γ rather than by the degree distribution. The new structure has wide implications for the control of the networks.

2. In a SF network, the hubs are both the strength and the weakness of the network. They insure global connectivity even if a large number of links are destroyed. But, when directly targeted, the network is deeply affected (targeted structural weakness). In a SF network propagation of ideas, opinions, fads (memes) are most effective if introduced to the hubs. However fast global establishment of a trend requires its introduction at many hubs.
3. A FR network is structurally very stable and resilient to attack. It is pointless or too expensive to disrupt the network. The hubs are no longer the controllers. The network itself is the HUB. Superdiffusion is both the strength and the weakness of the network. Well-crafted memes propagate very fast. But also do counter-memes. In SF networks the memes are most efficiently introduced at the hubs. Here they might be introduced anywhere.

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