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No-arbitrage, leverage and completeness in a fractional volatility model

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HIGHLIGHTS

- Mathematical consistency and no-arbitrage in a fractional volatility market model.
- Market not complete if the volatility process is independent from the price process.
- Market arbitrage free and complete when driven by a unique process.
- Arbitrage free complete market displays leverage properties as in empirical data.

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ABSTRACT

When the volatility process is driven by fractional noise one obtains a model which is consistent with the empirical market data. Depending on whether the stochasticity generators of log-price and volatility are independent or are the same, two versions of the model are obtained with different leverage behaviors. Here, the no-arbitrage and completeness properties of the models are rigorously studied.

1. Introduction

In liquid markets the autocorrelation of price changes decays to negligible values in a few ticks, consistent with the absence of long term statistical arbitrage. Because innovations of a martingale are uncorrelated, this strongly suggests that it is a process of this type that should be used to model the stochastic part of the returns process. As a consequence, classical Mathematical Finance has, for a long time, been based on the assumption that the price process of market securities may be approximated by geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dB(t). \tag{1}$$

Geometric Brownian motion (GBM) models the absence of linear correlations, but otherwise has some serious shortcomings. It does not reproduce the empirical leptokurtosis nor does it explain why nonlinear functions of the returns exhibit significant positive autocorrelation. For example, there is volatility clustering, with large returns expected to be followed by large returns and small returns by small returns (of either sign). This, together with the fact that autocorrelations of volatility measures decline very slowly [1–3] has the clear implication that long memory effects should somehow be represented

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in the process and this is not included in the GBM hypothesis. The existence of an essential memory component is also clear from the failure of reconstruction of a Gibbs measure and the need to use chains with complete connections in the phenomenological reconstruction of the market process [4].

As pointed out by Engle [5], when the future is uncertain investors are less likely to invest. Therefore uncertainty (volatility) would have to be changing over time. The conclusion is that a dynamical model for volatility is needed and σ in Eq. (1), rather than being a constant, becomes itself a process. This idea led to many deterministic and stochastic models for the volatility ([6,7] and references therein).

The stochastic volatility models that were proposed describe some partial features of the market data. For example leptokurtosis is easy to fit but the long memory effects are much harder. On the other hand, and in contrast with GBM, some of the phenomenological fittings of historical volatility lack the kind of nice mathematical properties needed to develop the tools of mathematical finance. In an attempt to obtain a model that is both consistent with the data and mathematically sound, a new approach was developed in Ref. [8]. Starting with some criteria of mathematical simplicity, the basic idea was to let the data itself tell what the processes should be.

The basic hypothesis for the model construction were:

(H1) The log-price process $\log S_t$ belongs to a probability product space $(\Omega_1 \times \Omega_2, P_1 \times P_2)$ of which the (Ω_1, P_1) is the Wiener space and the second one, (Ω_2, P_2), is a probability space to be reconstructed from the data. Denote by $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$ the elements (sample paths) in Ω_1 and Ω_2 and by $\mathcal{F}_{1,t}$ and $\mathcal{F}_{2,t}$ the σ-algebras in Ω_1 and Ω_2 generated by the processes up to time t. Then, a particular realization of the log-price process is denoted

$$\log S_t\left(\omega_1, \omega_2\right).$$

This first hypothesis is really not limitative. Even if none of the non-trivial stochastic features of the log-price were to be captured by Brownian motion, that would simply mean that S_t was a trivial function in Ω_1.

(H2) The second hypothesis is stronger, although natural. It is assumed that for each fixed ω_2, $\log S_t\left(\cdot, \omega_2\right)$ is a P_1-square integrable random variable in Ω_1.

These principles and a careful analysis of the market data led, in an essentially unique way, to the following model:

$$\mathrm{d}S_t = \mu_t S_t\,\mathrm{d}t + \sigma_t S_t\,\mathrm{d}B\left(t\right) \tag{2}$$

$$\log \sigma_t = \beta + \frac{k}{\delta}\left\{B_H\left(t\right) - B_H\left(t-\delta\right)\right\} \tag{3}$$

B_H being fractional Brownian motion with Hurst coefficient H. The data suggests [8] values of H in the range 0.8–0.9. In this coupled stochastic system, in addition to a mean value, volatility is driven by fractional noise. The empirical identification of the model is essentially unique in the sense that the empirically reconstructed volatility process is the simplest one, consistent with the scaling properties of the data. This is clear from Fig. 1 where the scaling properties of the residual of the integrated $\log \sigma\left(t\right)$ are exhibited

$$\sum_{n=0}^{t/\delta} \log \sigma\left(n\delta\right) = \beta t + R_\sigma\left(t\right).$$

Notice that this empirically based model is different from the usual stochastic volatility models which assume the volatility to follow an arithmetic or geometric Brownian process. Also in Comte and Renault [9] and Hu [10], it is fractional Brownian motion that drives the volatility, not its derivative (fractional noise). δ is the observation scale of the process. In the $\delta \to 0$ limit the driving process would be a distribution-valued process.

Eq. (3) leads to

$$\sigma_t = \theta \mathrm{e}^{\frac{k}{\delta}\left\{B_H(t) - B_H(t-\delta)\right\} - \frac{1}{2}\left(\frac{k}{\delta}\right)^2 \delta^{2H}} \tag{4}$$

with $E\left[\sigma_t\right] = \theta > 0$.

The model has been shown to describe well the statistics of price returns for a large δ-range as seen in Fig. 2, taken from Ref. [8], where the returns distribution of NYSE data for one and ten days is compared with the model.

A new option pricing formula, with ''smile'' deviations from Black–Scholes, was obtained. An agent-based interpretation [11] also led to the conclusion that the statistics generated by the model is consistent with the limit order book price setting mechanism.

In the past, several authors tried to describe the long memory effect by replacing in the price process Brownian motion by fractional Brownian motion with $H > 1/2$. However it was soon realized [12–15] that this replacement implied the existence of arbitrage. These results might be avoided either by restricting the class of trading strategies [16], introducing transaction costs [17] or replacing pathwise integration by a different type of integration [18,19]. However this is not free of problems because the Skorohod integral approach requires the use of a Wick product either on the portfolio or on the self-financing condition, leading to unreasonable situations from the economic point of view (for example positive portfolio with negative Wick value, etc.) [20].

The fractional volatility model in Eqs. (2)–(3) is not affected by these considerations, because it is the volatility process that is driven by fractional noise, not the price process. In fact a no-arbitrage result may be proven. This is no surprise because

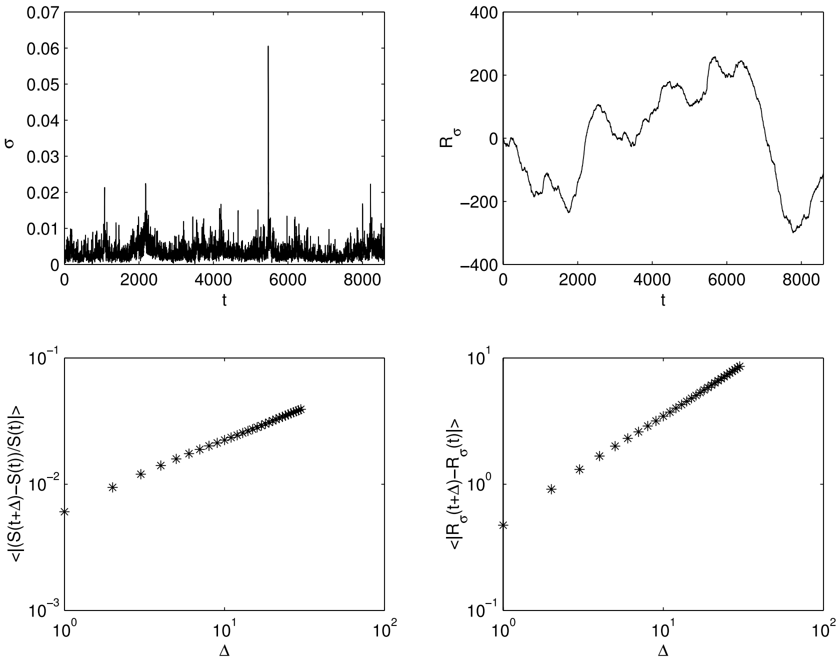

Fig. 1. Volatility, residual of integrated $\log \sigma\ (t)$ and its scaling properties (NYSE data).

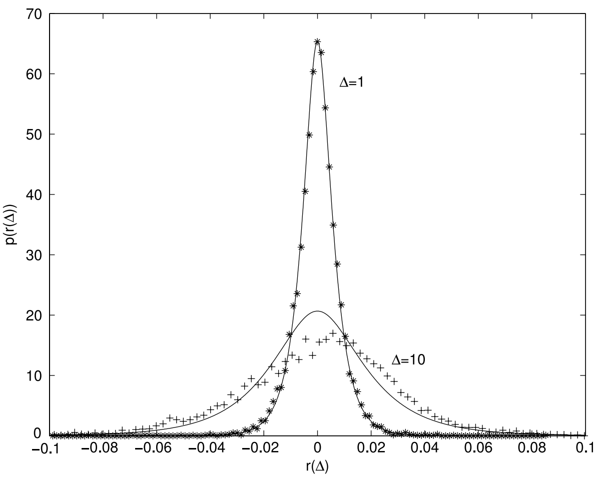

Fig. 2. One and ten days NYSE returns compared with the fractional volatility model predictions.

our requirement (H2) that, for each sample path $\omega_2 \in \Omega_2$, $\log S_t\ (\cdot, \omega_2)$ is a square integrable random variable in Ω_1 already implies that $\int \sigma_t \mathrm{d}B_t$ is a martingale. The square integrability is also essential to guarantee the possibility of reconstruction of the σ process from the data, because it implies [21]

$$\frac{\mathrm{d}S_t}{S_t}\ (\cdot, \omega_2) = \mu_t\ (\cdot, \omega_2)\ \mathrm{d}t + \sigma_t\ (\cdot, \omega_2)\ \mathrm{d}B_t. \tag{5}$$

As stated before, the empirical success of the fractional volatility model was already documented in Ref. [8]. The purpose of the present paper is to give a solid foundation for the fractional volatility model, discussing existence questions, arbitrage and market completeness.

2. Arbitrage, completeness and their tools

Modern finance theory uses a stylized set of market properties, which are rigorously implemented through the tools of stochastic analysis. For the sake of the reader less familiar with these tools we summarize here the main ideas. For a detailed treatment and proofs of the results we refer, for example, to Refs. [22–24].

A fundamental assumption is the *no-arbitrage principle* which states that a (perfect) market does not allow for risk-free profits with no initial investment or, equivalently, to profits without any risk. Let the value of a portfolio be

$$V(t) = \sum_{i=1}^{n} h_t^i S_t^i$$

$\{S_t^1, \ldots, S_t^n\}$ being the price processes of a set of n assets and h_t^i the amount of the asset S^i that the investor holds at time t. A portfolio is *self-financing* if there is no exogenous infusion or withdrawal of money, the purchase of a new asset must be financed by the sale of an old one. A self-financing portfolio would be an *arbitrage portfolio* if

$$V(0) = 0$$
$$P(V(T) > 0) > 0$$

for $T > 0$, P being the probability measure on the market scenarios.

The *no-arbitrage principle* means that no investor can obtain a profit without risk and with no initial endowment. In practice, situations where the no-arbitrage principle is violated are typically short-lived and the activities of attentive investors (arbitrageurs) pursuing arbitrage profits effectively make the market free of arbitrage opportunities. This situation which is true in normal market situations might be violated in crisis environments. For example, if a financial institution is able to borrow funds from a central bank at a low rate x, which may then invest in state bonds at a higher rate, this is an arbitrage situation.

Let A_t be the price process of a *risk-free asset* (a bank account or state bond). The *discounted price process* of a risky asset is

$$Z_t = \frac{S_t}{A_t}.$$

Closely related to the notion of no-arbitrage is the notion of *martingale process*. Z_t is a P-martingale if

$$\mathbb{E}_P(Z_t|Z_0) = Z_0$$

$\mathbb{E}_P$ denoting the expected value under the probability P. That is, if at time zero the process starts from Z_0 then the expected value at all later times is also Z_0.

A fundamental theorem [22] asserts that *a market is arbitrage free if and only if there is an equivalent martingale measure for the discounted price processes*. That existence of a martingale measure implies no-arbitrage is fairly obvious, the converse is not so obvious but it is nevertheless true. Sometimes it is not obvious that the price processes are martingales under the probability measure P. Therefore the whole question is the existence or non-existence of a measure Q that is a martingale and is equivalent to P. Equivalent measures are measures such

$$P(\omega) = 0 \Longrightarrow Q(\omega) = 0$$
$$Q(\omega) = 0 \Longrightarrow P(\omega) = 0$$

that is, if an event ω has zero probability under P it also has zero probability under Q and conversely.

Hence, the search for an equivalent martingale measure turns out to be a question of a stochastic change of variables and the central role here is played by the *Girsanov theorem* (see the next section for concrete examples of the application of this theorem).

Hedging is another issue of central importance for market investments. Given a value portfolio X, another portfolio H is said to be *an hedge* for X (or to *replicate* X) if

H is self-financing

$$V_H(T) = X(T), \qquad P\text{-almost surely}.$$

A market is *complete* if all X can be hedged. A second fundamental theorem [22] states that *a market is complete if and only if the martingale measure Q is unique*.

In the next section we make use of these notions to prove the consistency of the fractional volatility model.

3. The mathematical consistency of the fractional volatility model

3.1. No-arbitrage

According to the main hypothesis used to construct it, the fractional volatility model belongs to a probability product space $(\Omega_1 \times \Omega_2, P_1 \times P_2)$.

Let $(\Omega_1, \mathcal{F}_1, P_1)$ be a Wiener probability space, carrying a standard Brownian motion $B = (B_t)_{0\le t<\infty}$ and a filtration[1] $\mathbb{F}_1 = (\mathcal{F}_{1,t})_{0\le t<\infty}$. Let also $(\Omega_2, \mathcal{F}_2, P_2)$ be another probability space associated to a fractional Brownian motion B_H with

[1] For technical reasons, a probability measure can only be defined for a "nice" class F of events, which is closed under complements, countable intersections and countable unions (a σ-algebra). For a stochastic process evolving in time, a convenient set is a filtration, that is, an increasing set of σ-algebras $\{\mathcal{F}_t\}$ where each $\mathcal{F}_t$ is the set of events that the process may generate up to time t. In practical terms $\mathcal{F}_t$ represents the total amount of information that may be available up to time t.

Hurst parameter $H \in (0, 1)$ and a filtration $\mathbb{F}_2 = \left(\mathcal{F}_{2,t}\right)_{0\le t<\infty}$ generated by B_H. Denote by $\mathbb{E}_1$ and $\mathbb{E}_2$ the expectations with respect to P_1 and P_2 respectively.

Let us now embed these two probability spaces in a product space $\left(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P}\right)$, where $\overline{\Omega}$ is the Cartesian product $\Omega_1 \times \Omega_2$, $\overline{P}$ is the product measure $P_1 \otimes P_2$ and $\overline{\mathcal{F}}$ is the σ-algebra generated by the union of the σ-algebras $\mathcal{F}_1 \otimes \mathcal{F}_2$ and the null sets from $\mathcal{F}_1 \otimes \mathcal{F}_2$. $\overline{\mathbb{F}} = \left(\overline{\mathcal{F}}_t\right)_{0\le t<\infty}$ is the corresponding filtration. We also introduce π_1 and π_2, as the projections of $\overline{\Omega}$ onto Ω_1 and Ω_2.

Furthermore, we extend B and B_H to $\overline{\mathbb{F}}$-adapted processes on $\left(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P}\right)$ by $\overline{B}(\omega_1, \omega_2) = (B \circ \pi_1)(\omega_1, \omega_2)$ and $\overline{B}_H(\omega_1, \omega_2) = (B_H \circ \pi_2)(\omega_1, \omega_2)$ for $(\omega_1, \omega_2) \in \overline{\Omega}$. It is easy to prove that $\overline{B}$ and $\overline{B}_H$ are Brownian and fractional Brownian motions with respect to $\overline{P}$ and are independent. For notational simplicity, hereafter B and B_H will stand for $\overline{B}$ and $\overline{B}_H$.

We now consider a market with a risk-free asset with dynamics given by

$$\mathrm{d}A_t = rA_t\,\mathrm{d}t \qquad A_0 = 1 \tag{6}$$

with $r > 0$ constant and a risky asset with price process S_t given by Eqs. (2)–(3), with μ_t a $\overline{\mathbb{F}}$-adapted process with continuous paths, k the volatility intensity parameter and δ the observation time scale of the process.

The volatility σ_t is a measurable and $\overline{\mathbb{F}}$-adapted process satisfying for all $0 \le t < \infty$

$$\begin{aligned}\mathbb{E}_{\overline{P}}\left[\int_0^t \sigma_s^2 \mathrm{d}s\right] &= \int_0^t \theta^2 \mathrm{e}^{-\left(\frac{k}{\delta}\right)^2\delta^{2H}} \mathbb{E}_{\overline{P}}\left[\mathrm{e}^{\frac{2k}{\delta}\{B_H(s)-B_H(s-\delta)\}}\right]\mathrm{d}s \\ &= \theta^2 \exp\left\{\left(\frac{k}{\delta}\right)^2 \delta^{2H}\right\} t < \infty\end{aligned}$$

by Fubini's theorem and the moment generating function of the Gaussian random variable $B_H(s) - B_H(s-\delta)$.

Moreover $\int_0^t |\mu_s|\,\mathrm{d}s$ being finite $\overline{P}$-almost surely for $0 \le t < \infty$, an application of Itô's formula yields

$$S_t = S_0 \exp\left\{\int_0^t \left(\mu_s - \frac{1}{2}\sigma_s^2\right)\mathrm{d}s + \int_0^t \sigma_s \mathrm{d}B_s\right\}.$$

Additionally, we assume that investors are allowed to trade only up to some fixed finite planning horizon $T > 0$.

Lemma 3.1. *Consider the measurable process defined by*

$$\gamma_t = \frac{r - \mu_t}{\sigma_t}, \quad 0 \le t < \infty \tag{7}$$

with $\mu \in L^\infty\left([0, T] \times \overline{\Omega}\right)$. Then, for a continuous version of the B_H process

$$\exp\left[\frac{1}{2}\int_0^T \gamma_s^2(\omega_2)\,\mathrm{d}s\right] < A(\omega_2) < \infty \tag{8}$$

P_2-almost all $\omega_2 \in \Omega_2$.

We use the fact that P_2-almost surely the paths of a continuous version of fractional Brownian motion are Hölder continuous of any order $\alpha \ge 0$ strictly less than H, that is, there is a random variable $C_\alpha > 0$ such that for P_2-almost all $\omega_2 \in \Omega_2$ $|B_H(t) - B_H(s)| \le C_\alpha(\omega_2)\,|t-s|^\alpha$ for every $t, s \in [0, \infty)$

$$\begin{aligned}\exp\left[\frac{1}{2}\int_0^T \gamma_s^2(\omega_2)\,\mathrm{d}s\right] &\le \exp\left[\frac{\mathrm{e}^{k^2\delta^{2H-2}}}{2\theta^2}\int_0^T (r+|\mu_s|)^2\,\mathrm{e}^{-2\frac{k}{\delta}(B_H(s)-B_H(s-\delta))}\mathrm{d}s\right] \\ &\le \exp\left[\frac{(r+\|\mu_s\|_\infty)^2}{2\theta^2}\mathrm{e}^{k^2\delta^{2H-2}}\int_0^T \mathrm{e}^{-2\frac{k}{\delta}(B_H(s)-B_H(s-\delta))}\mathrm{d}s\right] \\ &\le \exp\left\{\frac{T(r+\|\mu_s\|_\infty)^2}{2\theta^2}\mathrm{e}^{k^2\delta^{2H-2}+2kC_\alpha(\omega_2)\delta^{\alpha-1}}\right\} < A(\omega_2) < \infty. \quad \blacksquare\end{aligned}$$

Proposition 3.2. *The market defined by (2), (3) and (6) is free of arbitrage.*

Proof. Restricting the process to a particular path ω_2 of the B_H-process, we construct the stochastic exponential of $\int_0^t \gamma_s(\omega_2)\,\mathrm{d}B_s$, that is

$$\eta_t(\omega_2) = \exp\left\{\int_0^t \gamma_s(\omega_2)\,\mathrm{d}B_s - \frac{1}{2}\int_0^t \gamma_s^2(\omega_2)\,\mathrm{d}s\right\}. \tag{9}$$

The bound proved on Lemma 3.1 is the Kallianpur condition [25] that insures that[2]

$$\mathbb{E}_{P_1}\left[\eta_t\left(\omega_2\right)\right]=1 \quad \omega_2\text{-a.s.} \tag{10}$$

Hence, we are in the framework of Girsanov theorem and each nonnegative continuous supermartingale $\eta_t\left(\omega_2\right)$ in (9) is a true P_1-martingale. Hence we can define for each $0 \leq T < \infty$ a new probability measure $Q_T\left(\omega_2\right)$ on $\mathcal{F}_1$ by

$$\frac{\mathrm{d}Q_T\left(\omega_2\right)}{\mathrm{d}P_1}=\eta_T\left(\omega_2\right), \quad P_1\text{-a.s.} \tag{11}$$

Then, by the Cameron–Martin–Girsanov theorem, for each fixed $T \in [0, \infty)$, the process

$$B_t^*=B_t-\int_0^t \frac{r-\mu_s}{\sigma_s\left(\omega_2\right)} \mathrm{d}s \quad 0 \leq t \leq T \tag{12}$$

is a Brownian motion on the probability space $\left(\Omega, \mathcal{F}_1, Q_T\left(\omega_2\right)\right)$.

Consider now the discounted price process

$$Z_t=\frac{S_t}{A_t} \quad 0 \leq t \leq T.$$

Under the new probability measure $Q_T\left(\omega_2\right)$, equivalent to P_1 on $\mathcal{F}_1$, its dynamics is given by

$$Z_t\left(\omega_2\right)=Z_0+\int_0^t \sigma_s\left(\omega_2\right) Z_s\left(\omega_2\right) \mathrm{d}B_s^* \tag{13}$$

and is a martingale in the probability space $\left(\Omega_1, \mathcal{F}_1, Q_T\left(\omega_2\right)\right)$ with respect to the filtration $\left(\mathcal{F}_{1,t}\right)_{0 \leq t<T}$. By the fundamental theorem of asset pricing, the existence of an equivalent martingale measure for Z_t implies that there are no arbitrages, that is, $\mathbb{E}_{Q_T(\omega_2)}\left[Z_t\left(\omega_2\right) | \mathcal{F}_{1,s}\right]=Z_s\left(\omega_2\right)$ for $0 \leq s<t \leq T$.

We have proved that there are no arbitrages for P_2-almost all ω_2 trajectories of the B_H process. But because this process is independent from the B process in (2), it follows that the no-arbitrage result is also valid in the probability product space. ■

3.2. Incompleteness

Another important concept is market completeness. We note that, in this financial model, trading takes place only in the stock and in the money market and, as a consequence, volatility risk cannot be hedged. Hence, since there are more sources of risk than tradable assets, in this model, the market is incomplete, as proved in the next proposition.

Proposition 3.3. *The market defined by (2), (3) and (6) is incomplete.*

Proof. Here we use an integral representation for the fractional Brownian motion [26,27]

$$B_H(t)=\int_0^t K_H(t, s) \mathrm{d}W_s \tag{14}$$

W_t being a Brownian motion independent from B_t and K is the square integrable kernel

$$K_H(t, s)=C_H s^{\frac{1}{2}-H} \int_s^t(u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} \mathrm{d}u, \quad s<t$$

($H>1/2$). Then the process

$$\eta_t^{\prime}=\exp \left(W_t-\frac{1}{2} t\right) \tag{15}$$

is a square-integrable P_2-martingale.

Defining a standard bi-dimensional Brownian motion,

$$W_t^*=\left(B_t, W_t\right)$$

the process $\eta_t^*\left(\omega_2\right)=\eta_t \eta_t^{\prime}\left(\omega_2\right)$

$$\eta_t^*\left(\omega_2\right)=\exp \left\{\int_0^t \Gamma_s\left(\omega_2\right) \bullet \mathrm{d}W_t^*-\frac{1}{2} \int_0^t\left\|\Gamma_s\left(\omega_2\right)\right\|^2 \mathrm{d}s\right\}$$

[2] Notice that this is different from $\mathbb{E}_{\overline{P}}\left[\eta_t\right]=1$ but what is needed for the construction of the equivalent martingale measure by Girsanov is Eq. (10) and not the latter stronger condition.

where, by Lemma 3.1, $\Gamma(\omega_2) = (\gamma(\omega_2), 1)$ satisfies the Novikov condition, is also a P_1-martingale. Then, by the Cameron–Martin–Girsanov theorem, the process

$$\widetilde{W}_t^* = \left(\widetilde{W}_t^{*(1)}, \widetilde{W}_t^{*(2)}\right)$$

defined by

$$\widetilde{W}_t^{*(1)} = B_t - \int_0^t \gamma_s(\omega_2)\,\mathrm{d}s$$
$$\widetilde{W}_t^{*(2)} = W_t - t$$

is a bi-dimensional Brownian motion on the probability space $\left(\Omega_1, \mathcal{F}_1, Q_T^*(\omega_2)\right)$, where $Q_T^*(\omega_2)$ is the probability measure

$$\frac{\mathrm{d}Q_T^*(\omega_2)}{\mathrm{d}P_1} = \eta_T^*(\omega_2). \tag{16}$$

Moreover, the discounted price process Z remains a martingale with respect to the new measure $Q_T^*(\omega_2)$. $Q_T^*(\omega_2)$ being an equivalent martingale measure distinct from $Q_T(\omega_2)$, the market is incomplete. ■

As stated above, incompleteness of the market is a reflection of the fact that there are two different sources of risk and only one of the risks is tradable. A choice of measure is how one fixes the volatility risk premium.

3.3. *Leverage and completeness*

The following nonlinear correlation of the returns

$$L(\tau) = \left\langle |r(t+\tau)|^2 r(t)\right\rangle - \left\langle |r(t+\tau)|^2\right\rangle \langle r(t)\rangle \tag{17}$$

is called *leverage* and the *leverage effect* is the fact that, for $\tau > 0$, $L(\tau)$ starts from a negative value whose modulus decays to zero whereas for $\tau < 0$ it has almost negligible values. In the form of Eqs. (2), (3) the volatility process σ_t affects the log-price, but is not affected by it. Therefore, in its simplest form the fractional volatility model contains no leverage effect.

Leverage may, however, be implemented in the model in a simple way. Using a standard representation for fractional Brownian motion [27] as a stochastic integral over Brownian motion and identifying the random generator of the log-price process with the stochastic integrator of the volatility, at least a part of the leverage effect is taken into account [28]. Here, for the generator of the volatility process we use a truncated version of a representation of Fractional Brownian motion [27],

$$\mathcal{H}(t) = \Pi^{(M)}\left[C_H\left\{\int_{-\infty}^0 \left((t-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}}\right)\mathrm{d}W_u + \int_0^t (t-u)^{H-\frac{1}{2}}\,\mathrm{d}W_u\right\}\right], \tag{18}$$

$\Pi^{(M)}$ meaning the truncation of the representation to an interval $[-M, M]$ with M arbitrarily large.

The identification of the two Brownian processes means that now, instead of two, there is only one source of risk. Hence it is probable that in this case completeness of the market might be achieved.

The new fractional volatility model would be

$$\mathrm{d}S_t = \mu_t S_t \mathrm{d}t + \sigma_t S_t \mathrm{d}W_t$$
$$\log\sigma_t = \beta + \frac{k'}{\delta}\left\{\mathcal{H}(t) - \mathcal{H}(t-\delta)\right\}. \tag{19}$$

Proposition 3.4. *The market defined by (19), (18) and (6) is free of arbitrage and complete.*

Proof. In this case because the two processes are not independent we cannot use the same argument as before to obtain the Kallianpur condition. However with the truncation in (18) the Hölder condition is trivially verified for all the truncated paths of σ_t and the construction of an equivalent martingale measure follows the same steps as in Proposition 3.2. Hence we have a P_1-martingale with respect to $\left(\mathcal{F}_{1,t}\right)_{0\le t<T}$

$$\eta_t = \exp\left\{\int_0^t \frac{r-\mu_s}{\sigma_s}\mathrm{d}W_s - \frac{1}{2}\int_o^t \left(\frac{r-\mu_s}{\sigma_s}\right)^2 \mathrm{d}s\right\}$$

and the probability measure Q_T, defined by $\frac{\mathrm{d}Q_T}{\mathrm{d}P_1} = \eta_T$ is an equivalent martingale measure.

The set of equivalent local martingale measures for the market being non-empty, let Q^* be an element in this set. Then, recalling that $\left(\mathcal{F}_{1,t}\right)_{0\le t<T}$ is the augmentation of the natural filtration of the Brownian motion W_t, by the Girsanov converse [29,22] there is a $\left(\mathcal{F}_{1,t}\right)_{0\le t<T}$ progressively measurable $\mathbb{R}$-valued process ϕ such that the Radon–Nikodym density

of Q^* with respect to P_1 equals

$$\frac{\mathrm{d}Q_T^*}{\mathrm{d}P_1} = \exp\left\{\int_0^T \phi_s \mathrm{d}W_s - \frac{1}{2}\int_0^T \phi_s^2 \mathrm{d}s\right\}.$$

Moreover the process W_t^* given by

$$W_t^* = W_t - \int_0^t \phi_s \mathrm{d}s$$

is a standard Q^*-Brownian motion and the discounted price process Z satisfies the following stochastic differential equation

$$\mathrm{d}Z_t = (\mu_t - r + \sigma_t\phi_t)Z_t\mathrm{d}t + \sigma_t Z_t \mathrm{d}W_t^*.$$

Because Z_t is a Q^*-martingale, it must hold $\mu(t,\omega) - r + \sigma(t,\omega)\phi(t,\omega) = 0$ almost everywhere w.r.t. $\mathrm{d}t \times P$ in $[0,T] \times \Omega$. It implies

$$\phi(t,\omega) = \frac{r - \mu(t,\omega)}{\sigma(t,\omega)}$$

a.e. $(t,\omega) \in [0,T] \times \Omega_1$. Hence $Q_T^* = Q_T$, that is, Q_T is the unique equivalent martingale measure. This market model is complete. ■

4. Remarks and conclusions

(1) Partially reconstructed from empirical data [8], the *fractional volatility model* yields an accurate description of the empirical market returns. The fact that, once the parameters are adjusted by the data for a particular observation time scale δ, the model also fits different time lags, is related to the fact that the volatility is driven not by fractional Brownian motion but by its increments.

Specific trader strategies and psychology should play a role in market crisis and bubbles. However, the fact that in the fractional volatility model the same set of parameters is appropriate for different markets [8] seems to imply that the market statistical behavior (in normal days) is more influenced by the nature of the financial institutions (the double auction process) than by the traders strategies [11]. Therefore some kind of universality of the statistical behavior of the bulk data throughout different markets would not be surprising.

(2) The identification of the Brownian process of the log-price with the one that generates the fractional noise driving the volatility, introduces an asymmetric coupling between σ_t and S_t that is also exhibited by the market data. In this sense the complete market, described in Section 3.3, is probably the more empirically accurate model.

(3) In this paper, mathematical consistency of the fractional volatility model has been established. This and its better consistency with the experimental data, make it a candidate to replace geometrical Brownian as the standard market model.

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