

# Fluctuations and control in the Vlasov-Poisson equation

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## Abstract

In this paper we study the fluctuation spectrum of a linearized Vlasov-Poisson equation in the presence of a small external electric field. Conditions for the control of the linear fluctuations by an external electric field are established.

## 1 Introduction

In the past, the fluctuation spectrum of charged fluids was studied either by the BBGKY hierarchy derived from the Liouville or Klimontovich equations, with some sort of closure approximation, or by direct approximations to the N-body partition function or by models of dressed test particles, etc. (see reviews in [1] [2]).

Alternatively, by linearizing the Vlasov equation about a stable solution and diagonalizing the Hamiltonian, a method has been developed [3] that uses the eigenvalues associated to the continuous spectrum and a canonical partition function to compute correlation functions. Here this approach will also be followed to study the control of the fluctuations. For simplicity we will consider the one-space dimensional case.

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A Vlasov-Poisson equation with an external electrical field control term is considered. Following the method developed by Morrison [4] we use an integral transform to solve the linearized equation. With a view to applications to more general kinetic equations (gyrokinetic, etc.) we also discuss in the appendix a generalization of Morrison's integral transform.

Control of the Vlasov-Poisson equation

$$\partial_t f + v \cdot \nabla_x f + \nabla_x \phi \cdot \nabla_v f = C(t, x, v)$$

in a periodic domain

$$(t, x, v) \in [0, T] \times T^n \times R^n$$

by means of an interior control located in a spatial subdomain has been discussed by Glass [5]. Conditions for controllability between two small distribution distributions  $f_0$  and  $f_1$  were established. However, to steer  $f_0$  to  $f_1$  a control  $C(t, x, v)$  that depends on the velocities is required and it is not clear how such a control could be implemented in practice. Therefore, we have restricted ourselves to the more realistic situation of a (small) controlling external electric field. In addition we concentrate on the problem of the damping of the small oscillations around an equilibrium distribution.

In Sect.2 the linearized Vlasov-Poisson equation with control is solved by an integral transform and in Sect.3 two controlling problems are studied, namely the control of the total energy of the fluctuations by a constant electric field and the dynamical damping of the fluctuating modes by a time-dependent electric field.

## 2 The linearized equation with control

Consider a Vlasov-Poisson system in 1 + 1 dimensions

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} - \frac{e}{m} \left( \frac{\partial \Phi_f}{\partial x} - E^c(x, t) \right) \frac{\partial f}{\partial v} = 0 \quad (1)$$

$$\frac{\partial^2}{\partial x^2} \Phi_f = -e \int f(v) dv + \rho_B \quad (2)$$

with an external (control) electric field  $E^c(x, t)$  and a background charge density  $\rho_B(x)$  chosen in a such a way that the total charge vanishes. Consider now the linearization about a homogeneous solution.

$$f(x, v, t) = f^{(0)}(v) + \delta f(x, v, t) \quad (3)$$

$$\Phi_f = \Phi_f^{(0)} + \delta\Phi_f$$

with

$$\Delta\Phi_f^{(0)} = -e \int f^{(0)}(v) dv + \rho_B$$

such that

$$\Delta\Phi_f^{(0)} = 0 = \nabla\Phi_f^{(0)}$$

Then

$$\frac{\partial}{\partial t} f^{(0)}(v) = 0$$

and  $f^{(0)}(v)$  is indeed a homogeneous static equilibrium. The linearized equation is

$$\frac{\partial\delta f}{\partial t} + v \frac{\partial\delta f}{\partial x} - \frac{e}{m} \frac{\partial\delta\Phi_f}{\partial x} \frac{\partial f^{(0)}}{\partial v} + \frac{e}{m} E^c(x, t) \frac{\partial f^{(0)}}{\partial v} = 0 \quad (4)$$

where we have assumed that

$$E^c(x, t) = O(\delta f) \quad (5)$$

that is,  $E^c(x, t)$  is a small external (control) electric field<sup>1</sup>.

Fourier transforming all the perturbations

$$\delta f(x, v, t) = \sum f_k(v, t) e^{ikx} \quad (6)$$

$$\delta\Phi(x, t) = \sum \phi_k(t) e^{ikx} \quad (7)$$

$$E^c(x, t) = \sum E_k(t) e^{ikx} \quad (8)$$

leads to

$$\partial_t f_k(v, t) + ikv f_k(v, t) - i \frac{e^2}{mk} \left( \int f_k(\mu) d\mu \right) f^{(0)'} + \frac{e}{m} E_k(t) f^{(0)'} = 0 \quad (9)$$

With a view to applications to more general kinetic equations (gyrokinetic, etc.) a more general equation is studied in the appendix. Equation (9) is then a particular case of Eq.(24) with

$$g_1(v) = ikv, \quad g_2(v) = -i \frac{e^2}{mk} f^{(0)'}(v), \quad g_3(v) = 1, \quad C(v, t) = \frac{e}{m} E_k(t) f^{(0)'}(v) \quad (10)$$

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<sup>1</sup>that is, the control electric field is of the same order as the fluctuations, not of order  $\int \delta f(v) dv$ , which would lead to a trivial control situation

the integral transform being (as in Morrison [4])

$$G_k(u, t) = (G_- f_k)(u) = \left(1 - \frac{\pi e^2}{mk^2} (H f^{(0)'})\right) f_k(u) + \frac{\pi e^2}{mk^2} f^{(0)'} (H f_k) \quad (11)$$

with left inverse

$$G_+ \circ \left\{ \left(1 - \frac{\pi e^2}{mk^2} (H f^{(0)'})\right)^2 + \left(\frac{\pi e^2}{mk^2} f^{(0)'}\right)^2 \right\}^{-1} \quad (12)$$

$G_+$  being

$$(G_+ f_k)(u) = \left(1 - \frac{\pi e^2}{mk^2} (H f^{(0)'})\right) f_k(u) - \frac{\pi e^2}{mk^2} f^{(0)'} (H f_k) \quad (13)$$

Applying the integral transform (11) to Eq.(9) it becomes

$$\partial_t (G_- f)(u) + iku (G_- f)(u) = -\frac{e}{m} E_k(t) f^{(0)'}(u) \quad (14)$$

with solution

$$G_k(u, t) = e^{-itku} \left( G_k(u, 0) - \frac{e}{m} f^{(0)'}(u) \int_0^t E_k(\tau) e^{i\tau ku} d\tau \right) \quad (15)$$

Then, according to (12), the Fourier modes solution is

$$\begin{aligned} f_k(v, t) = & \frac{\left(1 - \frac{\pi e^2}{mk^2} (H f^{(0)'})\right)(v) G_k(v, t)}{\left(1 - \frac{\pi e^2}{mk^2} (H f^{(0)'})\right)(v)^2 + \left(\frac{\pi e^2}{mk^2} f^{(0)'}(v)\right)^2} \\ & - \frac{\pi e^2}{mk^2} f^{(0)'}(v) H \left( \frac{G_k(u, t)}{\left(1 - \frac{\pi e^2}{mk^2} (H f^{(0)'})\right)(u)^2 + \left(\frac{\pi e^2}{mk^2} f^{(0)'}(u)\right)^2} \right)(v, t) \end{aligned} \quad (16)$$

## 2.1 Control of the linear modes by the electric field

Nonlinear stability of the steady states of the Vlasov-Poisson equation when the phase-space density is a decreasing function of the particle energy or depend on other invariants has been studied[6] [7] by the energy-Casimir method[8]. This means that deviations from the steady-state will remain bounded in time.

However, as expected from the non-dissipative nature of the Vlasov equation, the linear fluctuation modes of the uncontrolled equation are oscillatory and, once excited by a perturbation, they will not decay. As shown by Morrison [4] they may be used to obtain a statistical description of the fluctuations by the construction of a partition function. Here, one focus on the control of the fluctuations by the external electric field. Two situations will be considered. The first considers a constant in time electric field and tries to minimize the total energy associated to the fluctuations. The functional to be minimized is

$$F_1(E_k) = \lim_{T \rightarrow \infty} \int_0^T dt du |G_k(u, t)|^2 \quad (17)$$

In the second situation we allow the electric field to be time-dependent and chosen in a such a way as to introduce a damping effect in the solution (15).

For the first case ( $E_k$  independent of time), with the solution (15) one obtains a minimum for the functional  $F_1$  at

$$E_k = \frac{-\int \frac{f^{(0)'}(u)}{u} (-\text{Im}G_k(u, 0) + i\text{Re}G_k(u, 0)) du}{\frac{2e}{mk} \int \left(\frac{f^{(0)'}(u)}{u}\right)^2 du}$$

For this electrical field  $F_1$  is

$$F_{1 \min} = \int ((\text{Re}G_k(u, 0))^2 + (\text{Im}G_k(u, 0))^2) du - \frac{1}{2 \int \left(\frac{f^{(0)'}(u)}{u}\right)^2 du} \left( \int \left(\frac{f^{(0)'}(u)\text{Re}G_k(u, 0)}{u}\right)^2 + \left(\frac{f^{(0)'}(u)\text{Im}G_k(u, 0)}{u}\right)^2 du \right) \quad (18)$$

a smaller value as compared to the case  $E_k = 0$ , which would be  $F_1(E_k = 0) = \int ((\text{Re}G_k(u, 0))^2 + (\text{Im}G_k(u, 0))^2) du$ .

In the second case one allows the electric field to be time-dependent. One aims at controlling the fluctuation modes by an electric field induced dynamical damping. One looks for the solution of

$$\lim_{t \rightarrow \infty} \left( G_k(u, 0) - \frac{e}{m} f^{(0)'}(u) \int_0^t E_k(\tau) e^{i\tau ku} d\tau \right) = 0 \quad (19)$$

obtaining

$$E_k(t) = \frac{mk}{2\pi e} \int_{-\infty}^{\infty} \frac{G_k(u, 0)}{f^{(0)'}(u)} e^{-ikut} du \quad (20)$$

Then with this electric field

$$\begin{aligned} G_k(u, 0) &= \frac{e}{m} f^{(0)'}(u) \int_0^t E_k(\tau) e^{i\tau k u} d\tau \\ &= G_k(u, 0) - \frac{k}{2\pi} f^{(0)'}(u) \int_{-\infty}^{\infty} du' \frac{G_k(u', 0)}{f^{(0)'}(u')} \frac{e^{ik(u-u')t} - 1}{ik(u-u')} \end{aligned} \quad (21)$$

and from

$$\frac{k}{2\pi} \frac{e^{ik(u-u')t} - 1}{ik(u-u')} \xrightarrow{t \rightarrow \infty} \delta(u-u') \quad (22)$$

one sees that the electric field (20) induces a dynamical damping of the fluctuation modes.

### 3 Appendix. An integral transform for linearized kinetic equations

Morrison [4] solves the linearized Vlasov-Poisson equation by a Hilbert transform. However, for some practical applications, the linearized kinetic equations are more complex. For example the gyrokinetic Vlasov equation written in gyrocenter phase-space coordinates is [9]

$$\frac{\partial f}{\partial t} + \dot{X} \bullet \nabla f + \dot{U} \frac{\partial f}{\partial U} = 0 \quad (23)$$

where  $\dot{X} \neq U$

This is the motivation to study an equation more general than the linearized Vlasov-Poisson (9). Linearized Fourier kinetic equations are of the type

$$\frac{\partial f(v)}{\partial t} + g_1(v) f(v) + g_2(v) \int g_3(\mu) f(\mu) d\mu + C(v, t) = 0 \quad (24)$$

with  $g_1$  a monotone function of  $v$ . Let  $T$  be a transform such that

$$(Tg_1 f)(u) = \int g_3(\mu) f(\mu) d\mu + g_1(u) (Tf)(u) \quad (25)$$

namely

$$(Tf)(u) = P \int \frac{g_3(v) f(v)}{g_1(v) - g_1(u)} dv \quad (26)$$

Notice that, for invertible  $g_1$ , the  $T$ -transform may be written in terms of the Hilbert transform

$$(Tf)(u) = \pi \left( H \frac{g_3 f}{g_1'} \circ g_1^{-1} \right) (g_1(u)) \quad (27)$$

Then, one defines

$$(G_+ f)(u) = (1 + (Tg_2)(u)) f(u) + g_2(u) (Tf)(u) \quad (28)$$

and

$$(G_- f)(u) = (1 + (Tg_2)(u)) f(u) - g_2(u) (Tf)(u) \quad (29)$$

$\left\{ (1 + Tg_2)^2 + \pi^2 \left( \frac{g_2 g_3}{g_1'} \right)^2 \right\}^{-1} G_-$  is a left inverse of  $G_+$

$$(G_- G_+ f)(u) = \left\{ (1 + Tg_2)^2 + \pi^2 \left( \frac{g_2 g_3}{g_1'} \right)^2 \right\} f(u) \quad (30)$$

as may be checked using (27) and the properties of the Hilbert transform [10].

$G_-$ -transforming Eq.(24) one obtains

$$\begin{aligned} \partial_t (G_- f) + g_1(u) (G_- f) &= -C(u, t) + g_2(u) T(C)(u) - C(u, t) T(g_2) \quad (31) \\ &= \gamma(u, t) \end{aligned}$$

with solution

$$G_-(u, t) = e^{-tg_1(u)} \left( G_-(u, 0) + \int_0^t \gamma(u, \tau) e^{\tau g_1(u)} d\tau \right) \quad (32)$$

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