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The Geometry of Noncommutative Space-Time

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Abstract Stabilization, by deformation, of the Poincaré-Heisenberg algebra requires both the introduction of a fundamental lentgh and the noncommutativity of translations which is associated to the gravitational field. The noncommutative geometry structure that follows from the deformed algebra is studied both for the non-commutative tangent space and the full space with gravity. The contact points of this approach with the work of David Finkelstein are emphasized.

Keywords Spacetime · Noncommutative geometry · Gravity

1 Introduction

I first met David Finkelstein when, as a graduate student at Austin, went to a summer school where David was one of the lecturers. Further to his excellent lectures, I was deeply impressed by his warm readiness to meet and answer the questions of the students. Much later, through a common friend, Eric Carlen, I was introduced to his simplicity approach to physical theories and conversely he became aware of my approach to noncommutative space-time through deformation theory, to which he then gave generous reference in his papers.

We had planned to meet when he once passed by Lisbon, but at the time I was in France and unfortunately missed that chance. Nevertheless the correspondence we exchanged and the reading of his papers have been a constant source of inspiration for the exploration of uncharted and sometimes unpopular territory.

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In this paper, which I dedicate to the memory of David Finkelstein, the focus will be on the geometry of the noncommutative space time that follows from the (stable) deformed Poincaré-Heisenberg algebra. The many contact points with Finkelstein approach to these problems will be put into evidence. Among his many important contributions in different fields, our contact point came about in the study of modifications of the space-time algebra, which Finkelstein approached through the requirement of simplicity of the algebras [1, 2], whereas I have used a deformation-stability principle. Of course, in addition to stability, there are other arguments in favor of simple algebras, in particular the spectrum of its representations [3]. For quantum physics the deformation stability approach does indeed coincide with the simplicity approach, but deformation-stability may well go beyond the Lie algebra realm [4].

2 The Stable Poincaré-Heisenberg Algebra

The Poincaré-Heisenberg algebra is deformed [5] to the stable algebra $\Re_{\ell,R} = \{M^{\mu\nu}, p^\mu, x^\mu, \Im\}$ defined by the commutators

$$
\begin{aligned}
[M^{\mu\nu}, M^{\rho\sigma}] &= i(M^{\mu\sigma}\eta^{\nu\rho} + M^{\nu\rho}\eta^{\mu\sigma} - M^{\nu\sigma}\eta^{\mu\rho} - M^{\mu\rho}\eta^{\nu\sigma}) \\
[M^{\mu\nu}, p^\lambda] &= i(p^\mu\eta^{\nu\lambda} - p^\nu\eta^{\mu\lambda}) \\
[M^{\mu\nu}, x^\lambda] &= i(x^\mu\eta^{\nu\lambda} - x^\nu\eta^{\mu\lambda}) \\
[p^\mu, x^\nu] &= i\eta^{\mu\nu}\Im \\
[x^\mu, x^\nu] &= -i\epsilon_4\ell^2 M^{\mu\nu} \\
[p^\mu, p^\nu] &= -i\frac{\epsilon_5}{R^2}M^{\mu\nu} \\
[x^\mu, \Im] &= i\epsilon_4\ell^2 p^\mu \\
[p^\mu, \Im] &= -i\frac{\epsilon_5}{R^2}x^\mu \\
[M^{\mu\nu}, \Im] &= 0
\end{aligned}
\tag{1}
$$

which some authors now call the *stable Poincaré-Heisenberg algebra*

The stable algebra $\Re_{\ell,R}$, to which the Poincaré-Heisenberg algebra has been deformed, is the algebra of the 6-dimensional pseudo-orthogonal group with metric

$$\eta_{aa} = (1, -1, -1, -1, \epsilon_4, \epsilon_5), \;\; \epsilon_4, \epsilon_5 = \pm 1 \tag{2}$$

with the identifications

$$
\begin{aligned}
p^\mu &= \frac{1}{R}M^{\mu 4} \\
x^\mu &= \ell M^{\mu 5} \\
\Im &= \frac{\ell}{R}M^{45}
\end{aligned}
\tag{3}
$$

Both ℓ and R have dimensions of length. However they might have different physical status and interpretation. Whereas ℓ might be considered as a fundamental length and a constant of Nature, R, being associated to the non-commutativity of the generators of translation in the Poincaré group, seems to be associated to the local curvature of the space-time manifold and therefore is a dynamical quantity associated to the local intensity of the gravitational field (see Section 4).

In the tangent space one may take the limit $R \to \infty$ obtaining

$$[p^\mu, p^\nu]\big|_{R\to\infty} \to 0 \qquad \text{and} \qquad [x^\mu, \Im]\big|_{R\to\infty} \to 0 \tag{4}$$

all the other commutators being the same as in (1), leading to the tangent space algebra $\Re_{\ell,\infty} = \left\{x^\mu, M^{\mu\nu}, \overline{p}^\mu, \overline{\Im}\right\}^1$ whose consequences have been studied in a number of publications [6–12]. In this limit the operators $\left\{\overline{p}^\mu, \overline{\Im}\right\}$ are an Abelian set of derivations of the $\Re_{\ell,\infty}$ algebra.

Finkelstein, in line with his suggestion that Clifford algebra is the natural language for quantum physics [13, 14], identifies the world line of a spin $\frac{1}{2}$ particle with N Clifford cells, the usual Dirac spin being the growing tip of the world line [15, 16]. The space-time operators are then represented as sums of second-order elements in the spinor $6N$ space, for example

$$\begin{aligned} \smile{x}^\mu &= -\chi \sum_{n=1}^{N-1} \gamma^{\mu 4}(n) \\ \smile{p}^\mu &= \phi \sum_{n=1}^{N-1} \gamma^{\nu 5}(n) \end{aligned} \tag{5}$$

Then, essentially the same commutation relations as in (1) are obtained, the $\epsilon_4, \epsilon_5 = \pm 1$ metric choice being related to the real or imaginary nature of the simplifier (deformation) parameters χ and ϕ. The independent parameters are also two in number, N being constrained by

$$\chi\phi(N-1) = \frac{\hbar}{2} \tag{6}$$

In the setting of the stable $\Re_{\ell,R}$ algebra all variables are represented as operators with equal footing, the space-time coordinates themselves not having a special distinguished role. In particular the absence of nontrivial characters, implies that space-time has no points. Rather, the physical processes will be operations on a representation space (a module) over the algebra. The most economic way to construct a module would be to use free powers of the algebra itself, leading to the notion of physical processes as operations on free modules over the algebra. The view of physics as a process and an unifying status for the physical variables as operators was in several forms and places vigorously proposed by David Finkelstein.

In the following I will deal with the space-time geometry that follows from the stable Heisenberg-Poincaré algebra $\Re_{\ell,R}$ and its tangent space limit $\Re_{\ell,\infty}$.

3 The Space-Time Geometry and Some Consequences

In the classical (commutative) case the space-time coordinates $\{x^\mu\}$ are a commuting set whereas, in the deformed setting of (1), the space-time algebra becomes

$$\begin{aligned} [M^{\mu\nu}, M^{\rho\sigma}] &= i(M^{\mu\sigma}\eta^{\nu\rho} + M^{\nu\rho}\eta^{\mu\sigma} - M^{\nu\sigma}\eta^{\mu\rho} - M^{\mu\rho}\eta^{\nu\sigma}) \\ [M^{\mu\nu}, X^\lambda] &= i(X^\mu\eta^{\nu\lambda} - X^\nu\eta^{\mu\lambda}) \\ [X^\mu, X^\nu] &= -i\epsilon_4 M^{\mu\nu} \end{aligned} \tag{7}$$

[1] $\overline{p}^\mu, \overline{\Im}$ denote the tangent space ($R \to \infty$) limits of the operators, not be confused with the physical $p^\mu, \Im$ operators. According to the deformation-stability principle they are stable physical operators only when R is finite, that is, when gravity is turned on.

a non-commutative algebra, with $X^{\mu} = \frac{x^{\mu}}{\ell}$.

There is a clear relation between algebraic and geometric structures. Indeed, the way one explores a space S is by computing functions on it and functions on S form algebras. In the classical (commutative) case the Gelfand-Naimark theorem states that a C^{*}-algebra A is $*$-isomorphic to an algebra of functions $C_0(S_A)$ on its Gelfand spectrum S_A and the Serre-Swan theorem that continuous sections of a finite dimensional vector bundle $E \to M$ are finitely generated projective modules over $C(M)$ and every such module is a space of sections of a vector bundle over M. These and other correspondences were extended to non-commutative algebras, providing a framework for non-commutative geometry [17–19].

Given an algebra A, the standard way to obtain the correspondent geometry and in particular the differential algebra structure is by forming a triple $(H, \pi(A), D)$, where $\pi(A)$ is a representation of the algebra in the Hilbert space H and D is a Dirac operator. When a sufficient number of algebra derivations are available, the noncommutative generalization of the geometrical notions is a natural extension of the commutative case [17]. However, in general it might not be possible to use derivations to construct by duality the differential forms because many algebras have no derivations at all. The commutator with the Dirac operator is then used to generate the one-forms, the Dirac operator also providing the metric structure. Notice, however that, although the language of spectral geometry through the $(H, \pi(A), D)$ triple may be used as a guide, the assumptions of compactness and positive definite metric used in most rigorous constructions do not apply to the algebras studied here.

Depending on the sign of ϵ_5 the algebra in (7) is the algebra of $SO(3,2)$, $\epsilon_4 = +1$, or $SO(4,1)$, $\epsilon_4 = -1$. The simplest representation of these algebras would be as differential operators in a five-dimensional Euclidean space with coordinates $(\xi^1, \xi^2, \xi^3, \xi^0, \xi^4)$

$$\begin{aligned} M^{\mu\nu} &= i\left(\xi^{\mu}\frac{\partial}{\partial\xi_{\nu}} - \xi^{\nu}\frac{\partial}{\partial\xi_{\mu}}\right) \\ x^{\mu} &= \xi^{\mu} + i\ell\left(\xi^{\mu}\frac{\partial}{\partial\xi^{4}} - \epsilon_4\xi^{4}\frac{\partial}{\partial\xi_{\mu}}\right) \end{aligned} \tag{8}$$

In this setting the operators $\overline{p}^{\mu}, \overline{\Im}$ have a representation

$$\begin{aligned} \overline{p}^{\mu} &= i\frac{\partial}{\partial\xi_{\mu}} \\ \overline{\Im} &= 1 + i\ell\frac{\partial}{\partial\xi^{4}} \end{aligned}$$

generating, together with $M^{\mu\nu}$ and x^{μ} the algebras of the inhomogeneous $ISO(3,2)$ or $ISO(4,1)$. The minimal Abelian set of derivations generalizing those of the commutative case are associated to the operators $\overline{p}^{\mu}, \overline{\Im}$. Also in the commutative case, the derivations, used to construct the differential calculus are not inner derivations, as has been proposed in some versions of the matrix geometries [17], but operations of the Heisenberg algebra. The following maximal Abelian set $V = \{\overline{\partial}^{\mu}, \overline{\partial}^{4}\}$ of derivations in the enveloping algebras of the inhomogeneous groups are used

$$\begin{aligned} \overline{\partial}^{\mu}(x^{\nu}) &= \eta^{\mu\nu}\Im \\ \overline{\partial}^{4}(x^{\mu}) &= -\epsilon_4\ell p^{\mu}\Im \\ \overline{\partial}^{\sigma}(M^{\mu\nu}) &= \eta^{\sigma\mu}p^{\nu} - \eta^{\sigma\nu}p^{\mu} \\ \overline{\partial}^{4}(M^{\mu\nu}) &= 0 \end{aligned} \tag{9}$$

From this, by duality, the differential calculus is constructed [7]. Notice that although an extra dimension is used in the representation space, the space-time coordinates are still only four, noncommutative ones. However the derivations in V introduce, by duality, an additional degree of freedom in the exterior algebra. The Dirac operator is

$$\overline{D} = i\gamma^a \overline{\partial}_a \tag{10}$$

with $\overline{\partial}_a = \left(\overline{\partial}_\mu, \overline{\partial}_4\right)$, the γ's being a basis for the Clifford algebras $C(3,2)$ or $C(4,1)$

$$\begin{aligned} &\left(\gamma^0, \gamma^1, \gamma^2, \gamma^3, \gamma^4 = \gamma^5\right) \qquad \epsilon_4 = +1 \\ &\left(\gamma^0, \gamma^1, \gamma^2, \gamma^3, \gamma^4 = i\gamma^5\right) \qquad \epsilon_4 = -1 \end{aligned} \tag{11}$$

This was the approach followed in [7] and [20]. The representation (8) is an efficient tool for calculations, however it is not irreducible.

In the commutative case, the points of the geometry are the characters, the continuous algebra morphisms from the algebra to the complex numbers. Noncommutative algebras have no such characters and the most elementary geometric sets are the irreducible representations. Hence, the elementary space-time structures compatible with the (tangent space) deformed algebra are to be obtained from the irreducible representations of the groups $SO(3,2)$ or $SO(4,1)$. For future reference, those of $SO(3,2)$ are listed in the Appendix.

Deformation stability, or Lie algebra simplicity, as strongly proposed by Finkelstein, leads to modifications of the space-time algebra of the type described before, in particular the noncommutativity of the coordinates and a radical modification of the view of physical processes as happening in the background of a smooth space-time manifold. Experimental observation of the effects of these modifications will of course depend on the size of the deformation parameter ℓ. Two types of effects were predicted: those that depend simply on the noncommutativity of the variables and those that depend on the dimension of the differential algebra. Effects of the first type are associated for example to modifications of the phase-space volume [8, 10, 12] or to the measurement of the velocity of wave packets, because time and space being noncommuting operators their ratio can only be taken in the sense of expectation values [11].

The structure of the differential algebra and the associated Dirac operator (10) also implies the existence of two solutions for the "massless" Dirac equation, one massless and the other of very large mass (of order $1/\ell$) with the same quantum numbers [20]. Mixing of these solutions might, by the seesaw mechanism, endow neutrinos with small masses. Here it must be pointed out that Galiautdinov and Finkelstein [15], following a slightly different approach, also studied modifications to the Dirac equation. I do not feel comfortable with their interpretation of the relation to the Higgs mass, but the fact remains that they also pointed out the existence of large mass solutions.

Less explored is the fact that if the differential algebra has an additional dimension then, quantum fields that are connections should have an additional component[2] [7].

[2]Notice that additional components are not necessarily required for spinors because the Clifford algebras $C(3,2)$ or $C(4,1)$ both have four-dimensional representations.

4 Gravity as a Quantum Effect

I borrow the title of this section from the title of a preprint of David Finkelstein. The preprint itself was unpublished, I think, but the main ideas were published in [21] (see also [22]). According to Finkelstein "*the non-commutativity of parallel transport is a classical vestige of the quantum non-commutativity of event momentum-energy variables*".

Indeed, when the full stable $\Re_{\ell,R}$ algebra is considered, the generators of translations no longer commute,

$$[p^\mu, p^\nu] = -i\frac{\epsilon_5}{R^2}M^{\mu\nu} \tag{12}$$

Redefining $\frac{\epsilon_5}{R^2}$ as a new, gravity related, space-time dependent field ϕ

$$\frac{\epsilon_5}{R^2} \overset{\circ}{=} \phi \tag{13}$$

$$\begin{aligned} [p^\mu, p^\nu] &= -i\phi M^{\mu\nu} \\ [p^\mu, \Im] &= -i\phi x^\mu \end{aligned} \tag{14}$$

The algebra will be the same as before if ϕ commutes with all the generators, that is, if it is a function of the invariants. The invariants are those of the algebra of the 6-dimensional pseudo-orthogonal group with the metric in (2) and the identifications in (3). Then with indices $a, b, \dots \in \{0, 1, 2, 3, 4, 5\}$ the invariants are

$$\begin{aligned} C_1 &= \sum M_{ab}M^{ab} \\ C_2 &= \sum \epsilon_{abcdef}M^{ab}M^{cd}M^{ef} \\ C_3 &= \sum M_{ab}M^{bc}M_{cd}M^{da} \end{aligned} \tag{15}$$

with summation over repeated indices and $\epsilon_{012345} = +1$.

In this view, the scalar (operator) field ϕ appears, rather than the metric, as the primary gravitational field. Commutativity of ϕ with all the generators is an expression of the conformal covariance of gravity-related tensors. Notice however that this is not DeSitter or anti-DeSitter geometry. It would be if $\ell = 0$ and ϕ = constant, but here the coordinates are noncommuting operators ($\ell \neq 0$) and ϕ is also an operator-valued function of the invariants C_i.

With (14) and (1) a non-commutative geometry framework for $\Re_{\ell,R}$ may be developed along the same lines as done before [7] for the tangent-space $\Re_{\ell,\infty}$ algebra.

The basic spaces to be used are the enveloping algebra $U_\Re$ generated by x^μ, $M^{\mu\nu}$, p^μ, $\Im$ plus a unit and $\Im^{-1}$

$$U_\Re = \left\{x^\mu, M^{\mu\nu}, p^\mu, \Im, \Im^{-1}, \mathbf{1}\right\} \tag{16}$$

and free modules generated by $U_\Re$. Because of (14), that is, because the set $\{p^\mu, \Im\}$ is not closed under commutation, one should take into account the full space $\mathcal{V}$ of inner derivations corresponding to $\{x^\mu, M^{\mu\nu}, p^\mu, \Im\}$, namely

$$\begin{aligned} \partial^\mu &\longleftrightarrow \frac{1}{i}p^\mu \\ \partial^4 &\longleftrightarrow \frac{1}{i\ell}\Im \end{aligned}$$

$$\begin{aligned}\partial^{\mu\nu} &\longleftrightarrow \frac{1}{i}M^{\mu\nu}\\ \partial^{x_\mu} &\longleftrightarrow \frac{1}{i}x^{\mu}\end{aligned} \tag{17}$$

the correspondence symbol $\longleftrightarrow$ in (17) meaning that the derivations ∂ act on $U_{\Re}$ in the same way as the commutators with the operators on the right hand side.

Then the graded differential algebra $\Omega\left(U_{\Re}\right)$ is the complex of multilinear antisymmetric mappings from the space $\mathcal{V}$ of derivations to $U_{\Re}$. $\Omega^0\left(U_{\Re}\right)$ is identified with $U_{\Re}$. Defining a basis

$$\theta_a\left(\partial^b\right)=\delta_a^b \tag{18}$$

with $a, b \in \left\{\mu, 4, \mu\nu, x_\mu\right\}$, and an exterior derivative in $\Omega\left(U_{\Re}\right)$

$$\begin{aligned}d\omega\left(\partial^{a_0},\cdots,\partial^{a_k}\right) &= \sum_i (-1)^i\,\partial^{a_i}\left(\omega\left(\partial^{a_0}\cdots,\widehat{\partial^{a_i}},\cdots,\partial^{a_k}\right)\right)\\ &+\sum_{i<j}(-1)^{i+j}\omega\left(\left[\partial^{a_i},\partial^{a_j}\right],\partial^{a_0}\cdots,\widehat{\partial^{a_i}},\cdots,\widehat{\partial^{a_j}},\cdots,\partial^{a_k}\right)\end{aligned} \tag{19}$$

the differential of physical operators may be computed. For example:

$$\begin{aligned}dx^{\mu} &= \eta^{\mu\nu}\Im\theta_\nu-\epsilon_4\ell p^{\mu}\theta_4+\left(\eta^{\beta\mu}x^{\alpha}-\eta^{\alpha\mu}x^{\beta}\right)\theta_{\alpha\beta}-\epsilon_4\ell^2 M^{\alpha\mu}\theta_{x_\alpha}\\ dp^{\mu} &= -\phi M^{\nu\mu}\theta_\nu+\frac{\phi}{\ell}x^{\mu}\theta_4+\left(\eta^{\beta\mu}p^{\alpha}-\eta^{\alpha\mu}p^{\beta}\right)\theta_{\alpha\beta}-\eta^{\alpha\mu}\Im\theta_{x_\alpha}\end{aligned} \tag{20}$$

One also defines a contraction i_∂ as a mapping from $\Omega^p\left(U_{\Re}\right)$ to $\Omega^{p-1}\left(U_{\Re}\right)$

$$i_\partial\omega\left(\partial^{a_1},\cdots,\partial^{a_{p-1}}\right)=\omega\left(\partial,\partial^{a_1},\cdots,\partial^{a_{p-1}}\right) \tag{21}$$

and a Lie derivative

$$L_\partial = di_\partial + i_\partial d \tag{22}$$

Let now E be a $U_{\Re}$-left module generated by the identity $\mathbf{1}$

$$E=\left\{a\mathbf{1};\, a\in U_{\Re}\right\} \tag{23}$$

From this, other modules may be obtained by projection, a projection Π being a matrix with entries in $U_{\Re}$

$$E_{\Pi}=\left\{\psi\in E:\Pi\psi=\psi\right\} \tag{24}$$

with $\sum_{i=1}^{n}\psi_i\Pi_{ji}=\psi_j$. The modules E_Π are, in this non-commutative context, the $n-$dimensional quantum fields and

$$\Pi\psi-\psi=0 \tag{25}$$

the field equations.

In E one defines connections ∇ as mappings $\nabla: E\rightarrow\Omega^1\left(U_{\Re}\right)\otimes E$ such that

$$\nabla\left(a\chi\right)=a\nabla\left(\chi\right)+da\chi \tag{26}$$

with $a\in U_{\Re}$ and $\chi\in E$. Because of (26), if one knows how the connection acts on the algebra unit $\mathbf{1}$ one has the complete action. Let

$$\nabla\left(\mathbf{1}\right)=A^i\theta_i \tag{27}$$

Then

$$\nabla\left(x^{\mu}\right)=\nabla\left(\mathbf{1}x^{\mu}\right)=A^i\theta_i x^{\mu}+dx^{\mu} \tag{28}$$

The covariant derivative along ∂ is $\nabla_\partial = (\nabla, \partial)$ and the curvature is obtained by the commutator of two covariant derivatives. Using (28) one obtains

$$\left[\nabla_{\partial^\alpha}, \nabla_{\partial^\beta}\right]\left(x^\mu\right) = \left\{d_\alpha A^\beta - d_\beta A^\alpha - \left[A^\alpha, A^\beta\right]\right\} x^\mu + \phi\left(\eta^{\sigma\alpha}\eta^{\mu\beta} - \eta^{\sigma\beta}\eta^{\mu\alpha}\right) x_\sigma \quad (29)$$

with $d_\alpha A^\beta = \left(dA^\beta, \partial^\alpha\right)$. One sees that, in addition to the curvature of the (gauge) field A, there is a gravitational induced curvature associated to the stable Heisenberg-Poincaré algebra $\Re_{\ell,R}$, with curvature tensor,

$$R^{\sigma\alpha\mu\beta} = \phi\left(\eta^{\sigma\alpha}\eta^{\mu\beta} - \eta^{\sigma\beta}\eta^{\mu\alpha}\right) \quad (30)$$

the deformation field ϕ appearing as an operator-valued scalar curvature. Notice that, through (15), ϕ may depend not only on the coordinate operators but also on the momentum and angular momentum operators.

Here, because the algebra $\Re_{\ell,R}$ has a sufficiently rich set of derivations, the construction of the graded differential algebra was based on the derivations. However there are other standard ways to construct the differential algebra which do not rely on the existence of derivations. One of them uses an operator D and defines p-forms as

$$\omega = \sum a_0\left[D, a_1\right]\cdots\left[D, a_p\right] \quad (31)$$

Let Γ be the following set of 15 four-dimensional gamma matrices

$$\Gamma = \left\{\gamma^\mu, (i)^{\frac{1-\epsilon_4}{2}}\gamma^5, \gamma^{\mu\nu} = \frac{1}{2}\left[\gamma^\mu, \gamma^\nu\right], \gamma^\mu\gamma^5\right\} \quad (32)$$

and the operator

$$D = i\Gamma_a\partial^a \quad (33)$$

where $\{\partial^a\}$ is the set Γ of 15 derivations, listed in (17). Then from (31) and (33) one sees that the differential algebra structure obtained by (31) is identical to the one obtained from the derivations. The operator D may be identified as the Dirac operator of the theory and in the sector $\left(\partial^\mu, \partial^4\right)$ it coincides with the operator $\overline{D}$ defined in (10) for the noncommutative tangent space algebra $\Re_{\ell,\infty}$.

The search for stability of the physical models describes well the evolution of our understanding of physics. The transition from singular to generic stable structures seems to tells us how Nature organizes itself. From classical to relativistic mechanics we come from an unstable Galilean algebra to the stable Lorentz algebra. Quantum mechanics also, may be interpreted as the stabilization of the phase-space Poisson to the Moyal algebra or, equivalently to the Heisenberg algebra. Finally, stabilizing the Poincaré-Heisenberg algebra requires both a fundamental length (ℓ) and the noncommutativity of the translations (ϕ). There is a formal identity of all these processes and the last one relates to the emergence of gravity. Of course, this is only a sketch of my understanding of the deep intuition of David Finkelstein when stating that “gravity is a quantum effect”.

Appendix: Irreducible Representations of the Space-Time Algebra

For $SO\,(3,2)$ $(\epsilon_4 = +1)$ a way to characterize the irreducible representations of this groups is to consider its action on functions on a $V^{3,2}$ cone, with coordinates

$$\begin{aligned} y_1 &= e^s \cos\varphi_2 \\ y_2 &= e^s \sin\varphi_2 \cos\varphi_1 \\ y_3 &= e^s \sin\varphi_2 \sin\varphi_1 \\ y_4 &= e^s \sin\theta_1 \\ y_0 &= e^s \cos\theta_1 \end{aligned} \tag{34}$$

Then, on this cone, consider a space $S^{\sigma,\varepsilon}$ of functions satisfying the homogeneity conditions [23]

$$f\,(ax) = |a|^{\sigma}\, sign^{\varepsilon} a f\,(x) \tag{35}$$

$\sigma \in \mathbb{R}$ and $\varepsilon = \{0, 1\}$. In $M^{\sigma,\varepsilon}$ the group operators act as follows

$$T\,(g)\, f\,(x) = f\left(g^{-1}x\right) \tag{36}$$

Because of (35) the functions are uniquely characterized by their values in the $(s = 0)$ Γ_1 contour. This contour is topologically $S^2 \times S^1$. The spaces of homogeneous functions on this contour will be denoted S^{Γ_1}. Denote by $g_{ij}\,(\theta)$ a rotation in the plane ij and by $g'_{ij}\,(t)$ a hyperbolic rotations in the plane ij.

Given $f \in S^{\Gamma_1}$, using (35) and (36) one obtains for an hyperbolic rotation in the 1, 4 plane

$$\begin{aligned} T^{\sigma}\left(g'_{14}\,(t)\right) f\,(\varphi_1, \varphi_2, \theta_1) &= |a|^{\sigma/2} f\left(\varphi_1, \varphi'_2, \theta'_1\right) \\ |a| &= \left\{\sin^2\theta_1 + (\cos\theta_1 \cosh t - \cos\varphi_2 \sinh t)^2\right\}^{1/2} \\ \cos\varphi'_2 &= \frac{\cos\varphi_2 \cosh t - \cos\theta_1 \sinh t}{|a|} \\ \cos\theta'_1 &= \frac{\cos\theta_1 \cosh t - \cos\varphi_2 \sinh t}{|a|} \end{aligned} \tag{37}$$

Similar expressions are obtained for the other elementary operations. From these one obtains, as infinitesimal generators, a representation for the generators of the algebra $\left\{X_\mu, M_{\mu\nu}\right\}$ as operators in S^{Γ_1}

$$\begin{aligned} iX_1 &= iM_{14} = \sigma \cos\theta_1 \cos\varphi_2 - \sin\theta_1 \cos\varphi_2 \frac{\partial}{\partial\theta_1} - \cos\theta_1 \sin\varphi_2 \frac{\partial}{\partial\varphi_2} \\ iX_2 &= iM_{24} = \sigma \cos\theta_1 \sin\varphi_2 \cos\varphi_1 - \sin\theta_1 \sin\varphi_2 \cos\varphi_1 \frac{\partial}{\partial\theta_1} \\ &\quad + \cos\theta_1 \cos\varphi_2 \cos\varphi_1 \frac{\partial}{\partial\varphi_2} - \frac{\cos\theta_1 \sin\varphi_1}{\sin\varphi_2} \frac{\partial}{\partial\varphi_1} \\ iX_3 &= iM_{34} = \sigma \cos\theta_1 \sin\varphi_2 \sin\varphi_1 - \sin\theta_1 \sin\varphi_2 \sin\varphi_1 \frac{\partial}{\partial\theta_1} \\ &\quad + \cos\theta_1 \cos\varphi_2 \sin\varphi_1 \frac{\partial}{\partial\varphi_2} + \frac{\cos\theta_1 \cos\varphi_1}{\sin\varphi_2} \frac{\partial}{\partial\varphi_1} \\ iX_0 &= iM_{04} = \frac{\partial}{\partial\theta_1} \end{aligned}$$

$$
\begin{aligned}
iM_{12} &= -\cos\varphi_1 \frac{\partial}{\partial\varphi_2} + \frac{\cos\varphi_2 \sin\varphi_1}{\sin\varphi_2} \frac{\partial}{\partial\varphi_1} \\
iM_{13} &= -\sin\varphi_1 \frac{\partial}{\partial\varphi_2} - \frac{\cos\varphi_2 \cos\varphi_1}{\sin\varphi_2} \frac{\partial}{\partial\varphi_1} \\
iM_{23} &= -\frac{\partial}{\partial\varphi_1} \\
iM_{10} &= \sigma \sin\theta_1 \cos\varphi_2 + \cos\theta_1 \cos\varphi_2 \frac{\partial}{\partial\theta_1} - \sin\theta_1 \sin\varphi_2 \frac{\partial}{\partial\varphi_2} \\
iM_{20} &= \sigma \sin\theta_1 \sin\varphi_2 \cos\varphi_1 + \cos\theta_1 \sin\varphi_2 \cos\varphi_1 \frac{\partial}{\partial\theta_1} \\
&\quad + \sin\theta_1 \cos\varphi_2 \cos\varphi_1 \frac{\partial}{\partial\varphi_2} + \frac{\sin\theta_1 \sin\varphi_1}{\sin\varphi_2} \frac{\partial}{\partial\varphi_1} \\
iM_{30} &= \sigma \sin\theta_1 \sin\varphi_2 \sin\varphi_1 + \cos\theta_1 \sin\varphi_2 \sin\varphi_1 \frac{\partial}{\partial\theta_1} \\
&\quad + \sin\theta_1 \cos\varphi_2 \sin\varphi_1 \frac{\partial}{\partial\varphi_2} + \frac{\sin\theta_1 \cos\varphi_1}{\sin\varphi_2} \frac{\partial}{\partial\varphi_1}
\end{aligned}
\tag{38}
$$

These representations are irreducible for non-integer σ. There are also conditions for unitary of the representations, but this is not so important because only the $M_{\mu\nu}$ $(\mu, \nu = 0, 1, 2, 3)$ are generators of symmetry operations.

A similar construction is possible for $SO(4,1)$ $(\epsilon_4 = -1)$ with functions on a $V^{4,1}$ $(\epsilon_5 = -1)$ cone, with coordinates

$$
\begin{aligned}
y_1 &= e^s \cos\varphi_3 \\
y_2 &= e^s \sin\varphi_3 \cos\varphi_2 \\
y_3 &= e^s \sin\varphi_3 \sin\varphi_2 \cos\varphi_1 \\
y_4 &= e^s \sin\varphi_3 \sin\varphi_2 \sin\varphi_1 \\
y_0 &= e^s
\end{aligned}
\tag{39}
$$

the contour Γ_2 $(s = 0)$ in this case being topologically S^3.

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