The fractional volatility model and rough volatility

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Abstract

The question of the volatility roughness is interpreted in the framework of a data-reconstructed fractional volatility model, where volatility is driven by fractional noise. Some examples are worked out and, using the Malliavin calculus for fractional processes, an option pricing equation and its solution are obtained.

Keywords: Stochastic volatility, Fractional noise, Rough volatility

1 Introduction

The purpose of this paper is twofold: On one hand to show how the roughness of market volatility, observed by Gatheral and collaborators [1], may be simply explained in the context of a stochastic *fractional volatility model* (FVM) [2] without assuming that the volatility process itself has an Hurst index smaller than $\frac{1}{2}$. This is because, in the FVM, volatility is driven not by fractional Brownian motion but by fractional noise, that is, by the generalized derivative (or finite differences) of fractional Brownian motion. On the other hand to obtain the implications of the FVM for option pricing.

Section 2 provides an introduction and brief review of the FVM, a model that has been shown to be consistent with the bulk stochastic properties

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of several markets and has been proven to be arbitrage free and with well defined completeness or incompleteness properties, depending on the nature of the random processes used to drive the price process and the volatility [3].

Section 3 interprets the rough volatility in the framework of the FVM, using high frequency data of several markets. Finally, because the rough volatility paradigm has been used by other authors [4] to discuss option pricing, it seemed appropriate to also derive one such formula in the context of the FVM. A simplified result for option pricing had already been obtained in the original reference [2] using a construction of the Hull-White [5] type. Here, in Section 4, a new result is obtained using more rigorous mathematics.

2 The fractional volatility model

Some years ago, in collaboration with M. J. Oliveira [2], a program was started to reconstruct the market process from the data, using only minimal mathematical and theoretical prejudices. Consistency with the data was the main concern and only two general hypothesis were used, namely:

(**H1**) The log-price process $\log S_t$ belongs to a probability product space $\Omega \otimes \Omega^{'}$ of which the first one, Ω, is the Wiener space and the second, $\Omega^{'}$, is a probability space to be reconstructed from the data. Denote by $\omega \in \Omega$ and $\omega^{'} \in \Omega^{'}$ the elements (sample paths) in Ω and $\Omega^{'}$ and by $\mathcal{F}_t$ and $\mathcal{F}_t^{'}$ the $\sigma-$algebras in Ω and $\Omega^{'}$ generated by the processes up to t. Then, a particular realization of the log-price process would be denoted

$$\log S_t \left(\omega, \omega^{'} \right) .$$

This first hypothesis is really not limitative. Even if none of the non-trivial stochastic features of the log-price were captured by Brownian motion, that would simply mean that S_t is a trivial function in Ω.

(**H2**) The second hypothesis is stronger, although natural. It is assumed that for each fixed $\omega^{'}$, $\log S_t \left(\bullet, \omega^{'} \right)$ is a square integrable random variable in Ω.

From the second hypothesis it follows that, for each fixed $\omega^{'}$,

$$\frac{dS_t}{S_t} \left(\bullet, \omega^{'} \right) = \mu_t \left(\bullet, \omega^{'} \right) dt + \sigma_t \left(\bullet, \omega^{'} \right) dB \left(t \right) , \quad (1)$$

where $\mu_t\left(\bullet,\omega'\right)$ and $\sigma_t\left(\bullet,\omega'\right)$ are well-defined processes in Ω (Theorem 1.1.3 in Ref.[6]). The process associated to the probability space Ω' was then inferred from the data and this data-reconstructed σ_t process was called the *induced volatility*.

The scaling properties, of the data-reconstructed induced volatility process, were then carefully analyzed [2]. The conclusion was that the log-price, the volatility and the log-volatility are not self-similar processes and it is only after the log-volatility is integrated and the linear part extracted, that a self-similar process $R_\sigma(t)$ is obtained. This is an essential finding of the data-reconstructed model. That volatility is modeled by the finite differences of a self-similar process is an essential difference from other models that take into account the long-range correlation of the volatility. In some models [7] [8] [9] it is fractional Brownian motion (fBm) itself that drives the volatility, not a generalized derivative (or finite difference) of this process.

The analysis in [2], leads to the following stochastic volatility model

$$\begin{array}{rcl} dS_t &=& \mu S_t dt + \sigma_t S_t dB(t) \\ \log\sigma_t &=& \beta + \frac{k}{\delta}\left\{B_H(t) - B_H(t-\delta)\right\}. \end{array} \tag{2}$$

This fractional volatility model (FVM) is the minimal model that is consistent both with the mathematical hypothesis H1 and H2 and the scaling properties of the bulk market data.

$B_H(t)$, fractional Brownian motion, is a stochastic process with a continuous version, stationary increments, finite variance and covariance

$$\mathbb{E}\left[B_H(t)\,B_H(s)\right] = \frac{1}{2}\left\{t^{2H} + s^{2H} - |t-s|^{2H}\right\}.$$

$B_{\frac{1}{2}}(t) = B(t)$, the Brownian motion.

In (2) δ is the observation time scale (one day, for daily data). As stated above, in the model, the *volatility is not driven by fractional Brownian motion but by fractional noise*. For the volatility (at resolution δ)

$$\sigma(t) = \theta e^{\frac{k}{\delta}\left\{B_H(t)-B_H(t-\delta)\right\}-\frac{1}{2}\left(\frac{k}{\delta}\right)^2\delta^{2H}}, \tag{3}$$

the term $-\frac{1}{2}\left(\frac{k}{\delta}\right)^2\delta^{2H}$ insuring that $E(\sigma(t)) = \theta$. In (2) the constant k measures the strength of the volatility randomness. In the $\delta \to 0$ limit the driving process would be the distribution-valued process W_H

$$W_H = \lim_{\delta\to 0}\frac{1}{\delta}\left(B_H(t) - B_H(t-\delta)\right). \tag{4}$$

Explicit expressions for the distribution of price returns were obtained [2] and one interesting feature was the fact that, once the parameters were obtained from daily data in one market, then the model was also consistent with high-frequency data in a different market by simply changing the time scale δ. This seemed somewhat mysterious until comparison with agent based models [10] revealed that in business-as-usual days, the random fluctuations depend more on the limit-order-book price mechanism than on the individual actions of the market players.

Consistency of the FVM with the usual mathematical properties of market models was checked in Ref.[3] where proposition 3.2 proves that the market model (2) is arbitrage-free and propositions 3.3 and 3.4 characterize the conditions under which the market model is incomplete or complete. Completeness requires the two sources of randomness (in $B(t)$ and $B_H(t)$) to be related by an integral representation for $B_H(t)$[1].

3 Rough volatility

Some time ago Gatheral and collaborators [1], working in the context of the Comte-Renault model [7], suggested, by the analysis of the roughness of volatility data, a value $H < \frac{1}{2}$ for the Hurst index. In a fractional Brownian motion process the Hurst index characterizes both the roughness and the correlation or anticorrelation of the process. Hence, if the volatility is driven by fractional Brownian motion, a Hurst index smaller than $\frac{1}{2}$ would seem to contradict the market long-range dependent volatility (volatility clustering) [13] [14]. The way the realized volatility is measured at high frequency has however been criticized by several authors [15] [16] [17] suggesting that the origin of the roughness lies in the microstructure noise [18] rather than on the actual volatility process.

However there is no real contradiction in the framework of the fractional volatility model (2), because here the volatility is driven by fractional noise (fN), not by fractional Brownian motion (fBm). Fig.1 compares a simulated path of 10000 steps of fBm at $H = 0.8$ with the corresponding one-step fractional noise ($B_H(t+1) - B_H(t)$).

One sees that the apparent roughness of the fractional noise mimics fBm

[1]To associate the Brownian and fractional Brownian processes to the same underlying probability space is quite natural in the framework of white noise stochastic analysis [11] [12].

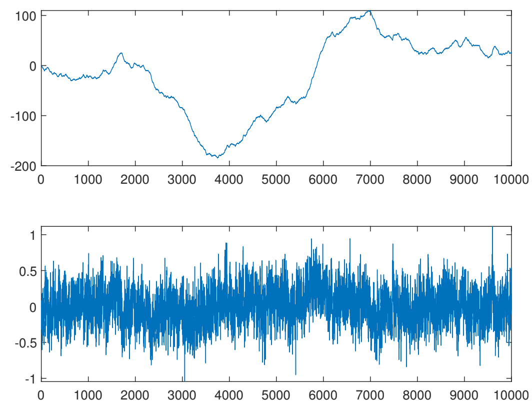

Figure 1: Fractional Brownian motion at $H = 0.8$ and the corresponding one-step fractional noise

at $H \simeq 0.1$. Therefore, using the hypothesis, as in Comte and Renault [7], that it is fBm that drives volatility, one would obtain the wrong Hurst index. What the data analysis performed in [2] implies is that, only when log-volatility is integrated and the linear part extracted, is a self-similar process $R_\sigma(t)$ obtained. Hence, long-range dependence and self-similarity are a property of integrated log-volatility, not of volatility itself.

As a check I have picked up some volatility data [19] and performed the same analysis as in [2]. The data that was analyzed was one-day volatility $\sigma(t)$ for the indexes DAX, Russel 2000, S&P500 and EURO STOXX 50 for the period 20/05/2021 to 22/05/2022 (Fig.2), as well as 6 minute data for S&P500 for the period 24/11/2021 to 24/05/2022.

Fig.3 displays the results for the DAX index.

The upper left panel displays $\log \sigma(t)$. Then, if this quantity were to follow fractional Brownian motion, as some authors have assumed, one would expect

$$\mathbb{E}\left\{(\log \sigma(t+\Delta) - \log \sigma(t))^2\right\} \sim \Delta^{2H}.$$

The upper right panel of Fig.3 clearly suggests that this is a bad hypothesis. In the figure $\langle \bullet \rangle$ stands for the empirical average, an empirical proxy for the expectation value. Next, I have formed the integrated log-volatility and after the extraction of the linear part βt one obtains the process $R(t)$ (in the lower

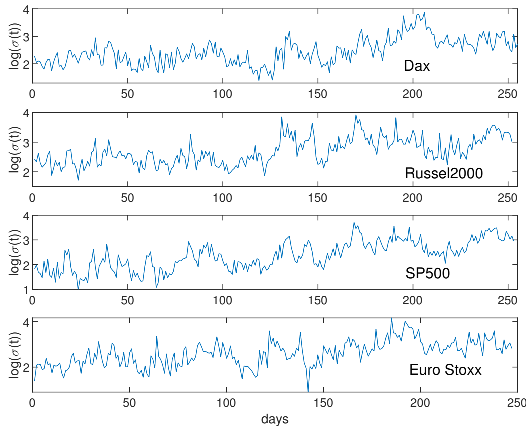

Figure 2: One-day volatility data for 4 indexes (May 2021-May 2022)

left panel)

$$\sum_{n=0}^{t/\delta} \log \sigma (n\delta) = \beta t + R(t) .$$

Computing $\left\langle (R(t+\Delta) - R(t))^2 \right\rangle \approx \mathbb{E}\left\{ (R(t+\Delta) - R(t))^2 \right\}$ one concludes (see the lower right panel of Fig.3) that

$$\mathbb{E}\left\{ (R(t+\Delta) - R(t))^2 \right\} \sim \Delta^{2H},$$

and the identification of $R(t)$ with fractional Brownian motion is a reasonable hypothesis. Hence

$$\log \sigma (t) = \beta + \frac{k}{\delta} \left\{ B_H(t) - B_H(t-\delta) \right\} .$$

The following table shows the values of β and H that are obtained for the indexes that were analyzed

	H	β
SP500_1d	0.85	2.35
SP500_6min	0.86	2.77
Russel2000_1d	0.84	2.68
Euro_Stoxx_1d	0.84	2.59
DAX_1d	0.88	2.40

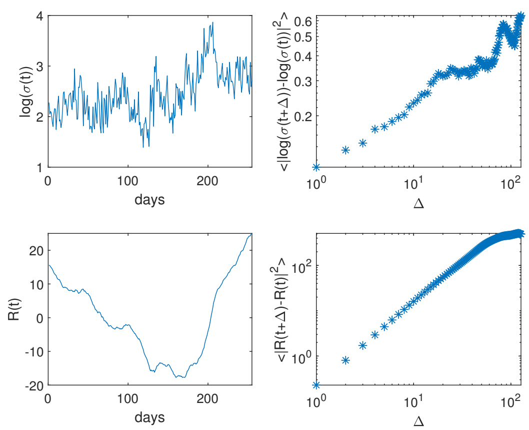

Figure 3: The fractional volatility analysis for the DAX index

Fig.4 compares the DAX volatility data with a simulated sample path of the fractional volatility model (FVM) with the H and β values listed in the table. The model, having the same statistical properties as the data, might be said to provide a "perfect simulation" [20] of the data. However it must be pointed out, that perfect simulation in the statistical sense only means the same statistical properties, it is not "perfect forecasting". An attempt was made in the past [21] to use the FVM to forecast volatility. The main conclusion was that, although having the good statistical properties, FVM was not optimal as a forecasting device, because the market seemed to have, in addition to the stochastic terms in (2), a deterministic mean-reverting component, for example

$$\log \sigma(t) = \beta e^{\alpha(\beta - \log \sigma(t))t} + \frac{k}{\delta} \left\{ B_H(t) - B_H(t-\delta) \right\}.$$

Such mean-reverting effect is in fact suggested by a close examination of the data plots and their comparison with the sample paths of the FVM.

Finally, to the question of whether volatility is rough, the answer is yes, but only because it is driven by fractional noise with $H > \frac{1}{2}$, not by fractional Brownian motion with $H < \frac{1}{2}$.

One of the motivations for the development of reliable volatility models lies on the problem of option pricing and on corrections to Black-Scholes. In particular the rough volatility framework has also been used for option pricing [4]. In [2], using a simple-minded extension of the Hull-White [5] reasoning, an approximate formula was already obtained with features similar

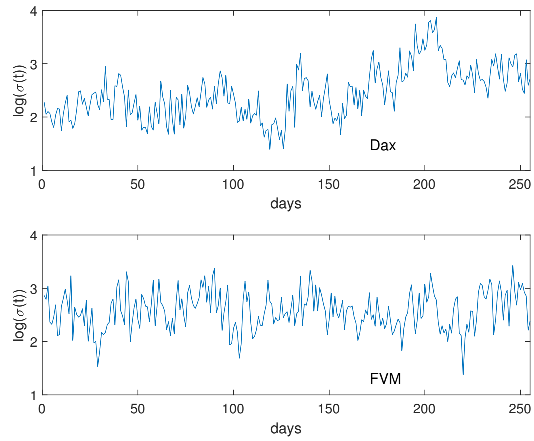

Figure 4: DAX volatility versus a corresponding fractional volatility model sample path

to those observed in the options data. Here a more rigorous derivation will be made, based on the mathematical framework that has been developed for the stochastic analysis of fractional processes.

4 Option pricing

A short historical note concerning the use of fractional processes as a tool for modeling in Finance: The first such suggestion goes back to Mandelbrot [22]. However, because it was pointed out [23] that markets based on $B_H(t)$ could have arbitrage, fractional Brownian motion (fBm) was not considered as a promising tool for mathematical modeling in Finance. The arbitrage result in [23] is a consequence of using pathwise integration. With a different definition [11],

$$\int_a^b f(t,\omega)\, dB_H(t) = \lim_{|\Delta|\to 0} \sum_k f(t_k,\omega) \diamond (B_H(t_{k+1}) - B_H(t_k)),$$

where $\Delta : a = t_0 < t_1 < \cdots < t_n = b$ is a partition of the interval $[a,b]$, $|\Delta| = \max_{0\leq k\leq n-1}(t_{k+1} - t_k)$ and $\diamond$ denotes the Wick product, the integral has zero expectation value and the arbitrage result is no longer true. This is, in fact, the most natural definition because it is the Wick product that is associated to integrals of Itô type, whereas the usual product is natural for integrals of Stratonovich type. An essentially equivalent approach constructs the stochastic integral through the divergence operator and Malliavin calculus [24]. Nevertheless, if fBm is included in the log return process [25] [26] it would contradict the empirical short autocorrelation of this process. The conclusion is that fractional processes might only be relevant to drive the volatility, as used here in the FVM, but not the log-price itself.

A fully consistent stochastic calculus has since been developed for fractional Brownian motion [11] [12] [24] [25] [27] [28] [29] [30] and this is the setting that will be used here to derive an option pricing equation.

Because volatility is not a tradable security, a pure arbitrage argument cannot completely determine the fair price of an option. On the other hand, because of the fractional nature of the volatility process, volatility follows a stochastic process different from the one of the underlying security. Therefore, we cannot apply the reasoning [31] that leads to uniform coefficients of the form $(\mu_i - \lambda_i \sigma_i)$ in the first derivative terms of the option pricing equation[2]. Hence, a first principles derivation, with clearly specified assumptions is required.

As in Black-Scholes [32] [33] form a portfolio,

$$\Pi(t) = V(S, \sigma, t) - \Delta(S, \sigma, t) S_t. \tag{5}$$

$V(S, \sigma, t)$ being the price of an European call option and $\Delta(S, \sigma, t)$ the number of units of stock. To compute the stochastic differential of $\Pi(t)$ one uses the Itô formula for the price process (S_t) and the fractional Itô formula [11] for the fractional processes. Namely, if $dX_t = c(t, \omega) dB_H(t)$, then

$$df(t, X_t) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial X} dX_t + \frac{\partial^2 f}{\partial X^2} c(t, \omega) D_t^{\phi}(X_t) dt, \tag{6}$$

$D_t^{\phi}(X_t)$ being the $\phi-$Malliavin derivative

$$D_s^{\phi}(X_t) = \int_0^t D_s^{\phi} c(u, \omega) dB_H(u) + \int_0^t c(u, \omega) \phi(s, u) du \tag{7}$$

where $\phi(s, u)$ is the kernel

$$\phi(s, u) = H(2H - 1) |s - u|^{2H-2} \tag{8}$$

for $\frac{1}{2} < H < 1$.

[2] μ_i, σ_i and λ_i would be the drift, volatility and market price of risk for each process

Then from (2), choosing $\Delta(S,\sigma,t) = \frac{\partial V}{\partial S}$, one obtains

$$\begin{aligned} d\Pi(t) &= \left\{ \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 \right\} dt + \frac{\partial V}{\partial(\log\sigma)}\frac{k}{\delta}\left(dB_H(t) - dB_H(t-\delta)\right) \\ &\quad + \frac{\partial^2 V}{\partial(\log\sigma)^2}\frac{k^2}{\delta^2} D_t^{\phi}\left(B_H(t) - B_H(t-\delta)\right) dt. \end{aligned}$$

To compute the Malliavin derivative only the second term in the r.h.s. of (7) intervenes and one obtains

$$\begin{aligned} D_t^{\phi}\left(B_H(t) - B_H(t-\delta)\right) &= H(2H-1)\int_{t-\delta}^{t} |t-u|^{2H-2}\, du \\ &= H\delta^{2H-1} \end{aligned}$$

yielding

$$\begin{aligned} d\Pi(t) &= \left\{ \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 \right\} dt + \sigma\frac{\partial V}{\partial \sigma}\frac{k}{\delta}\left(dB_H(t) - dB_H(t-\delta)\right) \\ &\quad + \left(\sigma^2\frac{\partial^2 V}{\partial \sigma^2} + \sigma\frac{\partial V}{\partial \sigma}\right)\frac{k^2}{\delta^2} H\delta^{2H-1} dt. \end{aligned} \tag{9}$$

In (9) one is still left with the stochastic term $\sigma\frac{\partial V}{\partial \sigma}\frac{k}{\delta}\left(dB_H(t) - dB_H(t-\delta)\right)$ and, because volatility is not a tradable security, this term cannot be eliminated by a portfolio choice. Instead one may assume as reasonable to equate the deterministic term in $d\Pi(t)$ to

$$\left(r\Pi(t) + \nu\frac{k}{\delta}\sigma\frac{\partial V}{\partial \sigma}\right) dt,$$

where r is the risk-free return and, with $\nu > 0$, the second term is a measure of the *market price of volatility risk.* One ends up with

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + \frac{k}{\delta}\left(kH\delta^{2H-2} - \nu\right)\sigma\frac{\partial V}{\partial \sigma} + Hk^2\delta^{2H-3}\sigma^2\frac{\partial^2 V}{\partial \sigma^2} = rV \tag{10}$$

as the general form for an option pricing equation consistent with the stochastic volatility model in (2).

One now obtains an integral representation for the solution of this equation with the change of variable

$$x = \log\frac{S}{K}, \tag{11}$$

K being the strike price. Using the two-dimensional Fourier transform

$$V(t,x,\sigma) = \int\int F(\phi,\rho,\sigma)\, e^{i(\phi t+\rho x)} d\phi d\rho, \tag{12}$$

$$Hk^2\delta^{2H-3}\sigma^2\frac{\partial^2 F}{\partial\sigma^2}+\frac{k}{\delta}\left(kH\delta^{2H-2}-\nu\right)\sigma\frac{\partial F}{\partial\sigma}+\left(i\left(\phi+\rho r-\frac{\sigma^2\rho}{2}\right)-\frac{\sigma^2\rho^2}{2}-r\right)F=0. \tag{13}$$

Defining new constants

$$\begin{array}{lcl} \chi(\rho) & = & \frac{\nu}{2Hk\delta^{2H-2}} \\ \xi^2(\rho,\phi) & = & \chi^2(\rho)-\frac{r-i(\phi+\rho r)}{Hk^2\delta^{2H-3}} \\ \zeta^2(\rho) & = & -\frac{i\rho+\rho^2}{2Hk^2\delta^{2H-3}} \end{array} \tag{14}$$

and making the replacement

$$F(\sigma) = \sigma^{\chi} Z_{\xi}(\zeta\sigma), \tag{15}$$

Eq.(13) reduces to a standard Bessel equation. Therefore the solution of (10) is

$$V(t,x,\sigma) = \int\int e^{i(\phi t+\rho x)}\sigma^{\chi(\rho)} Z_{\xi(\rho,\phi)}(\zeta(\rho)\sigma)\, d\phi d\rho, \tag{16}$$

$Z_{\xi}(\zeta\sigma)$ being a Bessel function. The Bessel function will be a linear combination

$$Z_{\xi}(\zeta\sigma) = c_1 J_{\xi}(\zeta\sigma) + c_2 N_{\xi}(\zeta\sigma)$$

of a Bessel function of first kind and a Neumann function, with coefficients c_1 and c_2 to be fixed by the boundary condition, which for call options is

$$V(T,x,\sigma) = \max(e^x - 1, 0)$$

Eq.(16) is an exact solution to the option pricing equation (10).

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