J. Phys. A: Math. Gen. 27 (1994) 8091–8104. Printed in the UK

Deformations, stable theories and fundamental constants

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Received 4 February 1994

Abstract. Models or theories that are stable, in the sense that they do not change in a qualitative manner under a small change of parameters, have a higher probability of having a wider range of validity. This also seems to be true for the fundamental theories of nature. Using the deformation theory of algebras, we review the stabilizing deformations leading from non-relativistic to relativistic and from classical to quantum mechanics. Unlike previous treatments, both deformations are carried out on a finite-dimensional algebra setting. One then finds that the resulting relativistic quantum algebra is itself unstable and admits a two-parameter stabilizing deformation. Taking into account reasonable physical constraints to identify the deformed variables, a new algebra is then proposed as the stable algebra of relativistic quantum mechanics in tangent space. This is isomorphic to the algebra of the pseudo-Euclidian group in five dimensions.

1. Deformations and stable theories

When, in the course of development of physical science, models are constructed for the natural world, it is reasonable to expect that only the robust properties of the models have a chance of being reproduced in the observed phenomena. Models are mere approximations of the natural world and it is highly unlikely that properties that are too sensitive to small changes in the model (i.e. that depend in a critical manner on particular values of the parameters) will ever be observed. Alternatively, if a fine tuning is needed to reproduce some natural phenomenon, then it is certain that the model is basically unsound and its other predictions are unreliable. It is therefore a good methodological approach to concentrate on the robust properties of the models or, equivalently, on models which are stable, in the sense that they do not change in a qualitative manner, when some parameter changes.

The stable-model point of view has come a long way in the field of non-linear dynamics, where it led to the rigorous notion of structural stability [1]. As emphasized by Flato [2] and Faddeev [3], the same pattern seems to occur in the fundamental theories of nature. In fact, the two physical revolutions of this century, namely the passage from non-relativistic to relativistic and from classical to quantum mechanics, may be interpreted as the transition from two unstable theories to two robust stable theories.

In general, a mathematical structure is said to be stable (or rigid†) for a class of deformations if any deformation in this class leads to an equivalent (isomorphic) structure. The idea of structures stability provides a guiding principle to test either the validity or the need for generalization of a physical theory. Namely, if the mathematical structure of a given

† For physical applications, it seems more natural to call these structures stable structures, however, in the mathematical literature it is always the term rigid that is used. In this paper, the two terms are used with equivalent meaning.

0305-4470/94/248091+14$19.50

theory is not stable, then one should try to deform it until it falls into a stable structure, which has a good chance of being a generalization of wider validity. The mathematical theory of deformations developed along several lines, the most developed being the deformations of analytic structures, of algebraic manifolds and of algebras. In all cases, the co-homology groups play a key role in characterizing the stability of the structures.

In physics, it is the theory of deformations of Lie algebras that has, so far, played the major role, although the deformations of other mathematical structures are potentially as useful as the deformations of algebras. For physics, it is useful to have an explicit representation of the deformation parameters, which play the role of fundamental constants in the deformed stable theories. I will therefore concentrate on the theory of formal deformations of Lie algebras [4–6]. A formal deformation of a Lie algebra $\mathcal{L}_0$, defined on a vector space V over a field K, is an algebra $\mathcal{L}_t$ on the space $V \otimes K[t]$ (where $K[t]$ is the field of formal power series), defined by

$$[A, B]_t = [A, B]_0 + \sum_{i=1}^{\infty} \phi_i(A, B) t^i \tag{1.1}$$

with $A, B, \phi_i(A, B) \in V$ and $t \in K$. The adjoint representation of $\mathcal{L}_0$ is

$$\rho(A)(B) = [A, B]_0. \tag{1.2}$$

An n-co-chain (relative to the adjoint representation) is a bilinear skew-symmetric mapping

$$V \times \cdots \times V \to V$$

and the n-co-chains form a vector space $C^n(\rho, V)$. In particular, $\phi_i(A, B)$ in equation (1.1) must be a 2-co-chain.

One also defines the following.

(i) The co-boundary operator

$$\begin{aligned} \mathrm{d}\phi(A_1, \ldots, A_{n+1}) &= \sum_{i=1}^{n+1} (-1)^{i-1} \rho(A_i) \phi(A_1, \ldots, \hat{A}_i, \ldots, A_{n+1}) \\ &\quad + \sum_{1 \leqslant i < j \leqslant n+1} (-1)^{i+j} \phi([A_i, A_j], A_1, \ldots, \hat{A}_i, \ldots, \hat{A}_j, \ldots, A_{n+1}). \end{aligned} \tag{1.3}$$

(ii) A co-cycle $\phi \in C^n(\rho, V)$ whenever $\mathrm{d}\phi = 0$. The set of all n-co-cycles is a vector space denoted $Z^n(\rho)$.

(iii) A co-boundary if $\phi \in \mathrm{d}(C^{n-1}(\rho, V))$. The set of all co-boundaries is a vector space denoted $B^n(\rho)$.

(iv) The quotient space

$$H^n(\rho) = \frac{Z^n(\rho)}{B^n(\rho)}$$

is the n-co-homology group (relative to the ρ-representation). From (1.3), it follows that $\mathrm{d}^2\phi = 0$. However, not all co-cycles need to be co-boundaries and the n-co-homology groups may be non-trivial.

The relevance of these concepts to the deformation problem formulated in equation (1.1) is as follows.

Using the deformed commutation relations (1.1) and differentiating the Jacobi identity

$$[A,[B,C]_t]_t+[B,[C,A]_t]_t+[C,[A,B]_t]_t=0 \tag{1.5}$$

in variable t and then setting $t=0$, one obtains

$$\mathrm{d}\phi_1(A,B,C)=0$$

that is, for the deformation in (1.1) to be a Lie algebra, ϕ_1 must be a 2-co-cycle.

A deformation of $\mathcal{L}_0$ is said to be *trivial* if the algebra $\mathcal{L}_t$ is isomorphic to $\mathcal{L}_0$. This means that there is an invertible linear transformation $T_t : V \to V$ such that

$$T_t([A,B]_t)=[T_tA,T_tB]_0. \tag{1.6}$$

If all deformations $\mathcal{L}_t$ are isomorphic to $\mathcal{L}_0$, then $\mathcal{L}_0$ is said to be stable or rigid. Suppose now that the second co-homology $H^2(\rho)$ is trivial. This means that all 2-co-cycles are 2-co-boundaries. Then there must be a 1-co-chain γ such that $\phi_1=\mathrm{d}\gamma$. Use the linear transformation $M_t'=\exp\{-t\gamma\}$ to transform the algebra $\mathcal{L}_t$

$$[A,B]_t'=M_t'^{-1}([M_t'A,M_t'B]).$$

From $\phi_1=\mathrm{d}\gamma$, one now obtains, by a simple calculation [7]

$$\phi_1'(A,B)=\phi_1(A,B)-[\gamma(A),B]-[A,\gamma(B)]+\gamma([A,B])=0.$$

Therefore, the power series expansion for $[A,B]_t'$ begins with terms of second order in t

$$[A,B]_t'=[A,B]_0+\phi_2'(A,B)t^2+\cdots$$

and from the Jacobi identity, as above, it follows that $\mathrm{d}\phi_2'(A,B)=0$.

By iterating the whole process, all powers of t are successively eliminated. This means that the limit

$$T_t=M_t'M_t''\cdots$$

is the transformation that establishes the equivalence of $\mathcal{L}_t$ and $\mathcal{L}_0$. In conclusion, the vanishing of the second co-homology group is a sufficient condition for the non-existence of non-trivial deformations, i.e. it is a sufficient condition for the stability (or rigidity) of the Lie algebra. This is the content of the 'rigidity theorem' of Nijenhuis and Richardson [5].

There is a nice geometrical interpretation of the role of co-cycles and co-boundaries in the rigidity of Lie algebra structures. The set L^n of all possible n-dimensional Lie algebras is an algebraic manifold embedded in $\mathbb{C}^N$ (with $N=(n^3-n^2)/2$), the defining algebraic relations being the Jacobi identity equations between the structure constants. Also, the natural topology in L^n is the topology induced by the structure constants. Isomorphism relation (1.6) is an action of the linear group $\mathrm{GL}(n,\mathbb{C})$

$$L^n\times \mathrm{GL}(n,\mathbb{C})\to L^n\colon (\mathcal{L},T)\to T^{-1}\circ\mathcal{L}\circ T\times T \tag{1.7}$$

where $\mathcal{L}\in L^n$ denotes the Lie algebra law.

Denoting $\mathcal{L}_0(A, B) \equiv [A, B]_0$, $\mathcal{L}_0$ will be a rigid algebra if its orbit $O(\mathcal{L}_0)$ under $GL(n, \mathbb{C})$ is open. Every vector in the tangent cone to L^n at $\mathcal{L}_0$ is in the 2-co-cycle space $Z^2(\rho)$ and the tangent space to the orbit $O(\mathcal{L}_0)$ at $\mathcal{L}_0$ is $B^2(\rho)$.

The rigidity theorem of Nijenhuis and Richardson establishes only a sufficient, and not a necessary, condition for stability. Semi-simple Lie algebras, for example, have a vanishing second co-homology group [8] and are stable. However, whenever there are non-trivial 2-co-cycles, these may still not be the infinitesimals of a deformation, i.e. they may not be integrable. Primary obstructions to integrability are to be found in the structure of the third co-homology group. Examples of rigid algebras with non-vanishing second co-homology group [9–11] were constructed and this fact led to the development of different non-co-homological techniques for classifying rigid Lie algebras [11–15].

Here an important simplifying role is played by the techniques of non-standard analysis. A Lie algebra law $\mathcal{L}_0$ is then said to be rigid if any perturbation $\mathcal{L}$ is isomorphic to $\mathcal{L}_0$. A perturbation of $\mathcal{L}_0$ is an algebra $\mathcal{L}$ such that

$$\mathcal{L}(A, B) \sim \mathcal{L}_0(A, B) \tag{1.8}$$

for A, B standard or limited. The symbol $\sim$ means infinitesimally close.

The decomposition of any perturbation of $\mathcal{L}_0$ is as follows

$$\mathcal{L} = \mathcal{L}_0 + \epsilon_1\phi_1 + \epsilon_1\epsilon_2\phi_2 + \cdots + \epsilon_1\epsilon_2\cdots\epsilon_k\phi_k \tag{1.9}$$

and is unique up to equivalence, where ϕ are standard antisymmetric bilinear mappings, ϵ are non-zero infinitesimals and $k \leqslant N$.

The most useful result for the characterization of rigid Lie algebras is the theorem that states that if $\mathcal{L}_0$ is rigid, there is a standard non-zero vector X such that $\mathrm{ad}_{\mathcal{L}_0} X$ ($\mathrm{ad}_{\mathcal{L}_0} X(Y) = [X, Y]$) is diagonalizable. The converse result is not true and to classify the rigid algebras in dimension n one still has to exclude the non-rigid algebras with a diagonalizable vector. A large number are simply excluded by checking the rank of the root system and, for the rest (which are a finite number), one has to check explicitly the isomorphism of the perturbation. This method allows, in principle, the classification of all rigid algebras in any dimension. For details I refer the reader to [12, 14, 15].

I will now review, briefly, the way in which deformation theory interprets the passage from non-relativistic to relativistic and from classical to quantum mechanics as stabilizing deformations of two unstable theories.

The Lie algebra of the homogeneous Galileo group, the kinematical group of non-relativistic quantum mechanics, is:

$$[J_i, J_j] = \mathrm{i}\epsilon_{ijk}J_k \tag{1.10a}$$

$$[J_i, K_j] = \mathrm{i}\epsilon_{ijk}K_k \tag{1.10b}$$

$$[K_i, K_j] = 0. \tag{1.10c}$$

The second co-homology group does not vanish because, for example, $\phi_1(K_i, K_j) = \mathrm{i}\epsilon_{ijk}J_k$, and $\phi_1 = 0$ for all other arguments, is a 2-co-cycle that is not a 2-co-boundary. In fact, the deformation

$$[K_i, K_j] = -\mathrm{i}\frac{1}{c^2}\epsilon_{ijk}J_k \tag{1.10d}$$

leads to the Lorentz algebra which, being semi-simple, has a vanishing second co-homology group and is stable.

For the deformation leading from classical to quantum mechanics, recall that the phase space of classical mechanics is a symplectic manifold $W = (T^*M, \omega)$ where T^*M is the co-tangent bundle over configuration space M and ω is a symplectic form. In local (Darboux) coordinates $\{p_i, q_i\}$, the symplectic form is

$$\mathrm{d}\omega = \sum \mathrm{d}p_i \wedge \mathrm{d}q_i .$$

The Poisson bracket gives a Lie algebra structure to the C^∞-functions on W

$$\{f, g\} = \sum_i \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \tag{1.11}$$

in local coordinates.

The transition to quantum mechanics is now regarded as a deformation of this Poisson algebra [16]. Let, for example, $T^*M = \mathbb{R}^{2n}$. Then

$$\begin{aligned}\omega &= \sum_{1\leqslant i,j\leqslant 2n} \omega_{ij}\mathrm{d}x^i \wedge \mathrm{d}x^j \\ &= \sum_{1\leqslant i\leqslant n} \mathrm{d}x^i \wedge \mathrm{d}x^{i+n}.\end{aligned}$$

Consider the following bidifferential operator:

$$P^r(f, g) = \sum_{\substack{i_1\cdots i_r \\ j_1\cdots j_r}} \omega^{i_1 j_1}\cdots\omega^{i_r j_r}\partial_{i_1}\cdots\partial_{i_r}(f)\partial_{j_1}\cdots\partial_{j_r}(g). \tag{1.12}$$

$P^1(f, g)$ is simply the Poisson bracket. $P^3(f, g)$ is a non-trivial 2-co-cycle and, barring obstructions, one expects the existence of non-trivial deformations of the Poisson algebra.

Existence of non-trivial deformations has indeed been proved in a very general context [17–20]. Non-trivial deformations always exist if W is finite-dimensional and, for a flat Poisson manifold, they are all equivalent to the Moyal [21] bracket

$$[f, g]_\mathrm{M} = \frac{2}{\hbar}\sin\left(\frac{\hbar}{2}P\right)(f, g) = \{f, g\} - \frac{\hbar^2}{4.3!}P^3(f, g) + \cdots. \tag{1.13}$$

Moreover $[f, g]_\mathrm{M} = \frac{1}{\mathrm{i}\hbar}(f *_\hbar g - g *_\hbar f)$ where $f *_\hbar g$ is an associative star-product

$$f *_\hbar g = \exp\left(\mathrm{i}\frac{\hbar}{2}P(f, g)\right). \tag{1.14}$$

Correspondence with quantum mechanics formulated in Hilbert space is obtained by the Weyl quantization prescription. Let $f(p, q)$ be a function in phase space and $\tilde{f}$ its Fourier transform. Then, if we associate the Hilbert space operator

$$\Omega(f) = \int \tilde{f}(x_i, y_i)\exp\left(-\mathrm{i}\frac{\sum x_i Q_i + y_i P_i}{\hbar}\right)\mathrm{d}x_i\,\mathrm{d}y_i$$

where $Q_i\psi = x_i\psi$ and $P_i = -i\hbar\frac{\partial}{\partial x_i}\psi$, to the function f, one finds

$$[\Omega(f), \Omega(g)] = -i\hbar\Omega([f, g]_M)$$

with, on the left-hand side, the usual commutator of Hilbert space operators. Therefore, quantum mechanics may be described either by associating self-adjoint operators in Hilbert space to the observables or, equivalently, remaining in the classical setting of phase-space functions but deforming their product to a $*_\hbar$-product and their Poisson brackets to Moyal brackets.

In both the Galileo and the Poisson algebra cases, the deformed algebras are all equivalent for non-zero values of $\frac{1}{c^2}$ and $\hbar$. This means that although we could have derived relativistic and quantum mechanics purely from considerations of the stability of their algebras, the exact values of the deformation parameters cannot be obtained from algebraic considerations. The deformation parameters are therefore the natural fundamental constants to be obtained from experiment. In this sense, deformation theory is not only the theory of stable theories, it is also the theory that identifies the fundamental constants.

There is a basic difference in the deformations leading from non-relativistic to relativistic and from classical to quantum mechanics. In the first case, one deals with the deformation of a finite-dimensional algebra and, in the second, with the more complex case of the deformation of an infinite-dimensional algebra of functions. With the benefit of hindsight, one may now simplify the presentation by using, for classical mechanics, instead of the Poisson algebra in phase space, a formulation in Hilbert space. Then, the transition appears in both cases as a simple deformation of finite-dimensional Lie algebras. This not only simplifies the presentation but is the appropriate setting for further analysis of the stability of relativistic quantum mechanics. This is the subject of section 2.

2. The stable finite-dimensional Lie algebra of relativistic quantum mechanics

A description of classical mechanics by operators in Hilbert space was proposed, soon after the discovery of quantum mechanics, by Koopman [22] and von Neumann [23]. A constant energy surface Ω_E in the phase space of N particles carries an invariant measure μ_E, which is the restriction of the Liouville measure $d^{3N}x\, d^{3N}p$ to Ω_E. In the space of square-integrable functions $L^2(\Omega_E, \mu_E)$, the Hamiltonian flow T_t induces a unitary operator by

$$(U_t f)(w) = f(T_t w) \tag{2.1}$$

where $w \in \Omega_E$ and $f \in L^2(\Omega_E, \mu_E)$. Unitarity is a consequence of the invariance of the measure $\mu(T_t^{-1}F) = \mu(F)$, for a measurable set $F \in \Omega_E$.

In the Hilbert space $L^2(\Omega_E, \mu_E)$, classical mechanics has an operator formulation. The time evolution is induced by a unitary operator U_t, as in quantum mechanics, and the observables are the smooth functions on Ω_E which act as multiplicative operators in $L^2(\Omega_E, \mu_E)$.

Considered as multiplicative operators in Hilbert space, the functions of coordinates and momenta are an infinite-dimensional Abelian algebra. However, in the Hilbert space formulation, we need not consider explicitly the infinite-dimensional algebra because the full content of the theory is obtained by selecting a finite set of paired observables (p_i, x_i)

and defining its transformation properties under U_t and its algebraic properties which, in classical mechanics, are

$$[p_i, x_j] = [p_i, p_j] = [x_i, x_j] = 0. \tag{2.2}$$

The transition to quantum mechanics is now effected by the replacement of this Abelian algebra by the Heisenberg algebra

$$[p_i, p_i] = [x_i, x_i] = 0 \tag{2.3a}$$

$$[x_i, p_i] = \mathrm{i}\hbar\mathcal{I} \tag{2.3b}$$

where $\mathcal{I}$ is the identity operator, a trivial centre of the algebra of observables.

The infinite-dimensional Moyal algebra is therefore replaced by the simpler finite-dimensional Heisenberg algebra. The role of this Heisenberg algebra, in the context of deformation theory, has however to be discussed carefully. Consider the one-dimensional case of a classical Abelian algebra $[x, p] = 0$. This Abelian algebra is clearly not stable and in its neighbourhood there is the algebra

$$[x, p] = \mathrm{i}\epsilon x \tag{2.4}$$

or the Heisenberg algebra

$$[x, p] = \mathrm{i}\hbar\mathcal{I} \tag{2.5}$$

which is the central extension of the Abelian algebra.

(2.4) is a stable algebra. Indeed, the only stable algebra in two dimensions is isomorphic to [15]

$$[Y, X_1] = X_1 \tag{2.6}$$

however, the Heisenberg algebra itself is not stable.

There are two ways of looking at the instability of the Heisenberg algebra. First, if we consider it as a tridimensional algebra $[X_2, X_3] = X_1$ (all the other commutators being zero), the complete structure of its neighbourhood, in the space of Lie-algebra laws, is known [14]. Namely, the Heisenberg algebra is a contraction of any algebra of the same dimension that carries a linear contact form. Conversely, any perturbation of the Heisenberg algebra supports a linear contact form. For example, from the Lie algebra of SO(3)

$$[X_1, X_2] = X_3 \qquad [X_2, X_3] = X_1 \qquad [X_3, X_1] = X_2$$

which is semi-simple and therefore stable, with the following linear change of coordinates

$$Y_1 = \epsilon X_1 \qquad Y_2 = \sqrt{\epsilon} X_2 \qquad Y_3 = \sqrt{\epsilon} X_3$$

one obtains

$$[Y_1, Y_2] = \epsilon Y_3 \qquad [Y_2, Y_3] = Y_1 \qquad [Y_3, Y_1] = \epsilon Y_2$$

and in the $\epsilon \to 0$ limit one obtains the Heisenberg algebra.

We could also have considered the Heisenberg algebra as a two-dimensional algebra with a trivial centre. That is, we restrict our deformations to those that preserve the zero commutator of X_1 with the other two elements. Consider in this case the deformation

$$[X_2, X_3] = X_1 + \alpha X_2 + \beta X_3.$$

With the linear change of variables

$$Y_2 = \alpha X_2 + X_1 + \beta X_3 \qquad Y_3 = \alpha^{-1} X_3$$

we now fall on the stable two-dimensional algebra (2.6) $[Y_2, Y_3] = Y_2$.

We conclude in both cases that the Heisenberg algebra is unstable and has a stable algebra in its neighbourhood. Therefore, it would seem, at first sight, that the Hilbert-space construction leads to conclusions different from the phase-space construction described in section 1, which interprets the transition from classical to quantum mechanics as a deformation from an unstable Poisson algebra to a stable Moyal–Vey algebra. A simple reasoning shows, however, that this is not the case and that the constructions are indeed equivalent and are both the transition from an unstable classical algebra to a stable quantum algebra. The apparent difference is merely an artefact of the singling out of x as the observable, when in fact the observables are all smooth functions of x (and p). Consider the explicit representation

$$p = \frac{\hbar}{\mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}x} \qquad x = x.$$

The physical content of the theory will be the same if, instead of the coordinate x, we consider any linear or non-linear function of x. In particular, considering $y = \exp(\mathrm{i}x)$, one obtains the algebra

$$[p, y] = \hbar y$$

which is isomorphic to the stable two-dimensional algebra (2.3). Hence, the Heisenberg algebra is equivalent, through a non-linear coordinate transformation that preserves the physical content, to a stable algebra. In this sense, the transition from classical to quantum mechanics is again seen to be a stabilizing deformation of an unstable algebra. The main reason why the coordinate choice leading to the Heisenberg algebra is physically convenient is that the observable p then has a simple interpretation as the generator of translations in x. This example also shows that, when selecting a finite subset of observables rather than an infinite-dimensional space of functions, the notion of linear equivalence of algebras, in the sense of equation (1.6), is not sufficient for the stability analysis and one should also consider non-linear transformations that preserve the physical content of the theory.

The transitions from non-relativistic to relativistic and from classical to quantum mechanics have thus been cast as deformations of a finite-dimensional Lie algebra of operators in Hilbert space. A trivial point in this construction, which, however, has non-trivial consequences, is the fact that, to have both these constructions in a finite-dimensional algebra setting, it is essential to include the coordinates as basic operators in the defining (kinematical) algebra of relativistic quantum mechanics. The full algebra of relativistic quantum mechanics will be the Poincaré algebra (2.7a, b, d), the Heisenberg algebra for the momenta and spacetime coordinates (P_μ, x_ν) in Minkowski space together with the commutators that define the vector nature (under the Lorentz group) of the P_μ and x_ν. Defining

$$M_{ij} = \epsilon_{ijk} J_k \qquad M_{0i} = K_i$$

and measuring velocities and actions in units of c and $\hbar$ (that is $c = \hbar = 1$), one obtains

$$[M_{\mu\nu}, M_{\rho\sigma}] = \mathrm{i}(M_{\mu\sigma} g_{\nu\rho} + M_{\nu\rho} g_{\mu\sigma} - M_{\nu\sigma} g_{\mu\rho} - M_{\mu\rho} g_{\nu\sigma}) \tag{2.7a}$$

$$[M_{\mu\nu}, P_\lambda] = \mathrm{i}(P_\mu g_{\nu\lambda} - P_\nu g_{\mu\lambda}) \tag{2.7b}$$

$$[M_{\mu\nu}, x_\lambda] = \mathrm{i}(x_\mu g_{\nu\lambda} - x_\nu g_{\mu\lambda}) \tag{2.7c}$$

$$[P_\mu, P_\nu] = 0 \tag{2.7d}$$

$$[x_\mu, x_\nu] = 0 \tag{2.7e}$$

$$[P_\mu, x_\nu] = \mathrm{i} g_{\mu\nu} \mathcal{I}. \tag{2.7f}$$

We know that the Lorentz algebra, being semi-simple, is stable and that each one of the two-dimensional Heisenberg algebras (P_μ, x_ν) is also stable in the nonlinear sense discussed above. When the algebras are combined through covariance commutators (2.7*b*) and (2.7*c*), the natural question to ask is whether the whole algebra is stable or whether there are any non-trivial deformations.

Actually the algebra $\Re_0 = \{M_{\mu\nu}, P_\mu, x_\mu, \mathcal{I}\}$ defined by equation (2.7) is not stable. This will be shown by exhibiting a two-parameter deformation of $\Re_0$ to a simple algebra which itself is stable. To understand the role of the deformation parameters, consider first the Poincaré subalgebra $\mathcal{P} = \{M_{\mu\nu}, P_\mu\}$. It is well known already that this subalgebra is not stable and may be deformed [2, 24] to the stable simple algebras of the De Sitter groups O(4, 1) or O(3, 2). Writing

$$P_\mu = \frac{1}{R} M_{\mu 4} \tag{2.8}$$

the commutation relations $[M_{\mu\nu}, M_{\rho\sigma}]$ and $[M_{\mu\nu}, P_\lambda]$ are the same as before, that is (2.7*a*) and (2.7*b*), and $[P_\mu, P_\nu]$ becomes

$$[P_\mu, P_\nu] = -\mathrm{i}\frac{\epsilon_4}{R^2} M_{\mu\nu}. \tag{2.9}$$

Equations (2.7*a*), (2.7*b*) and (2.9) together, are the algebra

$$[M_{ab}, M_{cd}] = \mathrm{i}(-M_{bd} g_{ac} - M_{ac} g_{bd} + M_{bc} g_{ad} + M_{ad} g_{bc}) \tag{2.10}$$

of the five-dimensional pseudo-orthogonal group with metric

$$g_{aa} = (1, -1, -1, -1, \epsilon_4) \qquad \epsilon_4 = \pm 1.$$

That is, the Poincaré algebra deforms to the stable algebras of O(3, 2) or O(4, 1), according to the sign of ϵ_4.

This instability of the Poincaré algebra is, however, physically harmless. It simply means that flat space is an isolated point in the set of arbitrarily curved spaces. As long as the Poincaré group is used as the kinematical group of the tangent space to the spacetime manifold, and not as a group of motions in the manifold itself, it is perfectly consistent to take $R \to \infty$ and this deformation goes away.

For the full algebra $\Re_0 = \{M_{\mu\nu}, P_\mu, x_\mu, \mathcal{I}\}$, the situation is more interesting. In this case, the stabilizing deformation is obtained by setting

$$P_\mu = \frac{1}{R} M_{\mu 4} \tag{2.11a}$$

$$x_\mu = \ell M_{\mu 5} \tag{2.11b}$$

$$\mathcal{I} = \frac{\ell}{R} M_{45} \tag{2.11c}$$

to obtain

$$[P_\mu, P_\nu] = -\mathrm{i}\frac{\epsilon_4}{R^2} M_{\mu\nu} \tag{2.12a}$$

$$[x_\mu, x_\nu] = -\mathrm{i}\epsilon_5 \ell^2 M_{\mu\nu} \tag{2.12b}$$

$$[P_\mu, x_\nu] = \mathrm{i} g_{\mu\nu} \mathcal{I} \tag{2.12c}$$

$$[P_\mu, \mathcal{I}] = -\mathrm{i}\frac{\epsilon_4}{R^2} x_\mu \tag{2.12d}$$

$$[x_\mu, \mathcal{I}] = \mathrm{i}\epsilon_5 \ell^2 P_\mu \tag{2.12e}$$

with $[M_{\mu\nu}, M_{\rho\sigma}]$, $[M_{\mu\nu}, P_\lambda]$ and $[M_{\mu\nu}, x_\lambda]$ being the same as before.

The stable algebra $\Re_{\ell,R}$, to which $\Re_0$ as been deformed, is the algebra of the six-dimensional pseudo-orthogonal group with metric

$$g_{aa} = (1, -1, -1, -1, \epsilon_4, \epsilon_5) \qquad \epsilon_4, \epsilon_5 = \pm 1.$$

As in the case of the Poincaré algebra discussed above, if one is mostly concerned with the algebra of observables in the tangent space, one may take the limit $R \to \infty$ and finally obtain

$$[M_{\mu\nu}, M_{\rho\sigma}] = \mathrm{i}(M_{\mu\sigma} g_{\nu\rho} + M_{\nu\rho} g_{\mu\sigma} - M_{\nu\sigma} g_{\mu\rho} - M_{\mu\rho} g_{\nu\sigma}) \tag{2.13a}$$

$$[M_{\mu\nu}, P_\lambda] = \mathrm{i}(P_\mu g_{\nu\lambda} - P_\nu g_{\mu\lambda}) \tag{2.13b}$$

$$[M_{\mu\nu}, x_\lambda] = \mathrm{i}(x_\mu g_{\nu\lambda} - x_\nu g_{\mu\lambda}) \tag{2.13c}$$

$$[P_\mu, P_\nu] = 0 \tag{2.13d}$$

$$[x_\mu, x_\nu] = -\mathrm{i}\epsilon_5 \ell^2 M_{\mu\nu} \tag{2.13e}$$

$$[P_\mu, x_\nu] = \mathrm{i} g_{\mu\nu} \mathcal{I} \tag{2.13f}$$

$$[P_\mu, \mathcal{I}] = 0 \tag{2.13g}$$

$$[x_\mu, \mathcal{I}] = \mathrm{i}\epsilon_5 \ell^2 P_\mu \tag{2.13h}$$

as our candidate for a stable algebra of relativistic quantum mechanics. The main features are the non-commutativity of the x_μ coordinates and the fact that $\mathcal{I}$, previously a trivial centre of the Heisenberg algebra, now becomes a non-trivial operator. These are, however, the minimal changes that seem to be required if the stability of the algebra of observables (in the tangent plane) is to be a good guiding principle. Two fundamental constants define this deformation. One is ℓ, a fundamental length, the other is the sign of ϵ_5.

The idea of modifying the algebra of the spacetime components x_μ in such a way that they become non-commuting operators has already appeared several times in the physical literature. Rather than being motivated (and forced) by stability considerations, the aim of these proposals has been to endow spacetime with a discrete structure, to be able, for example, to construct quantum fields free of ultraviolet divergences. Sometimes they simply postulate a non-zero commutator, at others they are guided by the formulation of field theory in curved spaces. Although the algebra we arrived at, in equations (2.13), is so simple and appears in such a natural way in the context of deformation theory, it seems that, strangely, it differs in some way or another from the past proposals. In one scheme, for example, the coordinates are assumed to be the generators of rotations in a five-dimensional space with constant negative curvature. This possibility was proposed by Snyder [25] and the consequences of formulating field theories in such spaces have been extensively studied by Kadishevsky and collaborators [26, 27]. The commutation relations of the coordinates $[x_\mu, x_\nu]$ are identical to (2.13*e*) however, because of the representation chosen for the momentum operators; the Heisenberg algebra is different and, in particular, $[P_\mu, x_\nu]$ has non-diagonal terms. Banai [28] also proposed a specific non-zero commutator which only operates between time and space coordinates, braking Lorentz invariance. Many other discussions exist concerning the emergence and the role of discrete or quantum spacetime, which, however, in general, do not specify a complete operator algebra [29–42].

Notice that there are other ways to deform algebra $\mathfrak{R}_0$ to the simple algebra of the pseudo-orthogonal group in six dimensions. These correspond to different physical identifications of the generators $M_{\mu 4}$, $M_{\mu 5}$ and M_{45}. For example, putting

$$P_\mu = \frac{1}{R'}(M_{\mu 4} + M_{\mu 5}) \tag{2.14a}$$

$$x_\mu = \frac{\ell'}{2}(M_{\mu 4} - M_{\mu 5}) \tag{2.14b}$$

$$\mathcal{I} = \frac{\ell'}{R'} M_{45} \tag{2.14c}$$

and $\epsilon_5 = -\epsilon_4 = 1$, the coordinates and momenta become commuting variables and the changes occur only in the Heisenberg algebra and the nature of $\mathcal{I}$, namely

$$[P_\mu, x_\nu] = \mathrm{i}\left(\frac{\ell'}{R'} M_{\mu\nu} + g_{\mu\nu}\mathcal{I}\right) \tag{2.15a}$$

$$[P_\mu, \mathcal{I}] = -\mathrm{i}\frac{\ell'}{R'} P_\mu \tag{2.15b}$$

$$[x_\mu, \mathcal{I}] = \mathrm{i}\frac{\ell'}{R'} x_\mu . \tag{2.15c}$$

However, this identification of the physical observables in the deformed algebra does not seem so natural as the previous one. In particular, equation (2.15*a*) implies a radical departure from Heisenberg algebra and the fundamental length scale is tied up to the large scale of the manifold curvature radius, in the sense that, if we take $R' \to \infty$, the whole deformation vanishes.

The $\Re_{\ell,\infty}$ algebra (2.13) has a simple representation by differential operators in a five-dimensional space with coordinates (ξ_μ, η)

$$P_\mu = \mathrm{i}\frac{\partial}{\partial \xi^\mu} \tag{2.16a}$$

$$M_{\mu\nu} = \mathrm{i}\left(\xi_\mu \frac{\partial}{\partial \xi^\nu} - \xi_\nu \frac{\partial}{\partial \xi^\mu}\right) \tag{2.16b}$$

$$x_\mu = \xi_\mu + \mathrm{i}\ell\left(\xi_\mu \frac{\partial}{\partial \eta} - \epsilon_5 \eta \frac{\partial}{\partial \xi^\mu}\right) \tag{2.16c}$$

$$\mathcal{I} = 1 + \mathrm{i}\ell\frac{\partial}{\partial \eta}. \tag{2.16d}$$

In this representation, the deformation has a simple interpretation. The spacetime coordinates x_μ, in addition to a usual (continuous spectrum) component, have a small angular-momentum component corresponding to a rotation (or hyperbolic rotation) in the extra dimension, and the centre of the Heisenberg algebra picks up a small momentum in the extra dimension.

Algebra (2.13) is seen to be the algebra of the pseudo-Euclidean groups $E(1,4)$ or $E(2,3)$, depending on whether ϵ_5 is -1 or $+1$. For the construction of quantum fields it is the representations of these groups that should be used. Notice however that only the Poincaré part of $E(1,4)$ or $E(2,3)$ corresponds to symmetry operations and only this part has to be implemented by unitary operators. The spacetime fields $\psi(x)$ are functionals over the auxiliary variables (ξ, η) the correspondence being established by representation (2.16). The construction of the generalized version of the usual quantum field-theory models may then be carried out in a fairly simple manner.

Being mostly concerned with the characterization of a general stable framework for relativistic quantum mechanics, it is outside the scope and the spirit of this paper to discuss specific models. I would like, however, to mention that a simple model-independent quantum mechanical sum rule follows from the double commutator

$$[x, [p, x]] = \epsilon_5 \ell^2 p. \tag{2.17}$$

Taking the expectation value of both sides in a normalized state ψ and using generalized momentum eigenvectors for the decompositions of the unit $\int \mathrm{d}k |k\rangle\langle k|$, one obtains

$$\int \mathrm{d}k\, k\{|\langle\psi|x|k\rangle|^2 - \mathrm{Re}(\langle\psi|x^2|k\rangle\langle k|\psi\rangle)\} = \frac{\epsilon_5}{2}\ell^2\langle\psi|p|\psi\rangle. \tag{2.18}$$

If the state ψ has a large momentum component, the right-hand side becomes large and this dipole momentum-type sum rule may lead to observable effects.

3. Remarks

(i) Faddeev [3] pointed out that, besides the stabilizing deformations leading from non-relativistic to relativistic and from classical to quantum mechanics, the Einstein theory of gravity might also be considered as a deformation in a stable direction. This theory is based on curved pseudo-Riemann manifolds. Therefore, in the set of Riemann spaces, Minkowski space is a kind of degeneracy, whereas a generic Riemann manifold is stable in the sense that, in its neighbourhood, all spaces are curved. The deformation parameter is the gravitation constant γ, which thus gains the same status, as a fundamental constant, as $\hbar$ and c.

We have, however, seen that a natural way of stabilizing the algebra is through a two-parameter deformation (ℓ, R). It seems that R is the parameter that relates predominantly to the curvature of the manifold, and this is the reason why I have called $\Re_{\ell,\infty}$ in equations (2.13) the deformed algebra in tangent space.

On the other hand, many of the authors that have concerned themselves with the issue of the fundamental length were aiming to obtain a natural scale for the masses of the elementary particles. However, the inverse of the mass scale of what are now called the elementary particles leads to such length values that, for example, the effect of the deformed commutators (2.13*e*) and (2.13*h*) should by now have been detected. So, in the end, it might well be that the deformation parameter ℓ, if it exists, is not directly related either to the mass scale of elementary particles or to the gravitational constant.

(ii) In this paper, all the algebra deformations that were considered are deformations in the classical sense of Gerstenhaber, Nijenhuis and Richardson. Another type of deformation that has received a great deal of attention lately, not only for algebras and groups [43, 44] but for other mathematical structures as well [45], is the notion of q-deformation. The q-deformations involve exponential functions of the algebra elements and, therefore, are deformations of the universal enveloping algebra, not deformations of the algebra in the classical sense. However, first steps have been given to establishing a stability theory for q-deformations [46] and, recently, a connection was also established between q-deformations and regular $*$-deformations in an enlarged phase space [47, 48]. Therefore, it is probably interesting to reanalyse the problem of stability of relativistic quantum mechanics in a q-deformation context as well.

References

[1] Smale S 1967 *Bull. Am. Math. Soc.* **73** 747

[2] Flato M 1982 *Czech. J. Phys.* B **32** 472

[3] Faddeev L D 1988 *Asia-Pacific Physics News* **3** 21; 1989 *Frontiers in Physics, High Technology and Mathematics* ed H A Cerdeira and S Lundqvist (Singapore: World Scientific) pp 238–46

[4] Gerstenhaber M 1964 *Ann. Math.* **79** 59

[5] Nijenhuis A and Richardson R W 1966 *Bull. Am. Math. Soc.* **72** 1; 1967 **73** 175; 1967 *J. Math. Mech.* **17** 89

[6] Gerstenhaber M and Schack S D 1988 *Deformation Theory of Algebras and Structures and Applications* ed M Hazewinkel and M Gerstenhaber (Dordrecht: Kluwer) pp 11–264

[7] Hermann R 1970 *Vector Bundles in Mathematical Physics* (New York: Benjamin) ch 3

[8] Jacobson N 1962 *Lie Algebras* (New York: Interscience)

[9] Richardson R W 1967 *Pac. J. Math.* **22** 339

[10] Rauch G 1972 *Ann. Inst. Fourier* **22** 239

[11] Carles R 1984 *Ann. Inst. Fourier* **33** 65; 1985 *Math. Ann.* **272** 477; 1985 *C. R. Acad. Sci., Paris* **300** 467

[12] Goze M and Ancochea Bermúdez J M 1985 *Indagationes Math.* **88** 397

[13] Carles R and Diakité Y 1984 *J. Algebra* **91** 53

[14] Goze M 1988 *Deformation Theory of Algebras and Structures and Applications* ed M Hazewinkel and M Gerstenhaber (Dordrecht: Kluwer) pp 265–355

[15] Ancochea Bermúdez J M 1988 *Deformation Theory of Algebras and Structures and Applications* ed M Hazewinkel and M Gerstenhaber (Dordrecht: Kluwer) pp 403–45

[16] Bayen F, Flato M, Fronsdal C, Lichnerowicz C and Sternheimer D 1977 *Lett. Math. Phys.* **1** 521; 1978 *Ann. Phys., NY* **111** 61, 111

[17] Vey J 1975 *Commun. Math. Helv.* **50** 421
[18] de Wilde M and Lecomte P 1983 *Lett. Math. Phys.* **7** 487
[19] Neroslavsky O M and Vlassov A T 1981 *C. R. Acad. Sci., Paris* **292** I 71
[20] Cahen M and Gutt S 1982 *Lett. Math. Phys.* **6** 395
[21] Moyal J 1949 *Proc. Camb. Phil. Soc.* **45** 99
[22] Koopman B O 1931 *Proc. Natl Acad. Sci. USA* **17** 315
[23] von Neumann J 1932 *Ann. Math.* **33** 587
[24] Barut A O and Brittin W E (ed) 1971 *De Sitter and Conformal Groups and Their Applications (Lectures on Theoretical Physics XIII)* (Boulder, CO: Colorado Associated University Press)
[25] Snyder H S 1947 *Phys. Rev.* **71** 38; 1947 **72** 68
[26] Kadyshevsky V G 1978 *Nucl. Phys.* B **141** 477
[27] Kadyshevsky V G and Fursaev D V 1990 *Teor. Mat. Phys.* **83** 474 and references therein
[28] Banai M 1984 *Int. J. Theor. Phys.* **23** 1043
[29] Das A 1966 *J. Math. Phys.* **7** 52
[30] Atkinson D and Halpern M B 1967 *J. Math. Phys.* **8** 373
[31] Gudder S P 1968 *SIAM J. Appl. Math.* **16** 1011
[32] Finkelstein D 1974 *Phys. Rev.* D **9** 2217
[33] Dineykhan M and Namsrai K 1986 *Mod. Phys. Lett.* **1** 183
[34] Lee T D 1983 *Phys. Lett.* **122B** 217
[35] Prugovecki E 1983 *Stochastic Quantum Mechanics and Quantum Spacetime* (Dordrecht: Reidel)
[36] Blokhintsev D I 1978 *Teor. Mat. Phys.* **37** 933
[37] Schild A 1948 *Phys. Rev.* **73** 414
[38] Veneziano G 1989 *CERN preprint* TH 5581/89
[39] 't Hooft G 1990 *Themes in Contemporary Physics* vol II, ed S Deser and R J Finkelstein (Singapore: World Scientific) p 77
[40] Jackson J C 1977 *J. Phys. A: Math. Gen.* **10** 2115
[41] Madore J 1992 *Ann. Phys.* **219** 187
[42] Maggiore M 1993 *Phys. Lett.* **319B** 83
[43] Doebner H D and Hennig J D (ed) 1990 Quantum Groups *Proc. 8th Int. Workshop on Math. Phys. (Springer Lecture Notes in Physics 370)* (Berlin: Springer)
[44] Kulish P P (ed) 1992 Quantum Groups *Proc. Workshop at the Euler International Institute (Springer Lecture Notes in Mathematics 1510)* (Berlin: Springer)
[45] Man'ko V I and Vilela Mendes R 1993 *Phys. Lett.* **180A** 39
[46] Gerstenhaber M and Shack S D 1990 *Proc. Natl Acad. Sci. USA* **87** 478
[47] Flato M and Lu Z 1991 *Lett. Math. Phys.* **21** 85
[48] Suzuki T 1993 *J. Math. Phys.* **34** 3453