

A consistent measure for lattice Yang-Mills

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Abstract

The construction of a consistent measure for Yang-Mills is a precondition for an accurate formulation of non-perturbative approaches to QCD, both analytical and numerical. Using projective limits as subsets of Cartesian products of homomorphisms from a lattice to the structure group, a consistent interaction measure and an infinite-dimensional calculus has been constructed for a theory of non-abelian generalized connections on a hypercubic lattice. Here, after reviewing and clarifying past work, new results are obtained for the mass gap when the structure group is compact.

Keywords: Yang-Mills, Euclidean measure, Mass gap

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1 Introduction: Non-perturbative QCD and the Euclidean measure

QCD, believed to be the theory of strong interactions, has the serious shortcoming that only a limited sector can be treated analytically, namely the one where short distance effects play a role. This is the perturbative approach or, in the language of functional integration, the domain of Gaussian measures. The perturbative approach to field theory is not entirely satisfactory because perturbation theory is only an asymptotic expansion. Nevertheless, in quantum electrodynamics, the coupling constant being small, perturbation theory is applied with practical success. For QCD the coupling is small

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only for distances which are much smaller than the radius of a proton and therefore it is hopeless to use perturbative methods to calculate light hadron masses, for example. Another aspect of QCD which is outside the realm of perturbation theory is confinement, the fact that asymptotically we do not observe the particles which correspond to the fundamental fields of QCD, the quarks and gluons, while when the Compton wavelength of the exchanged particles is very small the constituents of the hadrons behave nearly like free particles. Since one cannot treat analytically distances of the size of a hadron, confinement cannot be explained by perturbation theory.

To handle all these phenomena one has to rely on models or approximations, the lattice approach, the introduction of local condensates, chiral symmetry breaking models, the stochastic vacuum, etc. In all cases, in functional integral terms, one needs to go beyond the Gaussian measure. Therefore a precondition for the building of a reliable non-perturbative QCD, seems to be the construction of a rigorous Euclidean measure for the Yang-Mills theory. In particular a measure that handles in a consistent way both large and small distance limits. The use of a consistent measure (in the sense to be defined later) is also important for numerical calculations on the lattice to insure that, when the lattice spacing approaches zero, one is actually approaching the continuum limit.

First steps in the construction of such a measure were given in [1] where a space for generalized connections was defined using projective limits as subsets of Cartesian products of homomorphisms from lattice based loops to a structure group. In this space, non-interacting and interacting measures were defined as well as functions and operators. From projective limits of test functions and distributions on products of compact groups, a projective gauge triplet was obtained, which provides a framework for an infinite-dimensional calculus in gauge theories. A central role is played by the construction of an interacting measure which, satisfying a consistency condition, can be extended to a projective limit of decreasing lattice spacing and increasingly larger lattices.

Here the construction in [1] is further clarified and extended with a more detailed explanation on how the one-plaquette-at-a-time refinement is performed in dimensions higher than two. An essential point in this construction is the fact that the gauge covariant group elements that one associates to each edge are loops based on an external point x_0 . Therefore several distinct independent loops may be associated to the same edge.

In addition, some of the physical consequences of the constructed measure

will be explored, in particular the nature of the mass gap that it implies. Here the main tool to be used is the theory of small random perturbations of dynamical systems [2] [3]. This, as well as the stochastic representation of the principal eigenvalue of elliptic equations for arbitrary coupling intensities g , seems to be the most appropriate tool to handle non-perturbative problems because its leading term is of order $\exp\left(-\frac{C}{g^2}\right)$.

Although using rigorous mathematical concepts throughout the paper, the constructions are kept as simple as possible, with the necessary mathematical background explained both in the construction of the measure and in the use of the theory of small random perturbations of dynamical systems.

The basic setting, as used in [1], is the following:

In \mathbb{R}^d a sequence of hypercubic lattices is constructed in such a way that any plaquette of edge size $\frac{a}{2^k}$ ($k = 0, 1, 2, \dots$) is a refinement of a plaquette of edge $\frac{a}{2^{k-1}}$ (meaning that all vertices of the $\frac{a}{2^{k-1}}$ plaquette are also vertices in the $\frac{a}{2^k}$ plaquette). The refinement is made one-plaquette-at-a-time. Notice however that, when one plaquette of edge $\frac{a}{2^{k-1}}$ is converted into four plaquettes of edge $\frac{a}{2^k}$, $2 \times (d - 2)$ new plaquettes of edge $\frac{a}{2^{k-1}}$, orthogonal to the refined plaquette, are also added to the lattice. The additional plaquettes connect the new vertices of the refined $\frac{a}{2^k}$ plaquette to the middle points of $\frac{a}{2^{k-1}}$ plaquettes. Successive application of this process to all still unrefined $\frac{a}{2^{k-1}}$ plaquettes finally yields a full hypercubic $\frac{a}{2^k}$ lattice. See Fig.1 for a 3-dimensional projection of the process, where two of the additional four (in \mathbb{R}^d) plaquettes are shown, attached to the points A, B, C and D . This one-plaquette-at-a-time construction is used to check the consistency condition (see Section 2).

Finite volume hypercubes Γ in these lattices form a directed set $\{\Gamma, \succ\}$ under the inclusion relation \succ . $\Gamma \succ \Gamma'$ meaning that all edges and vertices in Γ' are contained in Γ , the inclusion relation satisfying

$$\begin{aligned} \Gamma &\succ \Gamma \\ \Gamma &\succ \Gamma' \text{ and } \Gamma' \succ \Gamma \implies \Gamma = \Gamma' \\ \Gamma &\succ \Gamma' \text{ and } \Gamma' \succ \Gamma'' \implies \Gamma \succ \Gamma'' \end{aligned} \tag{1}$$

Well-ordering of the directed set is insured by doing each new refinement in the previously refined lattice. After each complete refinement of a finite volume hypercube (from $\frac{a}{2^{k-1}}$ to $\frac{a}{2^k}$ size), the sequence is expanded to include larger and larger volume hypercubes which are likewise refined, etc..

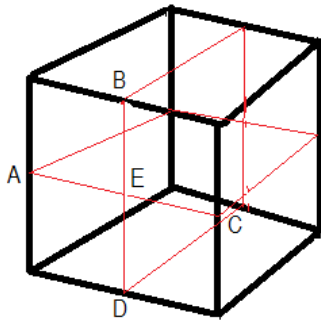


Figure 1: Partial 3-dimensional projection of the one-plaquette-at-a-time refinement process

Let \mathbb{G} be a compact group and x_0 a point that does not belong to any lattice point of the directed family. Assuming an analytic parametrization of each edge, associate to each edge l a x_0 -based loop and for each generalized connection A consider the holonomy $h_l(A)$ associated to this loop. For definiteness each edge is considered to be oriented along the coordinates positive direction and the set of edges of the lattice Γ is denoted $E(\Gamma)$. The set \mathcal{A}_Γ of generalized connections for the lattice hypercube Γ is the set of homomorphisms $\mathcal{A}_\Gamma = Hom(E(\Gamma), G) \sim G^{\#E(\Gamma)}$, obtained by associating to each edge the holonomies $h_l(\cdot)$ on the x_0 -based loops that are associated to that edge. $G^{\#E(\Gamma)}$ is a product group, $\#E(\Gamma)$ being the number of edges. The orientation of each $h_l(\cdot)$ associated to an edge is the one compatible with the orientation above defined for the edge. The set of gauge-independent generalized connections \mathcal{A}_Γ/Ad is obtained factoring by the adjoint representation at x_0 , $\mathcal{A}_\Gamma/Ad \sim G^{\#E(\Gamma)}/Ad$. However because, for gauge independent functions, integration in \mathcal{A}_Γ coincides with integration in \mathcal{A}_Γ/Ad , for simplicity, from now on one uses only \mathcal{A}_Γ . Finally one considers the projective limit $\mathcal{A} = \lim_{\leftarrow} \mathcal{A}_\Gamma$ of the family

$$\{\mathcal{A}_\Gamma, \pi_{\Gamma\Gamma'} : \Gamma' \succ \Gamma\} \quad (2)$$

$\pi_{\Gamma\Gamma'}$ and π_Γ denoting the surjective projections $\mathcal{A}_{\Gamma'} \rightarrow \mathcal{A}_\Gamma$ and $\mathcal{A} \rightarrow \mathcal{A}_\Gamma$.

The projective limit of the family $\{\mathcal{A}_\Gamma, \pi_{\Gamma\Gamma'}\}$ is the subset \mathcal{A} of the Cartesian product $\prod_\Gamma \mathcal{A}_\Gamma$ defined by

$$\mathcal{A} = \left\{ a \in \prod_\Gamma \mathcal{A}_\Gamma : \Gamma' \succ \Gamma \implies \pi_{\Gamma\Gamma'} a_{\Gamma'} = a_\Gamma \right\} \quad (3)$$

with $a_{\Gamma'} \in \mathcal{A}_{\Gamma'}$, $a_\Gamma \in \mathcal{A}_\Gamma$ and the projective topology in \mathcal{A} being the coarsest topology for which each π_Γ mapping is continuous.

For a compact group \mathbb{G} , each \mathcal{A}_Γ is a compact Hausdorff space. Therefore \mathcal{A} is also a compact Hausdorff space. In each \mathcal{A}_Γ one has a natural (Haar) normalized product measure $\nu_\Gamma = \mu_H^{\#E(\Gamma)}$, μ_H being the normalized Haar measure in \mathbb{G} . Then, according to a theorem of Prokhorov, as generalized by Kisynski [4] [5], if the following condition

$$\nu_{\Gamma'} (\pi_{\Gamma\Gamma'}^{-1} (B)) = \nu_\Gamma (B) \quad (4)$$

is satisfied for every $\Gamma' \succ \Gamma$ and every Borel set B in \mathcal{A}_Γ , there is a unique measure ν in \mathcal{A} such that $\nu (\pi_\Gamma^{-1} (B)) = \nu_\Gamma (B)$ for every Γ . In this way a sequence of measures is obtained that give the same weight to the sequence $\dots, B, \pi_{\Gamma\Gamma'}^{-1} (B), \dots$ of Borel sets. In particular a continuum limit measure is obtained that is consistent with the measures at each intermediate step of the lattice refinement.

2 The measure

As stated before, the essential step in the construction of the measure in the projective limit is the fulfilling of the consistency condition (4). One considers, on the finite-dimensional spaces $\mathcal{A}_\Gamma \sim G^{\#E(\Gamma)}$, measures that are absolutely continuous with respect to the Haar measure

$$d\mu_{\mathcal{A}_\Gamma} = p(\mathcal{A}_\Gamma) (d\mu_H)^{\#E(\Gamma)} \quad (5)$$

$p(\mathcal{A}_\Gamma)$ being a continuous function in \mathcal{A}_Γ with the following two simplifying assumptions:

- $p(\mathcal{A}_\Gamma)$ is a product of plaquette functions

$$p(\mathcal{A}_\Gamma) = p(U_{\square_1}) p(U_{\square_2}) \dots p(U_{\square_n}) \quad (6)$$

with $U_{\square}(A_{\Gamma}) = h_1 h_2 h_3^{-1} h_4^{-1}$, h_1 to h_4 being the holonomies of the x_0 -based loops associated to the edges of the plaquette, the orientation of the plaquette being uniquely defined by the positive orientation of the edge to which it is associated.

- $p(\cdot)$ is a central function, $p(xy) = p(yx)$ or, equivalently $p(y^{-1}xy) = p(x)$ with $x, y \in \mathbb{G}$.

Let p', p'' and p be the density functions associated respectively to the square plaquette with edges of size $\frac{a}{2^k}$, to the rectangular plaquette with edges of size $\frac{a}{2^k}$ and $\frac{a}{2^{k-1}}$ and, finally, to the square plaquette with edges of size $\frac{a}{2^{k-1}}$. Then

Proposition 1 [1] *A measure on the projective limit $\mathcal{A} = \varprojlim \mathcal{A}_{\Gamma}$ exists if a sequence of functions is found satisfying*

$$\begin{aligned} \int p'(G_i X) p'(X^{-1} G_j) d\mu_H(X) &\sim p''(G_i G_j) \\ \int p''(G_i X) p''(X^{-1} G_j) d\mu_H(X) &\sim p(G_i G_j) \end{aligned} \quad (7)$$

for plaquette subdivisions of all sizes.

Proof: In the directed set $\{\Gamma, \succ\}$ consider two elements Γ and Γ' which differ only in subdivision of a single plaquette from $\frac{a}{2^{k-1}}$ to $\frac{a}{2^k}$ size (see Fig.2) plus the additional $\frac{a}{2^{k-1}}$ plaquettes (based on the middle points A, B, C and D) as explained in the introduction.

To each edge one associates as many x_0 -based loops as the number of independent plaquettes that share that edge. For example to the edge connecting the points A and C in Fig.2 there are four (in \mathbb{R}^4) associated loops, two associated to the edges A-E and E-C, and two others associated to the full edge A-C corresponding to the additional plaquettes of size $\frac{1}{2^{k-1}}$. One associates the central function p' to the first two loops and p to the others. Notice that it is quite consistent to associate more than one independent loop to each edge. The integration is over the loops, not the edges.

Finally, the consistency condition (4) requires that

$$\begin{aligned} \frac{1}{Z'} \int p'(g_1^{-1} G_1 g_2 y_2^{-1} y_1^{-1}) p'(y_2 g_2^{-1} G_2 g_3^{-1} y_3^{-1}) p'(y_4 y_3 g_3 G_3^{-1} g_4^{-1}) \\ p'(g_1 y_1 y_4^{-1} g_4 G) \prod_{i=1}^4 d\mu_H(g_i) d\mu_H(y_i) d\mu_H(G_i) \prod_{k=1}^{2(d-2)} \left\{ p\left(G_1^{(k)} G_2^{(k)} G_3^{(k)-1} G_4^{(k)-1}\right) \right. \\ \left. \prod_{j=1}^4 d\mu_H\left(G_j^{(k)}\right) \right\} = \frac{1}{Z} \int p(G_1 G_2 G_3^{-1} G_4^{-1}) \prod_{i=1}^4 d\mu_H(G_i). \end{aligned} \quad (8)$$

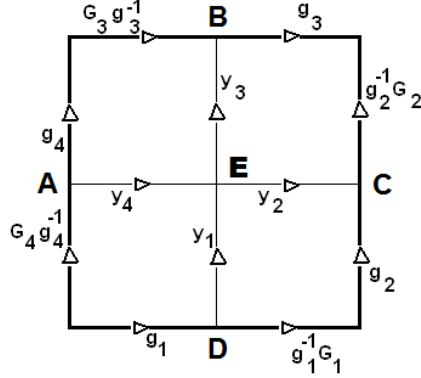


Figure 2: Subdivision of one plaquette

The last two factors in the left hand side

$$\prod_{k=1}^{2(d-2)} \left\{ p \left(G_1^{(k)} G_2^{(k)} G_3^{(k)-1} G_4^{(k)-1} \right) \prod_{j=1}^4 d\mu_H \left(G_j^{(k)} \right) \right\} \quad (9)$$

concern the integration over the additional $\frac{a}{2^k}$ plaquettes, the density function p used for these plaquettes being the one corresponding to edges of size $\frac{a}{2^{k-1}}$. Z and Z' are numerical constants related to normalization of the density functions.

Using centrality of p' , redefining

$$g_1 y_1 = X_1, \quad g_2 y_2^{-1} = X_2, \quad y_3 g_3 = X_3^{-1}, \quad y_4^{-1} g_4 = X_4^{-1} \quad (10)$$

and using invariance of the normalized Haar measure, one may integrate over y_1, y_2, y_3, y_4 and G_k , obtaining for the left hand side of (8) with the exclusion of the terms in (9)

$$\frac{1}{Z'} \int p' (X_1^{-1} G_1 X_2) p' (X_2^{-1} G_2 X_3) p' (X_3^{-1} G_3^{-1} X_4) p' (X_4^{-1} G_4^{-1} X_1) \prod_{i=1}^4 d\mu_H (X_i) d\mu_H (G_i)$$

Therefore if there is a sequence of central functions p', p'', p satisfying the

proportionality relations

$$\begin{aligned} \int p'(G_i X) p'(X^{-1} G_j) d\mu_H(X) &\sim p''(G_i G_j) \\ \int p''(G_i X) p''(X^{-1} G_j) d\mu_H(X) &\sim p(G_i G_j) \end{aligned} \quad (11)$$

the consistency condition (8) would be satisfied, because the terms in (9) dealing with integration over different loops may be absorbed in the proportionality constant of the measure normalization. The same procedure is then applied to all still unrefined plaquettes, meaning that a measure would exist in the projective limit, because all elements in the directed set $\{\Gamma, \succ\}$ may be reached by this method. \square

With the conditions (7) and the above construction the consistency condition is satisfied by means of the pairwise convolutions of the p functions for arbitrary dimensions. Notice also that by this refinement method all plaquettes of a full $\frac{1}{2^k}$ lattice are obtained. For example edges whose endpoints are at the center of a square of the coarser lattice are obtained when one of the additional plaquettes of the above process is also subdivided. Therefore, using a measure that satisfies the condition (4) one is sure that, in the continuum limit, a measure is obtained that is consistent with the physical premises used to postulate a measure for finite lattice spacing. This is an important feature, not only for rigorous analytical developments, but also for the consistency of numerical calculations at successively smaller lattice spacings.

If $p(U_\square)$ is a constant, $d\mu_{A_\Gamma}$ is factorizable and the consistency condition is trivially satisfied. $d\mu_{A_\Gamma}$ would be the Ashtekar-Lewandowski measure for generalized connections [6] [7]. A nontrivial solution that satisfies the consistency condition (8) is the choice of $p(U_\square)$ as the heat kernel

$$K(g, \beta) = \sum_{\lambda \in \Lambda^+} d_\lambda e^{-c(\lambda)\beta} \chi_\lambda(g) \quad (12)$$

with

$$\begin{aligned} \beta &\rightarrow \beta' = \frac{\beta}{4} \\ \beta &\rightarrow \beta'' = \frac{\beta}{2} \end{aligned} \quad (13)$$

β'', β' and β being the constants associated to p'', p' and p . In (12), $g \in \mathbb{G}$, $\beta \in \mathbb{R}^+$, Λ^+ is the set of highest weights, d_λ and $\chi_\lambda(\cdot)$ the dimension and character of the λ -representation and $c(\lambda)$ the spectrum of the Laplacian $\Delta_G := \sum_{i=1}^n \chi_i^2$, $\{\chi_i\}$ being a basis for the Lie algebra of \mathbb{G} .

Finally, one writes for the measure on the lattice Γ

$$d\mu_{\mathcal{A}_\Gamma} = \frac{1}{Z_\Gamma} \prod_{edges} d\mu_H(g_l) \prod_{plaquettes} \sum_{\lambda \in \Lambda^+} d_\lambda e^{-c(\lambda)\beta} \chi_\lambda(g_p) \quad (14)$$

and the consistency condition (4) being satisfied, the Prokhorov-Kisynski theorem [4] insures that a measure is also defined on the projective limit lattice, that is, on the projective limit generalized connections \mathcal{A} .

This measure has the required naive continuum limit, both for abelian and non-abelian theories (see [1]). Furthermore by defining infinite-dimensional test functionals and distributions, a projective triplet was constructed which provides a framework to develop an infinite-dimensional calculus over the hypercubical lattice. In particular, this step is necessary to give a meaning to the density $p(\mathcal{A}_\Gamma)$ in the $\beta \rightarrow 0$ limit, where $p(\mathcal{A}_\Gamma)$ would no longer be a continuous function. Thus $p(\mathcal{A}_\Gamma)$, a density that multiplies the Ashtekar-Lewandowski measure [6] [7] [8], gains a distributional meaning in the framework of the projective triplet.

A theory being completely determined whenever its measure is specified, the construction in [1] provides a rigorous specification of a projective limit gauge field theory over a compact group. Some of the consequences of this specification were already discussed in [1]. Here one analyses the nature of the mass gap which follows from the measure specification.

3 The mass gap

The experimental phenomenology of subnuclear physics provides evidence for the short range of strong interactions. Therefore, if unbroken non-abelian Yang-Mills is the theory of strong interactions, the Hamiltonian, associated to its measure, should have a positive mass gap. This important physical question has been addressed in different ways by several authors. An interesting research approach [9] [10] considers the Riemannian geometry of the (lattice) gauge-orbit space to compute the Ricci curvature. The basic inspiration for this approach is the Bochner-Lichnérowicz [11] [12] inequality which states that if the Ricci curvature is bounded from below, then so is

the first non-zero eigenvalue of the Laplace-Beltrami operator. The Laplace-Beltrami operator differs from the Yang-Mills Hamiltonian in that it lacks the chromo-magnetic term, but the hope is that in the relevant physical limit the chromo-electric term dominates the bound. An alternative possibility would be to generalize the Bochner-Lichnérowicz inequality.

Other approaches are based on attempts to solve the Dyson-Schwinger equation (see for example [13] [14] [15]) on a set of exact solutions to the classical Yang-Mills theory [16] or on the ellipticity of the energy operator of cut-off Yang-Mills [17] [18].

Once a consistent Euclidean measure is obtained, the nature of the mass gap may be found either by computing the distance dependence of the correlation of two local operators or from the lower bound of the spectrum in the corresponding Hamiltonian theory. Here the Hamiltonian approach will be used, using the fact that the Hamiltonian may be obtained from the knowledge of the ground state functional and the ground state functional may be obtained from the measure [19] [20] [21] [22] [23] [24].

By inserting a complete set of energy states on the Euclidean path integral and computing the integral pinned down at $t = 0$ to a fixed configuration the corresponding ground state may be obtained. This goes back to the work of Donsker and Kac [25] [26] and has been used and proved before in several contexts [27] [28]. At least for finite-dimensional quantum systems this provides a robust estimation of the ground state.

One of the axis directions in the lattice is chosen as the time direction. Denote by $\theta(0)$ the configuration of the system at time zero. Then, recalling that at each step in the projective limit construction one has a finite-dimensional system, the ground state wave functional $\Psi_0(\theta(0))$ at the particular configuration $\theta(0)$ may be written as

$$\begin{aligned} |\Psi_0(\theta(0))|^2 &= \int d\theta \Psi_0^*(\theta) \delta(\theta - \theta(0)) \Psi_0(\theta) \\ &= \int d\mu_{\mathcal{A}}(\theta) \delta(\theta - \theta(0)) \end{aligned} \quad (15)$$

where $\mu_{\mathcal{A}}(\theta)$ is the Euclidean measure and the integration is all variable configurations which in the time-zero slice coincide with $\theta(0)$. For the lattices considered in this paper θ and $\theta(0)$ stand respectively for the set of group configurations in the x_0 -based loops associated to the edges and for the set of group configurations in the time-zero slice, namely $\theta_j^\alpha(l)$'s will be the Lie algebra coordinates of the group elements $\exp(i\theta_j^\alpha(l)\tau_\alpha)$, see Eq. (29).

The ground state in (15) may be used to develop the usual Hamiltonian approach to lattice theory, for which one uses notations similar to those of Chapter 15 in Ref.[29], the main difference being that instead of constructing the Kogut-Susskind Hamiltonian from the Wilson action, one uses the ground state obtained from the measure.

The squared wave-function in (15) is the density of a *ground state measure* [19] [20]. Associated to this ground state measure, there is a stochastic process for which the measure is invariant. The canonical way [21] [22] [23] [24] to construct the elliptic operator generator of the process is

$$H'_g = \frac{g^2(\beta)}{2\beta} \sum_{l,j,\alpha} \left\{ -\frac{\partial}{\partial\theta_j^\alpha(l)} + L_j^\alpha(l) \right\} \left\{ \frac{\partial}{\partial\theta_j^\alpha(l)} + L_j^\alpha(l) \right\} \quad (16)$$

with

$$L_j^\alpha(l) = -\frac{1}{\Psi_0} \frac{\partial\Psi_0}{\partial\theta_j^\alpha(l)} \quad (17)$$

With the unitary transformation $H'_g \rightarrow H_g = \Psi_0 H'_g \Psi_0^{-1}$ the operator in (16) would have the familiar form of Laplacian plus potential. The $\theta_j^\alpha(l)$'s are the Lie algebra coordinates of the group element $\exp(i\theta_j^\alpha(l)\tau_\alpha)$ at each x_0 -based loop associated to the edge l of the time-zero slice of the lattice, the sum being over edges (l), lattice dimensions (j) and Lie algebra generators (α). $g(\beta)$ is a coupling constant to be adjusted consistently to obtain the continuum limit, to be discussed later. Recall that from (13) $\beta \rightarrow 0$ as the length of the lattice edges ($\frac{a}{2k}$) goes to zero. Eq.(16) implies that the ground state energy E_0 is adjusted to zero

In this way a Hamiltonian and a Hilbert space may be constructed from the Euclidean measure and estimations of the principal eigenvalue may be obtained from the theory of small random perturbations of dynamical systems [30] [31]. These steps are briefly summarized below and then applied to the lattices of the projective family.

Explicit computation of the integral in (15) is, in general, not easy. However, to study the nature of the mass gap a full calculation of the ground state functional is not required. The interpretation of elliptic operators as generators of a diffusion process [30] [31] may be used and, in the limit of small β , also the theory of small perturbations of dynamical systems [2] [3].

For simplicity, Eq.(16) applies to steps of the projective limit when the same uniform β exists throughout the lattice. For intermediate steps of the refinement process, a slightly more complex definition would apply. This

however will not change the main conclusions. At each step of the projective limit construction one deals with a finite dimensional quantum system. For the Hamiltonian Then, absence of zeros in the ground state allows the unitary transformation $H'_g = \Psi_0^{-1} H_g \Psi_0$, the ground state Ψ_0 is the unit function, the corresponding states of H_g being multiplied by Ψ_0^{-1}

$$-\beta H'_g = \frac{g^2(\beta)}{2} \sum_{l,j,\alpha} \frac{\partial}{\partial \theta_j^\alpha(l)} \frac{\partial}{\partial \theta_j^\alpha(l)} + \sum_{l,j,\alpha} b_j^\alpha(l) \frac{\partial}{\partial \theta_j^\alpha(l)} \quad (18)$$

with

$$b_j^\alpha(l) = -g^2(\beta) L_j^\alpha(l) = \frac{g^2(\beta)}{2\Psi_0^2} \frac{\partial \ln \Psi_0^2}{\partial \theta_j^\alpha(l)} \quad (19)$$

The second-order elliptic operator in (18) is the generator of the diffusion process

$$d\theta_j^\alpha(l) = b_j^\alpha(l) dt + g(\beta) dW_j^\alpha(l) \quad (20)$$

with drift $b_j^\alpha(l)$ and diffusion coefficient $g(\beta)$. $\{W_j^\alpha(l)\}$ is a set of independent Brownian motions. Ψ_0^2 is the invariant measure of this process. The question of existence of a mass gap for the Hamiltonian H'_g is closely related to the principal eigenvalue of the Dirichlet problem

$$\begin{aligned} \beta H'_g u &= \lambda u && \text{in } D \\ u &= 0 && \text{in } \partial D \end{aligned} \quad (21)$$

D being a bounded domain (in the space of the variables) and ∂D its boundary. The principal eigenvalue λ_1 , that is, the smallest positive eigenvalue of $\beta H'_g$ has a stochastic representation [3] [32]

$$\lambda_1 = \sup \left\{ \lambda \geq 0; \sup_{\theta \in D} \mathbb{E}_\theta e^{\lambda \tau} < \infty \right\} \quad (22)$$

\mathbb{E}_θ denoting the expectation value for the process started from the θ configuration and τ the time of first exit from the domain D . The validity of this result hinges on the following condition

(C1) The drift b and the diffusion matrix coefficient $\sigma(g(a)\delta_{ij}$ in this case) must be uniformly Lipschitz continuous with exponent $0 < \alpha \leq 1$ and σ positive definite.

(22) is a powerful result which may be used to compute by numerical means the principal eigenvalue for arbitrary values of g ¹. However, a particularly useful situation is the small noise (small g limit). That the small noise limit corresponds to the continuum limit of the lattice theory follows from a consistency argument. Under suitable conditions, to be discussed below, the small noise limit of the lowest nonzero eigenvalue (the mass gap) of the operator $\beta H'$ is

$$\beta m \sim \exp\left(-\frac{V}{g^2(\beta)}\right) \quad (23)$$

where V is the value of a functional. Hence, for the physical mass gap m to remain fixed when $\beta \rightarrow 0$, it should also be $g(\beta) \rightarrow 0$. Therefore the small noise limit is indeed the continuum limit.

In the small noise limit the mass gap may be obtained from the Wentzell-Freidlin estimates [2] [3]. Given a bounded domain D for the variables $\theta_j^\alpha(l)$ define the functional

$$I_{t_1, t_2}(\chi) = \frac{1}{2} \int_{t_1}^{t_2} \left(\frac{d\chi}{ds} - b(\chi(s)) \right)^2 ds \quad (24)$$

where $\chi(s \in [t_1, t_2])$ is a path from the configuration $\{\theta\}$ to the boundary ∂D of the domain D . Then let

$$I(t, \{\theta\}, \partial D) = \inf_{\chi} I_{0, t}(\chi) \quad (25)$$

be the infimum over all continuous paths that starting from the configuration $\{\theta\}$ hit the boundary ∂D in time less than or equal to t . A path is said to be a *neutral path* if $I(t, \{\theta\}, \partial D) = 0$.

The value of this functional is controlled by the nature of the deterministic dynamical system

$$\frac{d\theta_j^\alpha(l)}{dt} = b_j^\alpha(l) \quad (26)$$

Assume the following additional condition to be fulfilled:

(C2) There are a number r of ω -limit sets K_i of (26) in the domain D , with all points in each set K_i being equivalent for the functional I , that is, $I(t, x, y) = 0$ if both $x, y \in K_i$ and $b \cdot \nu > 0$, ν being the inward normal to ∂D .

¹See for example Ref. [33]

Then [30] [3] with

$$V_i = \inf I(t, x, \partial D) \quad \text{for } x \in K_i \quad (27)$$

and

$$\begin{aligned} V_* &= \max(V_1, \dots, V_r) \\ V^* &= \min(V_1, \dots, V_r) \end{aligned}$$

the lowest non-zero eigenvalue λ_1 satisfies

$$V_* \leq \lim_{g \rightarrow 0} (-g^2 \ln \lambda_1(g)) \leq V^*$$

In particular if there is only one V

$$\lambda_1(g) = \beta m(g) \asymp \exp\left(-\frac{V}{g^2(\beta)}\right) \quad (28)$$

the symbol \asymp meaning logarithmic equivalence in the sense of large deviation theory. If the drift is the gradient of a function, as in (19), the quasi-potential V is simply obtained from the difference of the function at the ω -limit set and the minimum at the boundary.

For details on the theory of small perturbations of dynamical systems as applied to the small β limit of lattice theory refer also to [34] where this technique was applied to an approximate ground state functional. Also [21] [22] [23] [24] provide details on how the ground state measure provides a complete specification of quantum theories both for local and non-local potentials. This theory developed for finite-dimensional systems follows earlier developments of Coester, Haag and Araki [35] [36] in the field theory context.

Now the existence of a mass gap associated to the Hamiltonian (18), obtained from the measure (14) by (15), hinges on checking the above conditions **(C1)** and **(C2)**. Inserting (14) into (15) one obtains

$$|\Psi_0(g_l(0))|^2 = \int \prod_{edges} d\mu_H(g_l) \delta(g_l - g_l(0)) \prod_{plaquettes} \sum_{\lambda \in \Lambda^+} d\lambda e^{-c(\lambda)\beta} \chi_\lambda(g_p) \quad (29)$$

g_l being the group element associated to the edge-associated loops and g_p those associated to the ordered product of group elements around a plaquette, $|\Psi_0(g_l(0))|^2$ being a function only of the group elements on the time slice.

For practical calculations one makes a global lattice gauge fixing in (29) but for the present considerations this is not important.

In (29) the only free variables are the edge variables in the time slice or, more precisely, the angles of the maximal torus of the group element associated to the corresponding plaquettes. Smoothness of the heat kernel implies that the Leibnitz rule for derivation under the integral can be applied and the drift $b_j^\alpha(l)$ in (26) is also a smooth function. Therefore condition **(C1)** is satisfied. As for condition **(C2)** one knows that the heat kernel satisfies the following two-sided Gaussian estimate

$$\frac{1}{|B(e, \beta^{\frac{1}{2}})|} c_1 \exp\left(\frac{-d^2(g)}{c_2 \beta}\right) \leq K(g, \beta) \leq \frac{1}{|B(e, \beta^{\frac{1}{2}})|} c_3 \exp\left(\frac{-d^2(g)}{c_4 \beta}\right) \quad (30)$$

$d(g)$ being the Carnot-Carathéodory distance of the group element g to the identity e and $|B(e, \beta^{\frac{1}{2}})|$ is the volume of a ball of radius $\beta^{\frac{1}{2}}$ centered at e [37] [38]. The estimate (30) holds if and only if

(A) the volume growth has the doubling property

$$\forall x \in \mathbb{G}, \forall r > 0, |B(x, 2r)| \leq c |B(x, r)|$$

(B) there is a constant γ such that

$$\forall x \in \mathbb{G}, \forall r > 0, \int_{B(x, r)} |f - Av_{B(x, r)} f|^2 dx \leq \gamma r^2 \int_{B(x, 2r)} |\nabla f|^2$$

$Av_{B(x, r)} f$ being the average of f over the ball $B(x, r)$. In particular if \mathbb{G} is unimodular (B) holds.

For a compact group (A) and (B) being satisfied, the two-sided estimate (30) holds. Therefore the dynamical system (26) has only one ω -limit set, the group identity, and one is in the situation of Eq.(28), V being obtained from the difference of the heat kernel at the identity and at the boundary of the domain. In conclusion:

Proposition 2 *If G is a compact group, the Hamiltonian (20) obtained from the heat-kernel measure has a positive mass gap.*

The Wentzell-Freitlin results that are used to reach this result apply to a Dirichlet problem with boundary. Therefore one is in fact considering some bounded domain in the group space containing the identity, not necessarily the full group space. This is probably consistent because for small β the

measure contributions are dominated by group elements close to the identity (see the comparison with the naïve continuum limit in Ref.[1]). However this is an issue that might deserve further consideration.

The result is obtained for the Hamiltonians constructed from the Euclidean measure constructed for each finite dimensional lattice in the directed set $\{\Gamma, \succ\}$. By itself, the result depends on the nature of the central functions p chosen in (12), and not on the consistency property and the existence of the projective limit measure. However, what *the specific form of the mass gap (28) and the consistency of the measure together imply is that there is a choice $g(\beta)$ that allows the construction of a continuum limit theory (at $\beta \rightarrow 0$) with a finite mass gap.*

This mass gap result, being based on the small random perturbations (Wentzell-Freidlin (WF)) estimates, is of a strictly non-perturbative (NP) nature. The WF estimates are in fact a tool of choice for NP reasoning because they have at all orders an essential singularity on the coupling (noise) constant.

The projective limit, being the subset of the direct product of all lattice refinements that satisfies a consistent condition (Eq.(3)), it describes a framework for all length scales, with a consistent measure down to the vanishing lattice space limit. Therefore, this construction may then be considered as a scaffold for the physical theory, which is embedded in the projective limit structure by a Hamiltonian constructed from the measure up to a (coupling) constant. Uniform physical results are obtained by the choice at each length scale of the free parameter (the coupling constant). In particular, to obtain a finite mass gap at all length scales, it is indeed needed to make the coupling constant approach zero with the lattice spacing. However, this weak coupling limit is fully non-perturbative because based on an estimate with an essential singularity.

The existence of the projective limit measure, the projective triplet, consistency with the required physical continuum limit as shown in [1], as well as the characterization of the nature of the mass gap obtained here, might provide a consistent constructive definition of a theory that might serve the physical purposes aimed at by the Yang-Mills action. Of course, to scale up these results to a full understanding of QCD the role of fermions as well as of the non-generic strata [39] would be required. In particular to clarify the importance of these strata for the structure of low-lying excitations.

References

- [1] R. Vilela Mendes; *An infinite-dimensional calculus for generalized connections in hypercubic lattices*, J. Math. Phys. 52 (2011) 052304.
- [2] A. D. Wentzell and M. I. Freidlin; *On small random perturbations of dynamical systems*, Russian Math. Surveys 25 (1970) 1–55.
- [3] M. I. Freidlin and A. D. Wentzell; *Random perturbations of dynamical systems*, Springer, Berlin 2012.
- [4] J. Kisynski; *On the generation of tight measures*, Studia Math. 30 (1968) 141-151.
- [5] K. Maurin; *General eigenfunction expansions and unitary representations of topological groups*, PWN - Polish Scient. Publ., Warszawa 1968.
- [6] A. Ashtekar and J. Lewandowski; *Differential geometry on the space of connections via graphs and projective limits*, J. Geom. Phys. 17 (1995) 191-230.
- [7] A. Ashtekar and J. Lewandowski; *Projective techniques and functional integration for gauge theories*, J.Math. Phys. 36 (1995) 2170-2191.
- [8] C. Fleischhack; *On the support of physical measures in gauge theories*, arXiv:math-ph/0109030.
- [9] M. S. Laufer and P. Orland; *The metric of Yang-Mills orbit space on the lattice*, Phys.Rev. D88 (2013) 065018
- [10] M. S. Laufer; *The Geometry of Lattice-Gauge-Orbit Space*, Ph. D. Thesis The City University of New York, 2011.
- [11] S. Bochner; *Vector fields and Ricci curvature*, Bull. Amer. Math. Soc. 52 (1946) 776-797.
- [12] A. Lichnérowicz; *Géométrie des groupes de transformations*, Dunod, Paris 1958.
- [13] R. Alkofer, A. Hauck, L. von Smeckal; *Infrared Behavior of Gluon and Ghost Propagators in Landau Gauge QCD*, Physical Review Letters 79 (1997) 3591-3594.

- [14] V. Gogokhia; *How to demonstrate a possible existence of a mass gap in QCD*, arXiv:hep-th/0604095v4
- [15] B. Holdom; *Soft asymptotics with mass gap*, Physics Letters B 728 (2014) 467–471.
- [16] M. Frasca; *Exact solutions for classical Yang-Mills fields*, arXiv:1409.2351
- [17] A. Dynin; *Quantum Yang–Mills–Weyl dynamics in the Schroedinger paradigm*, Russian Journal of Mathematical Physics 21 (2014) 169–188.
- [18] A. Dynin; *On the Yang–Mills Mass Gap Problem*, Russian Journal of Mathematical Physics 21 (2014) 326–328.
- [19] S. Albeverio, Yu. G. Kondratiev, R. A. Minlos and G. V. Shchepa'uk; *Ground state Euclidean measures for quantum lattice systems on compact manifolds*, Rep. Math. Phys. 45 (2000) 419-429.
- [20] F. Hiroshima; *Ground state measure and its applications*, RIMS pub. 1156, pp. 74-84, 2000.
- [21] S. Albeverio, R. Høegh-Krohn and L. Streit; *Energy Forms, Hamiltonians, and Distorted Brownian Paths*, J. Math. Phys. 18 (1977) 907-917.
- [22] S. Albeverio, R. Høegh-Krohn and L. Streit; *Regularization of Hamiltonians and Processes*, J. Math. Phys. 21 (1980) 1636-1642.
- [23] L. Streit; *Energy forms: Schroedinger theory, processes*, Physics Reports 77 (1981) 363-375.
- [24] R. Vilela Mendes; *Reconstruction of dynamics from an eigenstate*, J. of Math. Phys. 27 (1986) 178-184.
- [25] M. D. Donsker and M. Kac; *Sampling method for determining the lowest eigenvalue and the principal eigenfunction of Schrödinger equation*, J. Res. Nat. Bur. Standards 44 (1950) 551-557.
- [26] M. Kac; *On some connections between probability theory and differential and integral equations*, in Proc. of the second Berkeley Symposium on Math. Stat. and Probability, 189-215, Berbeley Press 1951.

- [27] G. C. Rossi and M. Testa; *Ground State Wave Function from Euclidean Path Integral*, Annals of Physics 148 (1983) 144-167.
- [28] E. Fradkin; *Wave functionals for field theories and path integrals*, Nuclear Physics B389 (1993) 587-600.
- [29] M. Creutz; *Quarks, gluons and lattices*, Cambridge U. P., Cambridge 1983.
- [30] A. Friedman; *Stochastic differential equations and applications, vol. 2*, Academic Press, New York 1976.
- [31] M. Freidlin; *Markov processes and differential equations: Asymptotic problems*, Birkhäuser, Basel 1996.
- [32] R. Z. Khas'minskii; *On positive solutions of the equation $\mathcal{L}u + V \cdot u = 0$* , Theory Probab. Appl. 4 (1959) 309-318.
- [33] S. M. Eleutério and R. Vilela Mendes; *Numerical predictions from a stochastic model for $SU(2)$ lattice gauge fields*, Phys. Lett. B173 (1986) 332-336.
- [34] R. Vilela Mendes; *Stochastic processes and the non-perturbative structure of the QCD vacuum*, Z. Phys. C - Particles and Fields 54 (1992) 273-281.
- [35] F. Coester and R. Haag, *Representation of states in a field theory with canonical variables*, Phys. Rev. 117 (1970) 1137-1145.
- [36] H. Araki, *Hamiltonian formalism and the canonical commutation relations in quantum field theory*, J. Math. Phys. 1 (1960) 492-504.
- [37] L. Saloff-Coste; *Aspects of Sobolev-type inequalities*, Cambridge Lect. Notes 289, Cambridge Univ. Press, Cambridge 2002.
- [38] N. Th. Varopoulos, L. Saloff-Coste and T. Coulhon; *Analysis and geometry on groups*, Cambridge Tracts on Math. 100, Cambridge Univ. Press, Cambridge 1992.
- [39] R. Vilela Mendes; *Stratification of the orbit space in gauge theories. The role of nongeneric strata*, J. Phys. A: Math. Gen. 37 (2004) 11485-11498.