Ergodic parameters and dynamical complexity

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Using a cocycle formulation, old and new ergodic parameters beyond the Lyapunov exponent are rigorously characterized. Dynamical Renyi entropies and fluctuations of the local expansion rate are related by a generalization of the Pesin formula. How the ergodic parameters may be used to characterize the complexity of dynamical systems is illustrated by some examples: clustering and synchronization, self-organized criticality and the topological structure of networks. © *2011 American Institute of Physics*. [doi:10.1063/1.3634008]

Characterization of the invariant measures is central in ergodic theory but Lyapunov and conditional exponents only provide a partial characterization of these measures. Using a unified cocycle formulation, ergodic parameters beyond (and independent from) the Lyapunov and the conditional exponents are obtained, which not only characterize the local fluctuations of the expansion rate but also the multi-point correlations. The generalized ergodic parameters are related to the dynamical Rényi entropies by a generalization of Pesin's formula and may provide an ergodic measure of complexity. Finally, a section of a review nature illustrates how a careful use of the ergodic parameters may give useful insights on the complex features of dynamical systems.

I. INTRODUCTION

Most successful approaches to characterize the complexity of existing dynamical systems have been based on the construction of minimal mathematical models able to reconstruct the time series of symbols generated by the system. In this context, the natural tools to address such modeling task are those of information theory. However, when the emphasis shifts from time series reconstruction to a characterization of the states of the system and to the occupation probabilities of these states, it is ergodic theory that comes into play.

Ergodic theory (M, f, μ) deals with μ–measure preserving actions of a measurable map f on a measure space M. Characterization of the measure is central in ergodic theory because it is the invariant measure that characterizes the occupation probability of each state in asymptotic conditions. Some parameters such as the Lyapunov[1] and the conditional exponents[2,3] have been used to obtain a partial characterization of the invariant measure. These exponents are global functions of the invariant measure. However, the invariant measure itself contains much more information. Ruelle,[4] for example, has emphasized this situation by pointing out that the ergodic parameters being averages of local fluctuating quantities, the quantities describing the fluctuations would again be ergodic parameters, and the same for the fluctuations of the fluctuations, etc.

In this paper, I will address the question of how to obtain ergodic parameters beyond (and independent from) the Lyapunov and the conditional exponents. A unified cocycle framework is used. Some of the generalized ergodic parameters have already appeared in several forms in the literature, others are new. A complete characterization of the invariant measure should not only include fluctuations but also correlations. The cocycle formulation is extended to deal with correlations as ergodic parameters. Also a relation between fluctuation parameters and dynamical Rényi entropies is established.

The final section, of a review nature, illustrates how a careful use of the ergodic parameters may give useful insights on several complex features of dynamical systems.

II. THEORETICAL DEVELOPMENTS

A. Generalized ergodic parameters: A cocycle formulation

The cocycle version of the Oseledets theorem[5] provides a unified way to construct ergodic parameters beyond the usual Lyapunov and conditional exponents.

Let f: $M \to M$ be a measure preserving transformation of a Lebesgue space $(M, \mathcal{B}, \mu)$ and, for any measurable function $g : M \to GL(N,\mathbb{R})$, define

$$C(x,n) = g\left(f^{n-1}(x)\right)\cdots g(x)$$
$$C(x,n+k) = C\left(f^{k}(x),n\right)C(x,k)$$

$C : M \times \mathbb{Z} \to GL(N,\mathbb{R})$ is called a *cocycle* (over f). Furthermore, any cocycle has this form. g is called the *generator* of the cocycle.

Theorem (Oseledets): If $\ln_{+} \| g(x) \| \in L^1(M,\mu)$

(i) There is a decomposition $\mathbb{R}^N = \oplus_{i=1}^{k(x)} E_i(x)$ invariant under $C(x, n)$,

(ii) $\lim_{n\to\infty}\frac{1}{n}\ln\frac{\|C(x,n)v\|}{\|v\|} = \chi_i(x)$ with $\chi_1(x) < \cdots < \chi_{k(x)}(x)$, exists uniformly in $v \in E_i(x)\backslash\{0\}$.

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For the usual **Lyapunov exponent**, the cocycle generator would be

$$g_1(x) = Df(x) = \exp(\ln(Df(x))), \tag{1}$$

and for the **conditional exponents**, instead of *Df*, one uses partial blocks of the Jacobian *Df*. The conditional exponents were first proposed by Pecora and Carroll[2] as a tool to deal with synchronization in chaotic systems and their existence under the same conditions as the **Lyapunov exponents** proved in Ref. 3.

Beyond the Lyapunov and conditional exponents, one may define the **Lyapunov fluctuation moments** $\chi_i^{(p)}(x)$ as the following limit:

$$\lim_{n\to\infty}\frac{1}{n}\ln\frac{||C(x,n)v||}{||v||} = \chi_i^{(p)}(x), \tag{2}$$

when the cocycle generator is

$$g_p(x) = \exp(\ln_+^p(Df(x))). \tag{3}$$

Notice that in the above definition, the logarithm is understood in the framework of the Oseledets-Pesin ε–reduction theorem. That is, for any $\varepsilon > 0$, there is an invertible map $\Gamma_\varepsilon(x): M \to GL(N,\mathbb{R})$ such that $g_\varepsilon(x) = \Gamma^{-1}(f(x))g(x)\Gamma(x)$ has block form and in each block $e^{\chi_i(x)-\varepsilon} \leq \| g_\varepsilon^i(x)v \| \leq e^{\chi_i(x)+\varepsilon}$. Then $g_\varepsilon(x)$ generates a cocycle $C_\varepsilon(x, n)$ equivalent to $C(x, n)$.

$\ln_+$ in Eq. (3) is therefore computed without ambiguity in each block and

$$\chi_i^{(p)}(x) = \lim_{n\to\infty}\frac{1}{n}\ln\frac{||g_p(f^{n-1}(x))\ldots g_p(x)v||}{||v||}, \tag{4}$$

is an ergodic average of the *p*–moment of the local expansion rate.

As a consequence of the Oseledets multiplicative ergodic theorem, the Lyapunov fluctuation moments $\chi_i^{(p)}(x)$ exist whenever

$$\ln_+||g_p(x)|| \in L^1(M,\mu). \tag{5}$$

This cocycle construction provides a unified description of some of the fluctuation ergodic parameters previously considered by several authors.[6–15]

From Eq. (5) follows that the existence of the fluctuation moments depends on the integrability of

$$\exp\left(\sum k_i \lambda_i^p(x)\right),$$

$\lambda_i(x)$ being the local expansion rate at x and k_i, respectively, the multiplicity of this rate. In general, non-Gaussian random variables fail to have moments of arbitrarily large order. If that is the case for the expansion rate variable, a better characterization is obtained by the characteristic function.

Definition 1. The Lyapunov characteristic fluctuation function $C(\alpha)$ is defined as the $\lim_{n\to\infty}\frac{1}{n}\ln\frac{||C(x,n)v||}{||v||}$ when the generator of the cocycle is

$$g_\alpha(x) = \exp(\exp(i\alpha\ln_+(Df(x)))). \tag{6}$$

As before, existence of $C(\alpha)$ depends on integrability of $\ln_+\|g_\alpha(x)\|$ and, because $\exp(i\alpha\ln_+(Df(x)))$ is bounded, this is always fulfilled.

B. Dynamical Rényi entropy and fluctuations of the local expansion rate

For f: $M \to M$ a measure preserving dynamical system in $(M, \mathcal{B}, \mu)$, let Φ be a partition of M and $\left\{\phi_i^{(n)}\right\}$ the elements of refined partition $\Phi_n = \vee_{i=0}^{n-1} f^{-i}(\Phi)$. Define the **dynamical Rényi entropy of order** α as

$$K(\alpha) = \sup_\Phi\left\{\lim_{n\to\infty}\frac{1}{1-\alpha}\frac{1}{n}\ln\sum_i \mu\left(\phi_i^{(n)}\right)^\alpha\right\}. \tag{7}$$

The $K(\alpha)$ entropy is related to what some authors[6–8] call generalized Lyapunov exponents.

An easier way to compute (not necessarily equivalent) definition is

$$K_B(\alpha) = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{1-\alpha}\frac{1}{n}\ln\sum_{i_0\cdots i_{n-1}}(p(i_0\ldots i_{n-1}))^\alpha, \tag{8}$$

$p(i_0\ldots i_{n-1})$ being the joint probability to be at the box i_0 at time 0, to be at box i_1 at time 1 … and to be at box i_{n-1} at time $n-1$, the sum being over all different blocks of length n.

For invariant measures absolutely continuous with respect to Lebesgue, the Rényi entropies may be estimated from the local expansion rate. The fact that the local expansion rate is $\Lambda(x) = \prod_{\lambda_i>0} e^{\lambda_i(x)}$ implies that if the system is in box i_0 at time 0, it can go to $\Lambda(i_0)$ boxes in the next step, then to $\Lambda(i_0)\Lambda(i_1)$ boxes, etc., ($\Lambda(i_k)$ is the average expansion rate in the box i_k and $\mu(i_0)$ the measure of the box i_0). Then,

$$p(i_0\ldots i_{n-1}) = \frac{\mu(i_0)}{\Lambda(i_o)\cdots\Lambda(i_{n-1})},$$

and

$$K_B(\alpha) = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{1-\alpha}\frac{1}{n}\ln\sum_{i_0\cdots i_{n-1}}\left(\frac{\mu(i_0)}{\Lambda(i_o)\cdots\Lambda(i_{n-1})}\right)^\alpha, \tag{9}$$

Taking averages and normalizing to obtain $\sum_{i_0\cdots i_{n-1}} p(i_0\cdots i_{n-1}) = \mu(i_0)$ in the $\alpha \to 1$ limit

$$K_B(\alpha) = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{1-\alpha}\frac{1}{n}\ln\left\langle \mu(i_0)^\alpha\left(\frac{1}{\Lambda(i_o)\cdots\Lambda(i_{n-1})}\right)^{\alpha-1}\right\rangle, \tag{10}$$

and in the $\lim_{n\to\infty}$

$$K_B(\alpha) = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{1-\alpha}\frac{1}{n}\ln\left\langle \exp\left((1-\alpha)\sum_{k=0}^{n-1}\ln\Lambda(i_k)\right)\right\rangle, \tag{11}$$

$(1-\alpha)K(\alpha)$ is the *pressure function* for the random variable $Y_n = \frac{1}{n}\sum_{k=0}^{n-1}\ln\Lambda(i_k)$ and the Legendre transform

$$I(y) = \sup_\alpha\{(1-\alpha)y - (1-\alpha)K(\alpha)\}, \tag{12}$$

is the *deviation function* for the large deviations of the random variable $Y_n = \frac{1}{n}\sum_{k=0}^{n-1} \ln\Lambda(i_k)$, that is

$$P_n\left\{\frac{1}{n}\sum_{k=0}^{n-1} \ln\Lambda(i_k) \in (y, y+dy)\right\} \asymp \exp(-nI(y))dy. \quad (13)$$

In conclusion,

Proposition 2. (i) The Legendre transform of the (box) dynamical Rényi entropy is the deviation function for the local expansion rate.

(ii) If a weak correlation condition is verified, namely,

$$\left\langle \exp\left((1-\alpha)\sum_{k=0}^{n-1} \ln\Lambda(i_k)\right)\right\rangle \prod_{k=0}^{n-1} \langle \exp((1-\alpha)\ln\Lambda(i_k))\rangle^{-1} \leq c_1 e^{c_2 n^\gamma},$$

$c_2 > 0$ and $\gamma < 1$

$$K_B(\alpha) = \lim_{\varepsilon\to 0} \frac{1}{1-\alpha} \ln\langle \exp((1-\alpha)\ln\Lambda(i))\rangle, \quad (14)$$

$$K_B(\alpha) = \lim_{\varepsilon\to 0} \sum_{s=1}^{\infty} k_s(\ln\Lambda)(1-\alpha)^{s-1}, \quad (15)$$

where the $k_s(\ln\Lambda)$ are the cumulants of the local expansion rate.

In its range of validity, Eq. (15) is a generalization of Pesin's formula.[16]

C. Correlation parameters

The Lyapunov fluctuation moments $\chi_i^{(p)}(x)$ or the Lyapunov characteristic fluctuation function $C(\alpha)$, defined before, contain complete information on the statistical properties of the local fluctuation rate. However, a full ergodic characterization of the dynamics should also contain information about correlations at different points. An ergodic characterization of the correlations may be obtained by the construction of **correlation cocycles**.

Let f: $M \to M$ be a measure preserving transformation of a Lebesgue space $(M, \mathcal{B}, \mu)$). For a measurable function $g : M \times M \to GL(N, \mathbb{R})$ let

$$C_k(x, n) = g\left(f^{nk}(x), f^{(n-1)k}(x)\right) \cdots g\left(f^k(x), x\right)$$

Then,

$$C_k(x, n+p) = C_k\left(f^{pk}(x), n\right) C_k(x, p)$$

and $C_k : M \times \mathbb{Z} \to GL(N, \mathbb{R})$ is called a *correlation cocycle* (over f). If $\ln_+ \| g(f^k(x), x) \| \in L^1(M, \mu)$ Oseledets' theorem applies and choosing appropriate functions $g : M \to GL(N, \mathbb{R})$, one obtains *correlation ergodic parameters*.

For example,

$$g\left(f^k(x), x\right) = Df\left(f^k x\right) Df(x) - (Df(x))^2.$$

Notice, also, that a different construction of ergodic parameters[17] through a variational formulation of general maps[18] also contains some information on the correlations, beyond the local fluctuations.

D. Ergodic parameters and measures of complexity

Measures and characterizations of complexity are to a great extent based on the tools and notions of the information theory.[19–21] Here, I will show that complementary notions may be obtained from the invariant parameters of ergodic theory.

A well-known characterization of complexity is the *excess entropy* or *effective measure complexity*. Let, in the time series generated by a dynamical system, $p_N(s_1 \cdots s_n)$ be the probability to find the block $s_1 \cdots s_n$ of size n. Then,

$$H(n) = -\sum_{\{s_i\}} p_n(s_1 \cdots s_n) \ln p_n(s_1 \cdots s_n), \quad (16)$$

and $h_s = \lim_{n\to\infty} \frac{1}{n} H(n)$ is the *Shannon entropy*.

The excess entropy E

$$E = \sum_n (H(n) - H(n-1) - h_s) = \lim_{n\to\infty} (H(n) - nh_s), \quad (17)$$

measures the deviation of the Shannon entropy from its finite n estimates. It may be interpreted either as the effort needed to construct a model of the system or as a measure of the diversity of its correlation structures.

Finite-time fluctuations in the calculation of the Lyapunov exponents are a symptom of the diversity of dynamical structures. This suggests that these fluctuations might provide information on dynamical diversity that is complementary to the correlation diversity provided by the excess entropy. This is achieved by a large deviation reasoning applied to the dynamical Rényi entropy.

As seen before, the Legendre transform $I(y)$ of $(1-\alpha)K(\alpha)$ (Eq. (12)) is the deviation function of the random variable $Y_n = \frac{1}{n}\sum_{k=0}^{n-1} \ln\Lambda(i_k)$. The asymptotic value (when $n \to \infty$) is the value that minimizes $I(y)$ and the probability of a deviation from this value for a sample of size n is

$$P_n(y) = \frac{e^{-nI(y)}}{\int e^{-nI(y)} dy}, \quad (18)$$

the integral in the denominator being over the domain of y, [min ln $\Lambda(i_k)$, max ln $\Lambda(i_k)$]. Outside this interval $I(y) = \infty$ by definition. As far as dynamical diversity is concerned, Y_n plays the same role as $\frac{1}{n}H(n)$ for the entropy. Therefore, the deviation function $I(y)$ contains all the required information about the dynamical diversity and on the nature of the convergence of Y_n to the characteristic exponents. From the analogy $Y_n \leftrightarrow \frac{1}{n}H(n)$, it is a simple matter to write, using Eq. (17), a dynamical version of the excess entropy. However, the deviation function itself is already a compact way to characterize the dynamical structures.

A simple example shows that the information provided by $I(y)$ is complementary to the one provided by the excess entropy. Consider the following map of the unit interval:[22]

$$f(x) = \begin{cases} \dfrac{1-|1-2x|}{1+c} & \text{for} \left|x - \dfrac{1}{2}\right| \geq \dfrac{1-c}{4} \\ 1 - \dfrac{|1-2x|}{1-c} & \text{for} \left|x - \dfrac{1}{2}\right| < \dfrac{1-c}{4} \end{cases}, \quad (19)$$

which has breaks at the first preimages of the maximum point $\frac{1}{2}$ and an invariant measure with density

$$\rho(x) = \begin{cases} 1+c & \text{for} \quad x < \dfrac{1}{2} \\ 1-c & \text{for} \quad x > \dfrac{1}{2} \end{cases}. \tag{20}$$

If one now considers a symbolic dynamics with two symbols,[22] which distinguishes whether the trajectory is below or above $\frac{1}{2}$, one obtains $H(1) = h_s$ and the excess entropy vanishes. However, the deviation function $I(y)$ is sensitive to the fact that there are two different expansion rates $\Lambda_1 = \frac{2}{1+c}$ and $\Lambda_2 = \frac{2}{1-c}$. Therefore,

$$\lim_{n\to\infty} Y_n = h_s = \frac{1+c}{2}\ln\frac{2}{1+c} + \frac{1-c}{2}\ln\frac{2}{1-c}, \tag{21}$$

but the deviation function is non-trivial

$$I(y) = \frac{y - \ln\Lambda_1}{\ln\Lambda_2 - \ln\Lambda_1}\ln\frac{2(y-\ln\Lambda_1)}{(1-c)(\ln\Lambda_2 - \ln\Lambda_1)} + \frac{\ln\Lambda_2 - y}{\ln\Lambda_2 - \ln\Lambda_1}\ln\frac{2(\ln\Lambda_2 - y)}{(1+c)(\ln\Lambda_2 - \ln\Lambda_1)}, \tag{22}$$

allowing to characterize the fluctuations of the expansion rate.

For correlated systems, the deviation function is in general harder to compute analytically, but there is a robust algorithm to compute it numerically from the data generated by the system.[23]

III. APPLICATIONS

To illustrate how the ergodic parameters may be used to characterize the complexity and other features of the dynamical systems, I will briefly review here some of the applications that have been done in the past. These tools are now being also used to characterize the clustering and metastability in systems with long-range interactions. The applications covered here concern mostly the spectrum of Lyapunov and conditional exponents. Full exploration of the new ergodic parameters proposed in this paper will be carried out in the future.

A. Structure and self-organization

The Lyapunov and conditional exponents spectra provide tools to characterize the creation of structures and self-organization.[3,24–26]

A structure (in a collective system) is a phenomenon that operates at a scale very different from the scale of the component units in the system. A structure in space is a feature at a length scale larger than the characteristic size of the components and a structure in time is a phenomenon with a time scale larger than the cycle time of the individual components.

Structures are collective motions of the system. Therefore, their characteristic times are the characteristic times of the separation dynamics, that is, the inverse of the positive Lyapunov exponents. Hence, for a N–dimensional system with N_+ positive Lyapunov exponents a (temporal) **structure index** S may be defined by

$$S = \frac{1}{N}\sum_{i=1}^{N_+}\left(\frac{\lambda_0}{\lambda_i} - 1\right), \tag{23}$$

the sum being over the positive Lyapunov exponents λ_i. λ_0 is the largest Lyapunov exponent of an isolated component or some other reference value.

Under a change of parameters, the temporal structure index diverges whenever a Lyapunov exponent approaches zero from above. Therefore, the index diverges at the points where long time correlations develop.

In a multi-agent system, a characterization of self-organization is obtained by the comparison of the global dynamics with their local image, that is, by comparing Lyapunov and conditional exponents.

For an invariant measure μ absolutely continuous with respect to the Lebesgue measure of M or for measures that are smooth along unstable directions (BRS measures), Pesin's identity[16] states that the sum over positive Lyapunov exponents coincides with the Kolmogorov-Sinai entropy. By analogy, one defines the *conditional exponent entropies* associated with the splitting $R^k \times R^{m-k}$ of the phase space as

$$h_k(\mu) = \sum_{\xi_i^{(k)}>0} \xi_i^{(k)}, \tag{24}$$

$$h_{m-k}(\mu) = \sum_{\xi_i^{(m-k)}>0} \xi_i^{(m-k)}, \tag{25}$$

the ξ_i's being the conditional exponents. These quantities, defined in terms of the conditional exponents, are well-defined ergodic invariants. In information theory, the mutual information $I(A: B)$ is

$$I(A:B) = S(A) + S(B) - S(A+B) \tag{26}$$

By analogy, one defines a **measure of dynamical self-organization** $I(S, \Sigma, \mu)$

$$I(S,\Sigma,\mu) = \sum_{k=1}^{N}\{h_k(\mu) + h_{m-k}(\mu) - h(\mu)\} \tag{27}$$

the sum being over all relevant partitions $R^k \times R^{m-k}$ and $h(\mu)$ the sum of the positive Lyapunov exponents

$$h(\mu) = \sum_{\lambda_i>0}\lambda_i \tag{28}$$

$I(S, \Sigma, \mu)$ is also a well-defined ergodic invariant for the measure μ.

The Lyapunov exponents of a dynamical system measure the rate of information production or, from an alternative point of view, they define the dynamical freedom of the system, in the sense that they control the amount of change that is needed today to have an effect on the future. In this sense, the larger a Lyapunov exponent is, the freer the system is in that particular direction, because a very small change in the present state will induce a large change in the future. From the point of view of the unit k and of the remaining subsystem, the quantity $h_k(\mu)+h_{m-k}(\mu)$ is therefore the apparent dynamical freedom that they possess (or the apparent rate of information production). The actual rate is in fact $h(\mu)$. Hence, $I(S, \Sigma, \mu)$ is a measure of the apparent excess dynamical freedom.

The behavior of the structure index S and the measure of dynamical self-organization $I(S, \Sigma, \mu)$ is now illustrated in a simple example. Let

$$x_i(t+1) = (1-c)f(x_i(t)) + \frac{c}{N-1}\sum_{k\neq i} f(x_k(t)), \tag{29}$$

with $f(x) = 2x$ (mod.1). As shown in Fig. 1, there is a synchronization transition at $c = 0.5$. In Fig. 2, one sees that this mode change is well identified by S and $I(S, \Sigma, \mu)$.

B. Synchronization and clustering

Ergodic parameters, in particular, the Lyapunov spectrum, characterize the synchronization in multi-agent systems, but, even more importantly, provide a way to understand the subtler correlation and clustering effects that occur before synchronization. This is illustrated by a discrete-time oscillators model with piecewise linear interactions.[27]

$$x_i(t+1) = x_i(t) + \omega_i + \frac{k}{N-1}\sum_{j=1}^{N} f_\alpha(x_j - x_i), \tag{30}$$

with a heavy-tailed distribution of oscillator frequencies

$$p(\omega) = \frac{\gamma}{\pi\left[\gamma^2 + (\omega-\omega_0)^2\right]}, \tag{31}$$

and

$$f_\alpha(x_j - x_i) = \alpha(x_j - x_i)(mod.1). \tag{32}$$

As in the classical Kuramoto model, one defines a synchronization order parameter

$$r(t) = \left|\frac{1}{N}\sum_{j=1}^{N} e^{i2\pi x_j(t)}\right|. \tag{33}$$

When the parameter k changes, one sees by direct numerical simulation that the behavior is very similar to the one in the Kuramoto model. For small k values, the system looks disorganized and the order parameter is very small (Fig. 3), whereas for large k there is synchronization of most oscillators (Fig. 4) and the order parameter is close to one.

However, and in contrast with the Kuramoto model, the Lyapunov spectrum may be computed analytically, namely,

$$\begin{aligned} &\lambda_1 = 0. \\ &\lambda_i = \log\left(1-\alpha k\left(\frac{N}{N-1}\right)\right) \qquad (N-1) \text{ times.} \end{aligned} \tag{34}$$

That is, there are $N-1$ contracting directions for any $k \neq 0$. From the ergodic point of view for any arbitrarily small value of k, the system converges to an effective *one-dimensional system*. In this sense, synchronization is not the most

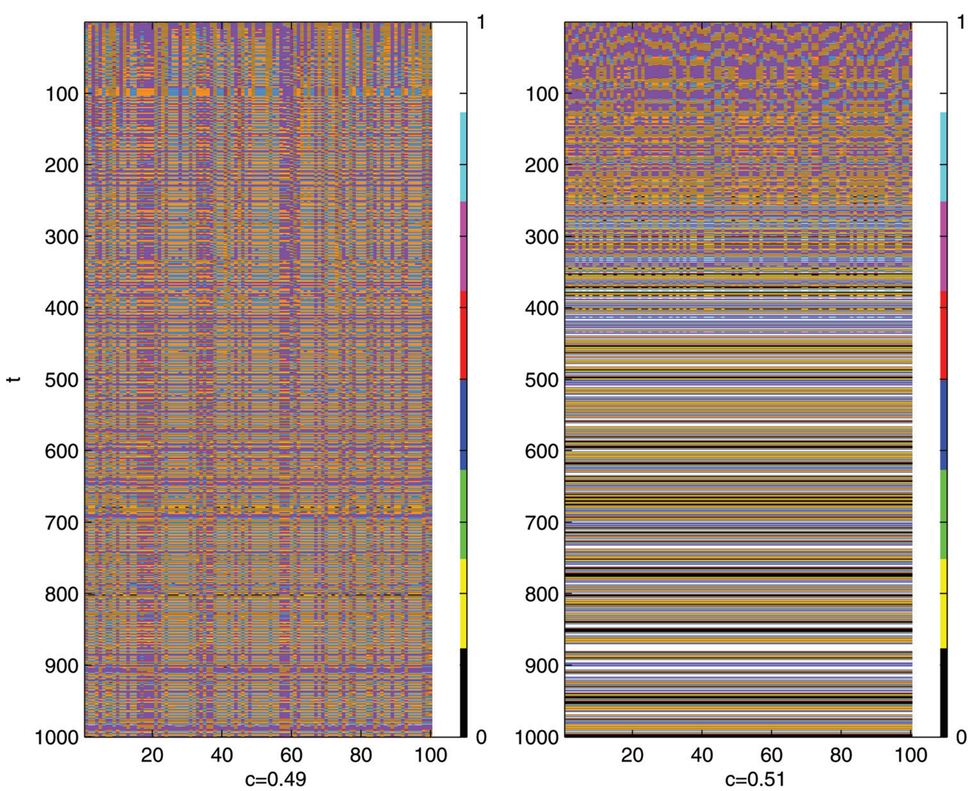

FIG. 1. (Color) The time evolution, starting from random initial conditions, of the multi-agent system (29) for two parameter values.

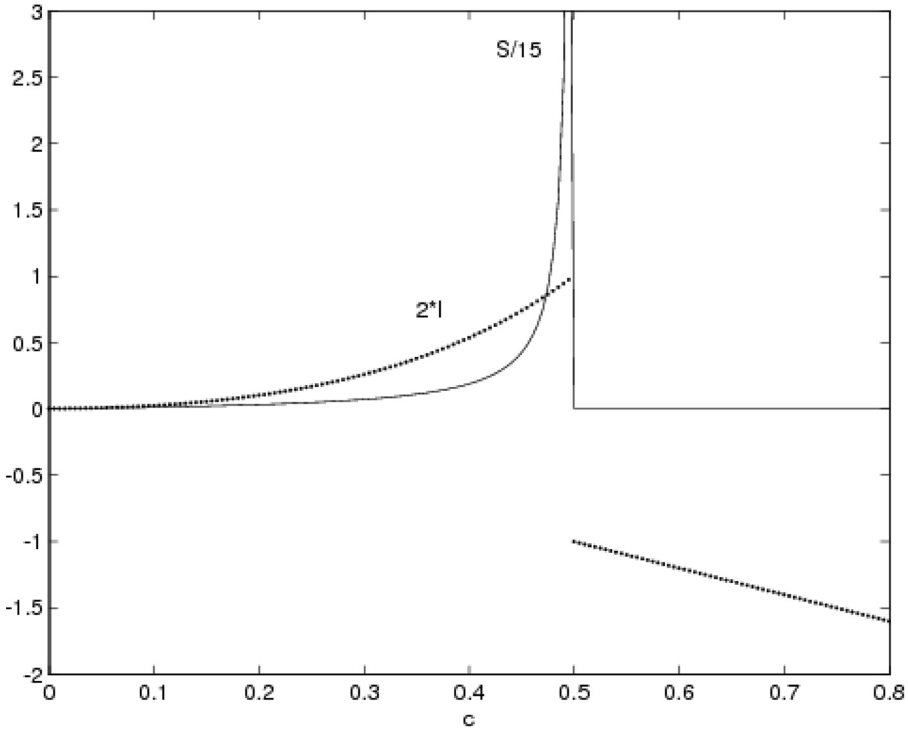

FIG. 2. The structure index and the measure of dynamical self-organization for the multi-agent system (29).

important effect, being simply a rough version of the kind of subtler correlations that already occur for any small k. This is illustrated in the Fig. 5, which shows the time evolution of $x_1 - x_2$ and $x_1 + x_2 - 2x_3$. One sees that these compound variables trace periodic trajectories, complex but neither chaotic nor disorganized. In conclusion, the oscillator system is fully enslaved by dynamical correlations even before synchronization.

C. Dynamical characterization of network topology

In contrast with purely random networks, many networks in nature display both short path lengths and high clustering. Watts and Strogatz[28] have illustrated this phenomenon with a simple model (Fig. 6). They start from a regular network which by successive rewiring becomes increasingly randomized. They notice that for a randomness range up to $\beta \approx 0.4$, the network displays both short path

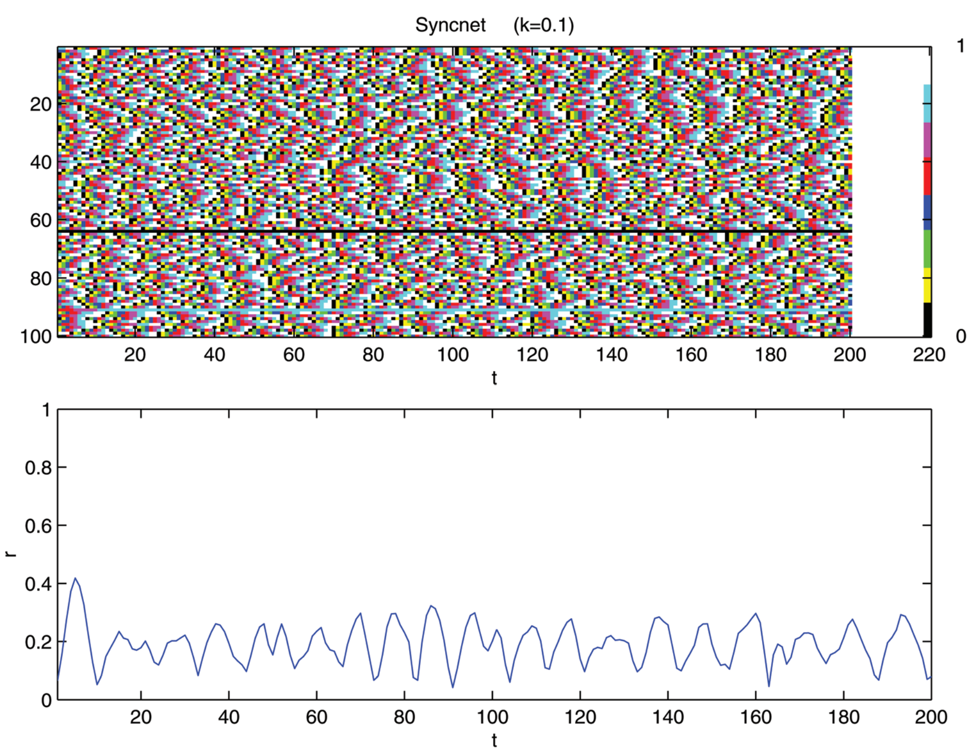

FIG. 3. (Color) Time evolution and order parameter of the coupled oscillators model (30) for $k = 0.1$.

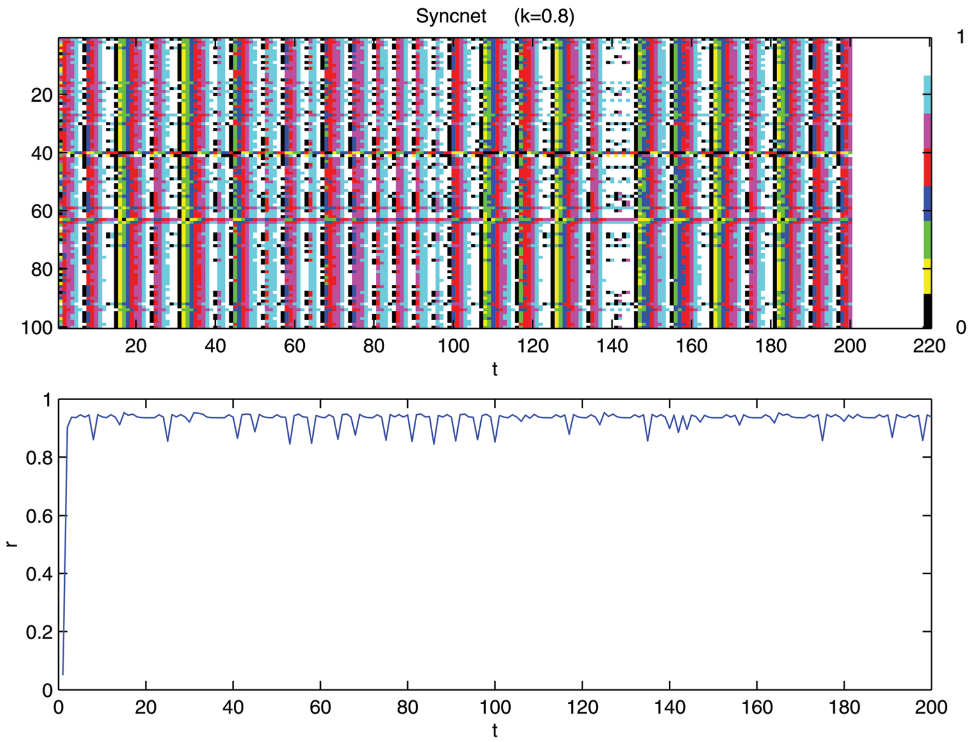

FIG. 4. (Color) Time evolution and order parameter of the coupled oscillators model (30) for $k = 0.8$.

length and high clustering. This region has been called the "small-world" region. For higher values of β, the clustering has a fast decrease and the network displays all the attributes of a random network. A natural question arising from this model is whether the small-world region may be considered as a "phase" in the statistical mechanics sense and whether the small-world to random passage is a "phase transition." This question may be addressed by defining a dynamical system on the network and studying the behavior of its ergodic parameters.[29]

On each one of the β–networks, a dynamical system is defined, with a map at each node and convex-coupling interactions defined by the network connections

$$x_i(t+1) = \sum_{j=1}^{N} W_{ij} f\left(x_j(t)\right), \tag{35}$$

where

$$W_{ij} = \begin{cases} 1 - \frac{n_v(i)}{2v} c & \text{if } i = j \\ \frac{c}{2v} & \text{if } i \neq j \quad \text{and} \quad i \quad \text{is connected to} \quad j, \\ 0 & \text{otherwise} \end{cases} \tag{36}$$

$n_v(i)$ is the number of agents connected to i and c is a control parameter.

For the agent dynamics, one chooses

$$f(x) = \alpha x \qquad mod.1, \tag{37}$$

Typically $\alpha = 2$.

The transition from the small-world to the random region is now studied by comparing the Lyapunov exponents entropy with the conditional exponents entropies. For each agent i, we consider a subblock of dimension $d_i \times d_i$ formed by itself and those that are connected to it. The positive conditional exponents $\lambda^*_\beta(j)$ associated with each subblock are computed and a dimension-weighed sum is performed over all subblocks. This gives a version of what in subsection III A has been called a *conditional exponents entropy*.

$$h^*_\beta = \sum_{i=1}^{N} \left(\frac{1}{d_i} \sum_{\lambda^*_\beta > 0} \lambda^*_\beta(j) \right) \tag{38}$$

Subtracting h^*_β from the sum of the positive Lyapunov exponents, $h_\beta = \sum_{\lambda_\beta > 0} \lambda_\beta(j)$, one defines the coefficient

$$C_\beta = \left| \frac{h^*_0 - h_0}{h^*_\beta - h_\beta} \right| \tag{39}$$

This coefficient has the following dynamical interpretation: The Lyapunov exponents measure the rate of information production and also define the dynamical freedom of the system, in the sense that they control the amount of change that is needed today to have an effect on the future. In this sense, the larger a Lyapunov exponent is, the freer the system is in that particular direction, because a very small change in the present state will induce a large change in the future. The conditional exponents have a similar interpretation concerning the dynamics as seen from the point of view of each agent and his neighborhood. However, the actual information production rate is given by the sum of the positive Lyapunov exponents, not by the sum of the conditional exponents. Therefore, the quantity $h^*_\beta - h_\beta$ is a measure of apparent dynamical freedom (or apparent rate of information production).

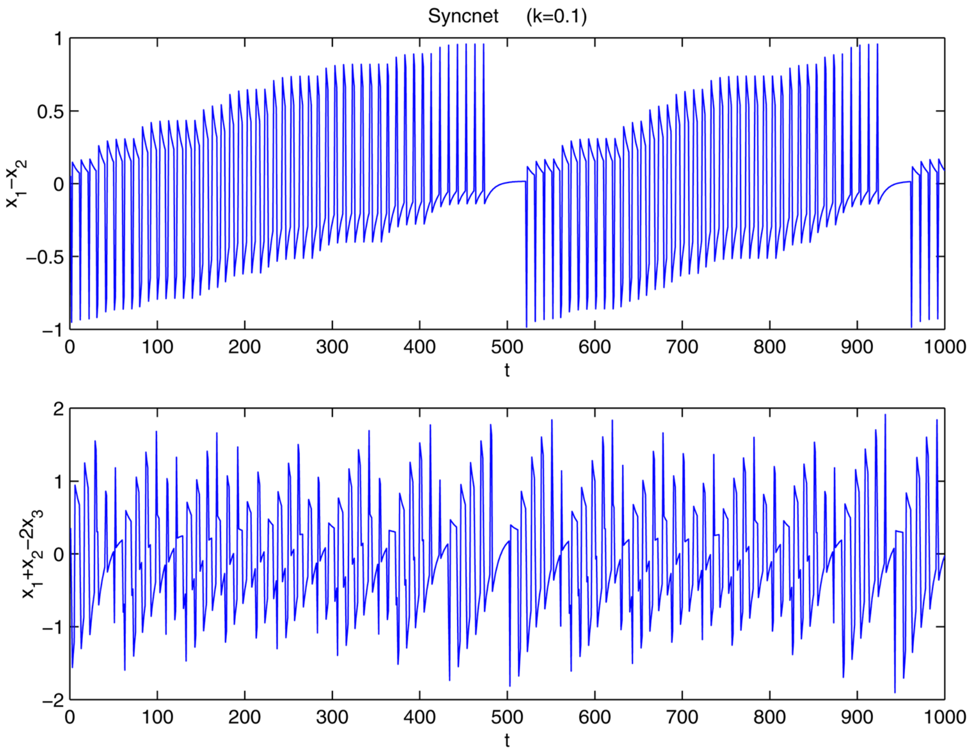

FIG. 5. (Color online) Time evolution of $x_1 - x_2$ and $x_1 + x_2 - 2x_3$ in the oscillators model (30) for $k = 0.1$.

In Fig. 7, one shows the average values of C_β taken over 100 different samples for each β (with $2\nu = 6$ as the average degree of the network and $N = 100$, 200, 400, 600, and 800). Notice that the N-independence of C_β which follows from the fact that, in Eq. (39), it is defined as a ratio of two quantities with the same N-dependence. For small β values, the difference between the entropy and the conditional exponents entropy is a small quantity. It means that each agent may have exact information on the global behavior from observation of his own neighborhood. When β increases the difference changes sign and becomes very large, meaning that the neighborhood information has ceased to provide reliable information on the global dynamics of the network. This is the dynamical correlate of the decreasing cluster properties and

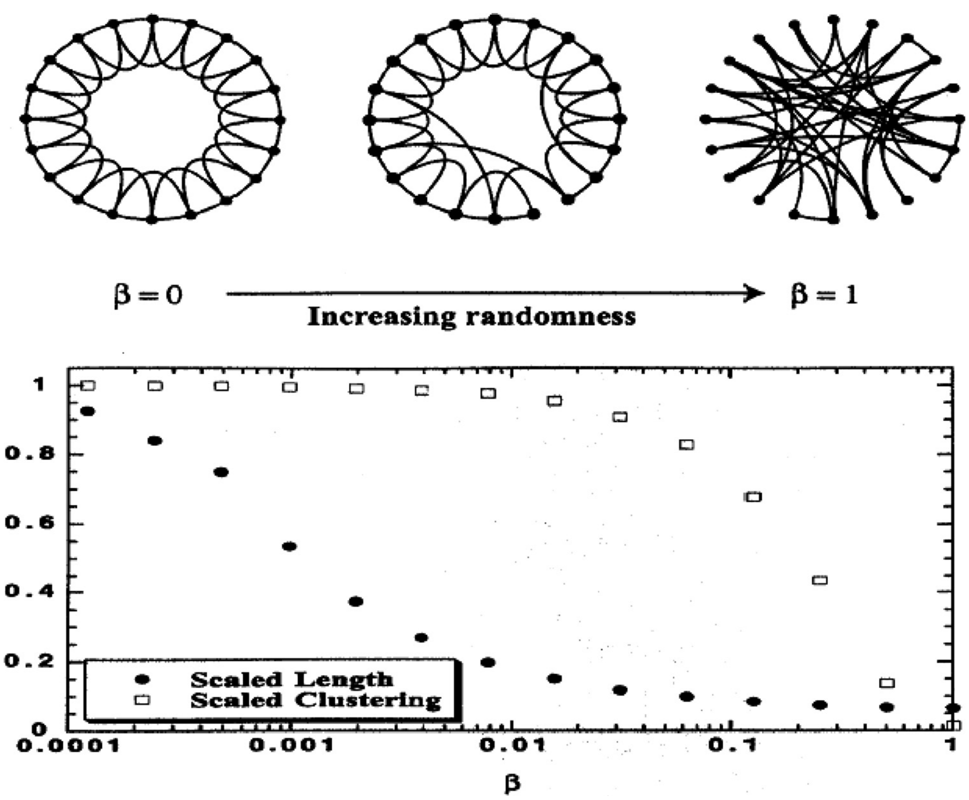

FIG. 6. The "small-world" model of Watts and Strogatz.

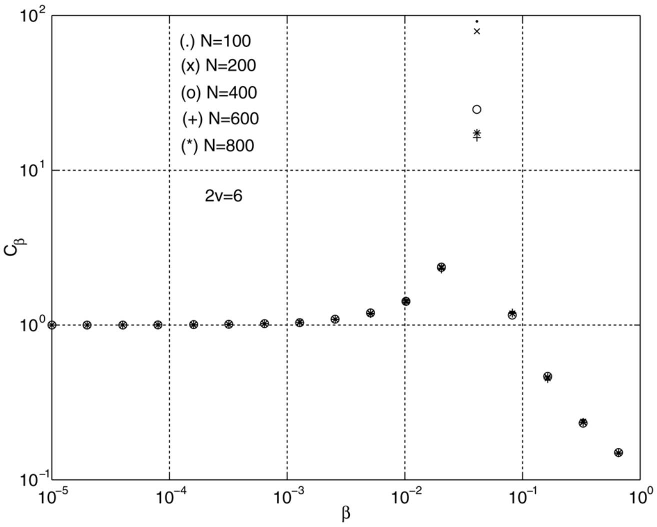

FIG. 7. The small world—random transition as seen by the mismatch of Lyapunov and conditional exponents.

allows us to define the transition at the divergence point β_{c2} of C_β. One finds

$$\beta_{c_2} \simeq 0.04. \tag{40}$$

Near the transition region

$$C_\beta \sim \left|\beta - \beta_{c_2}\right|^{-\eta_2}, \tag{41}$$

with $\eta_2 \simeq 1.14$ below the transition and $\eta_2 \simeq 0.93$ above it.

It is the mismatch between local and global dynamics that defines the transition. For other details and also for the characterization of the regular—small world transition refer to Ref. 29.

D. Ergodic parameters and self-organized criticality

Self-organized criticality (SOC) is a property of dynamical systems, which, without the need to tune control parameters to precise values, display the spatial or the temporal behavior characteristic of phase transition critical points. Here, only the absence of natural time scales is emphasized, without much concern about space scales, nature of the driving, separation of time scales and other relevant issues, useful for a precise characterization of SOC.

Absence of time scales, as a property of dynamics, is naturally related to the Lyapunov spectrum. Time scales disappear whenever the Lyapunov exponents vanish. This may lead to power laws for both time and space correlation.

Most SOC models display unstable behavior of the local dynamics and extremal dynamics. In fact, these two features are sufficient conditions for SOC behavior.

Theorem: *If the single-agent dynamics has positive Lyapunov exponents and the global dynamics is extremal with finite range, then, in the* $N \to \infty$ *limit, the Lyapunov spectrum converges to* 0^+

Proof: In the $T \to \infty$ limit, used to compute the Lyapunov spectrum, the tangent maps have only a nontrivial finite size block during an average time of order $(2r+1)\frac{T}{N}$, $2r+1$ being the number of agents which move at each time step. Hence, in the $N \to \infty$, limit the Lyapunov spectrum converges to 0^+, that is, there are no dynamical scales. Thus in the $N \to \infty$ limit, the system is "tuned" to SOC.

This sufficient condition for SOC has been illustrated[30] in a "detuned" version of the Bak-Sneppen model,[31] which is a deterministic version of a model proposed by Head.[32] The dynamics is defined by

$$x_i(t+1) = \Gamma_i(\vec{x})x_i(t) + (1-\Gamma_i(\vec{x}))f(x_i(t)), \tag{42}$$

where $f(x) = kx\ (mod.\ 1)$ and $\Gamma_i(\vec{x})$ is nearly zero if i corresponds to the minimum x value or to one of its $2n_v$ neighbors and is nearly one otherwise. For example,

$$\Gamma_i(\vec{x}) = \prod_{j=i-n_v}^{j=i+n_v}\left(1 - \frac{e^{-\frac{x_j}{T}}}{\sum_{k=1}^{N} e^{-\frac{x_k}{T}}}\right). \tag{43}$$

This model may be considered as a "finite temperature" Bak-Sneppen model. Direct simulation shows that it displays scaling laws only in the $T \to 0$ limit. Computation of the Lyapunov spectrum, as shown in the Fig. 8, illustrates the theorem. As $T \to 0$, the Lyapunov spectrum tends to 0^+ and it only then that scaling laws are obtained. The small deviation from 0^+ and the dispersion of the Lyapunov spectrum that one sees in the figure, arise from the fact that the calculation is made with a finite number of agents, emphasizing the fact that, also according to the theorem, exact SOC is a $N \to \infty$ effect.

Ergodic theory also provides some insight on the nature of the SOC "attractor." Let the multi-agent system be formulated as a measure-preserving dynamical system. If the return set A, which serves as reference for the counting of the return time (avalanche size) k, has non-zero measure ($\mu(A) \neq 0$), the measure $\mu(A)$ itself serves as a natural time scale. Then, the large time behavior of the return times distribution would be dominated by an exponential factor $\exp(-k\mu(A))$. Therefore, for cases that fit in the ergodic dynamical systems setting, power laws may occur only if the return set has vanishing measure. In the $\mu(A) \to 0$ limit, the return times distribution is dominated by the pre-factor that multiplies the exponential and this one might be or not be a power law.

When μ is an ergodic measure, by Kac's lemma, the mean return time to a set A is $1/\mu(A)$. Therefore when $\mu(A) = 0$, the numerical evaluation of the return time (avalanche) law has to be carried out for a set slightly larger than A. This implies that it may be difficult to disentangle the prefactor dependence from the exponential one, leading to some uncertainty about the exact values of numerically measured scaling exponents.

The dynamical zero-measure nature of the Bak-Sneppen return set as $T \to 0$ has been put in evidence in Ref. 30. The Bak-Sneppen self-organized return set is an N–dimensional hypercube of space-volume $(1\text{–}0.667)^N$. This set has repelling directions corresponding to the agents that are active and neutral directions for all others. Not being an invariant set, it falls outside the usual definition of "weak repeller." It has been called a "ghost weak repeller" in Ref. 30.

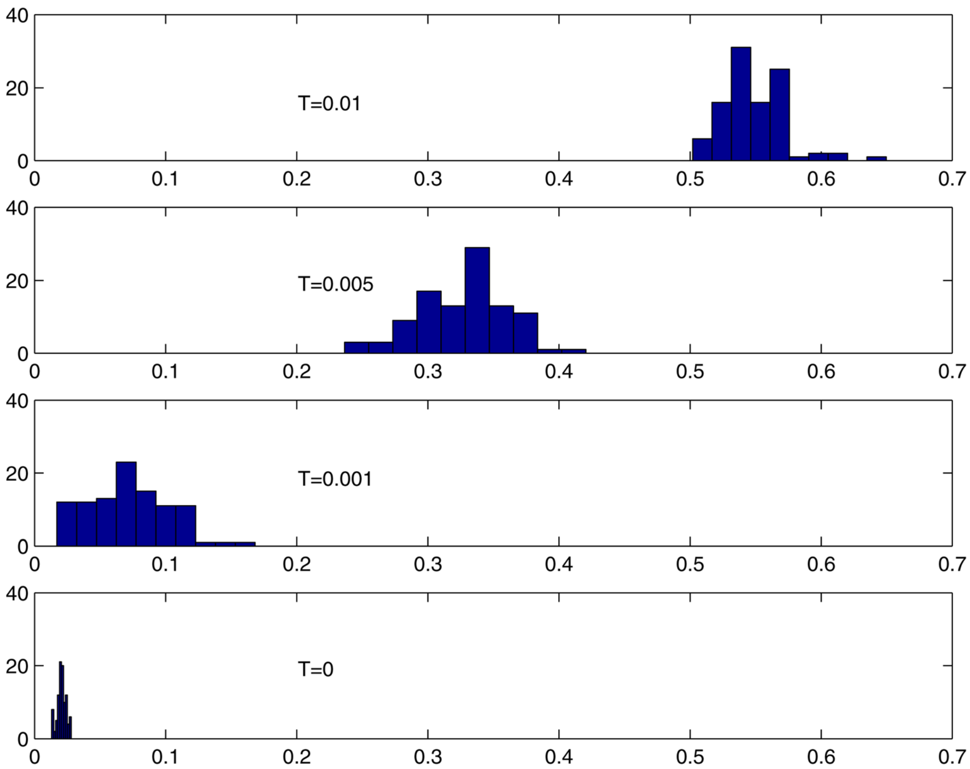

FIG. 8. (Color online) Lyapunov spectrum of the "finite-temperature" Bak-Sneppen model.

IV. FINAL REMARKS

In this paper, two different issues were addressed. First, how to develop a unified formulation to obtain ergodic parameters beyond the Lyapunov and conditional exponents, not only to characterize the local fluctuations of the expansion rate but also the multi-time correlations. Second, how the ergodic parameters may be used to characterize the relevant features of dynamical systems such as clustering, synchronization, criticality, and topological structure.

The applications that were reviewed concern mostly the use of the Lyapnov and conditional exponents spectrum. Practical exploration of the ergodic parameters related to fluctuations of the expansion rate deserves further study, in particular in relation to the role they might play as complexity indicators complementary to those obtained from information theory.

Dealing with fluctuations of the expansion rate and the extension of the usual ergodic parameters, the emphasis is clearly on the complexity of the time evolution of the system. However, for spatially extended systems, the statistics of spatial nonuniformities is a subject of great interest. The structure of space-time measures and the extension of the corresponding ergodic parameters is also a subject that deserves exploration.

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[1] A. M. Lyapunov, *General Problem of Stability of Motion* (Annals of Mathematics Studies no. 17, Princeton University, Princeton, NJ, 1947).
[2] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **64**, 821 (1990); Phys. Rev. A **44**, 2374 (1991).
[3] R. Vilela Mendes, Phys. Lett. A **248**, 167 (1998).
[4] D. Ruelle, *Theory and Experiment in the Ergodic Study of Chaos and Strange Attractors*, in *Proceedings of the VIII International Congress on Mathematical Physics*, edited by M. Mebkhout and R. Sénéor (World Scientific, Singapore, 1987), pp. 273–282.
[5] V. I. Oseledets, Trans. Moscow Math. Soc. **19**, 197 (1968).
[6] H. Fujisaka, Prog. Theor. Phys. **70**, 1264 (1983).
[7] R. Benzi, G. Paladin, G. Parisi, and A. Vulpiani, J. Phys. A **18**, 2157 (1985).
[8] P. Grassberger and I. Procaccia, Physica D **13**, 34 (1984).
[9] P. Grassberger and I. Procaccia, Phys. Rev. A **28**, 2591 (1983).
[10] J.-P. Eckmann and I. Procaccia, Phys. Rev. A **34**, 659 (1986).
[11] A. Crisanti, G. Paladin, and A. Vulpiani, J. Stat. Phys. **53**, 583 (1988).
[12] G. Paladini and S. Vaienti, J. Phys. A: Math. Gen. **21**, 4609 (1988).
[13] C. Froeschlé, Ch. Froeschlé, and E. Lohinger, Class. Mech. Dyn. Astron. **56**, 307 (1993).
[14] J. Vanneste, Phys. Rev E **81**, 036701 (2010).
[15] M. J. de Oliveira and A. Petri, Phys. Rev. E **53**, 2960 (1996).
[16] Y. B. Pesin, Russian Math. Surveys **32**, 55 (1977).
[17] A. Carreira, M. O. Hongler, and R. Vilela Mendes, Phys. Lett. A **155**, 388 (1991).
[18] R. Vilela Mendes, Phys. Lett. A **104**, 391 (1984).
[19] C. R. Shalizi and J. P. Crutchfield, J. Stat. Phys. **104**, 817 (2001).
[20] C. J. Ellison, J. R. Mahoney, and J. P. Crutchfield, J. Stat. Phys. **136**, 1005 (2009).
[21] C. R. Shalizi, K. L. Shalizi, and R. Haslinger, Phys. Rev. Lett. **93**, 118701 (2004).
[22] G. Györgyi and S. Szépfaluzy, Phys. Rev. A **31**, 3477 (1985).
[23] J. Seixas and R. Vilela Mendes, Nucl. Phys. B **383**, 622 (1992).
[24] R. Vilela Mendes, Phys. Lett. A **257**, 132 (1999).
[25] R. Vilela Mendes, Physica A **276**, 550 (2000).
[26] R. Vilela Mendes, Physica A **295**, 537 (2001).
[27] R. Vilela Mendes, Int. J. Bifurcation Chaos **15**, 1185 (2005).
[28] D. J. Watts and S. H. Strogatz, Nature (London) **393**, 409 (1998).
[29] T. Araújo, J. Seixas, and R. Vilela Mendes, Phys. Lett. A **319**, 285 (2003).
[30] R. Vilela Mendes, Physica D **214**, 182 (2006).
[31] P. Bak and K. Sneppen, Phys. Rev. Lett. **71**, 4083 (1993).
[32] D. Head, Eur. Phys. J. B **17**, 289 (2000).