Statistical analysis of time series: Gibbs measures and chains with complete connections

Rui Vilela Mendes

Statistical analysis of time series data

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$Y^{\mathbb{Z}}$: the *path space*

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  (2) Probability measures on state space Y
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- Examples:
  Hydrodynamic turbulence
  Market fluctuations:
  (there are analogies but the statistical indicators are different)

Statistical analysis of time series data

- **Working hypothesis**: statistical methods are an appropriate tool to describe and reconstruct the market fluctuation process
  Related to modern view of the *efficient market* (expected value of abnormal returns is zero - Fama)
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  However: Behavioral trends not inconsistent with statistical description if the different reaction times of the market components as well as secondary reactions are taken into account (Olsen et al.)

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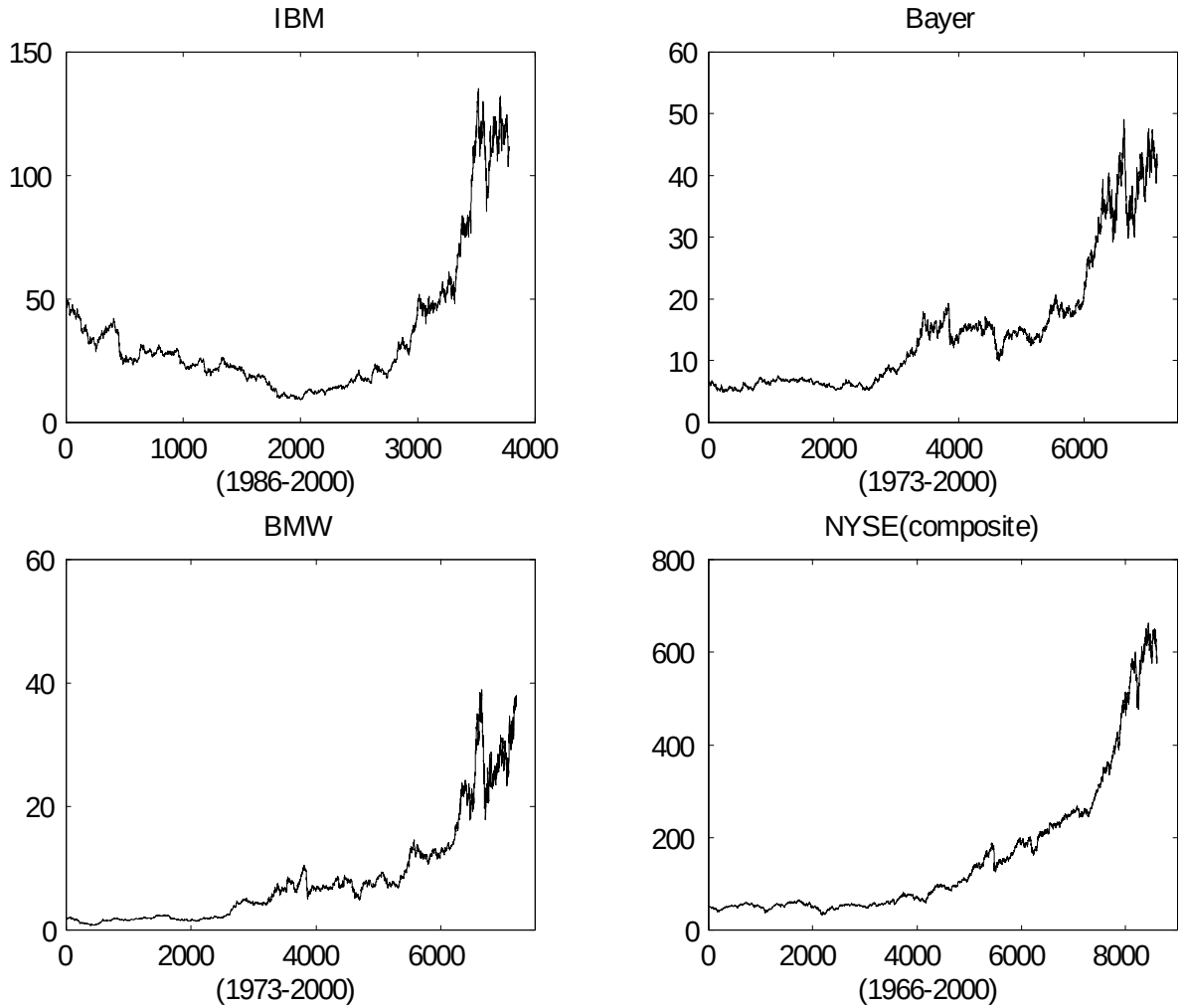
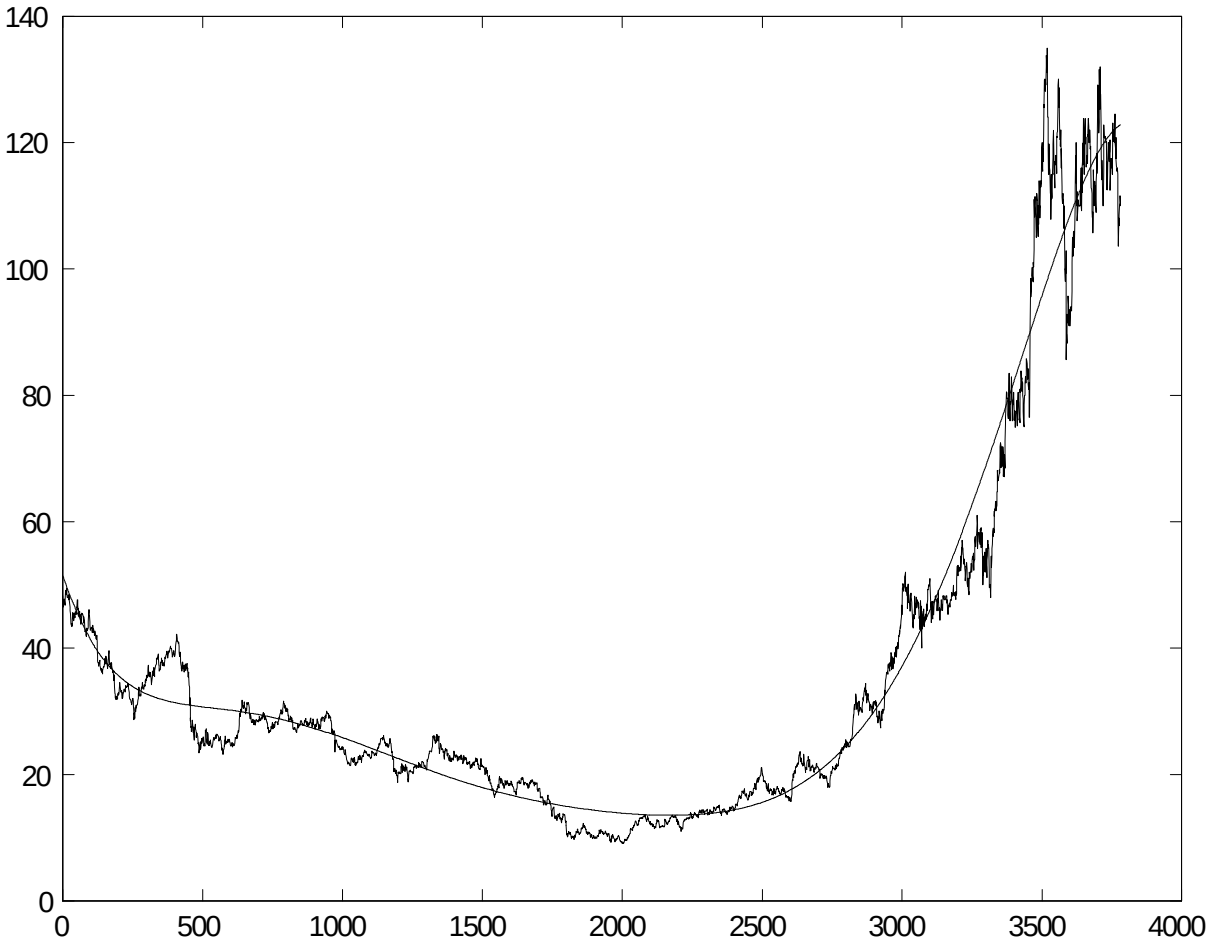
  (ii) Typical samples
- Stocks as experimental probes revealing the mechanisms of the market process
  (i) $\Rightarrow$ preprocessing of the data
  High-frequency versus low-frequency data
  Complexity versus statistics trade-off

Statistical analysis of time series data

Daily data $p(t)$

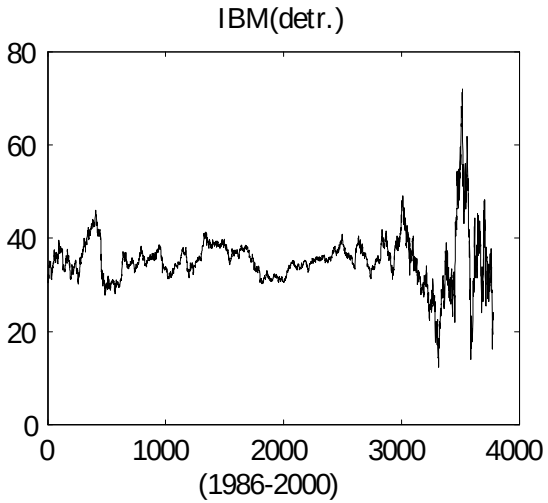


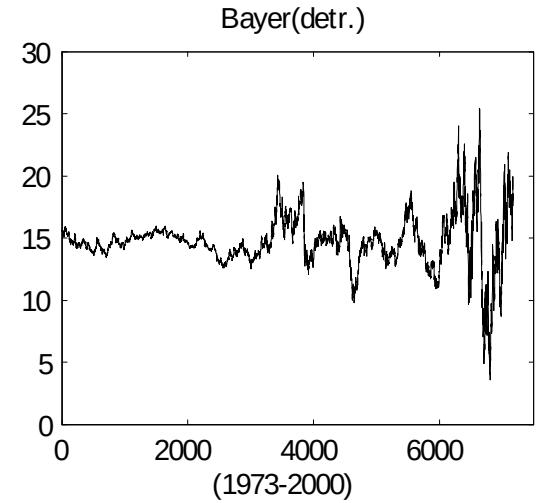
Detrending by a polynomial

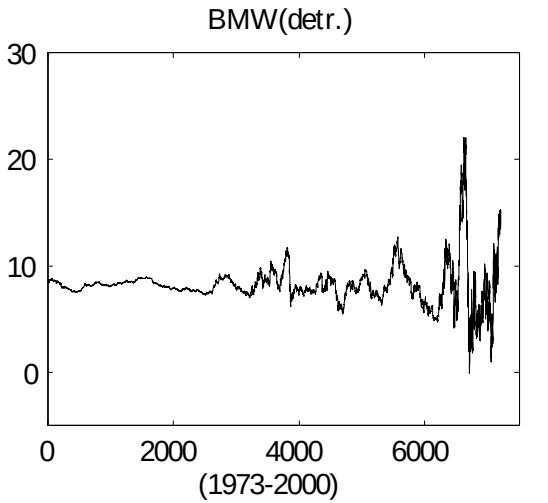


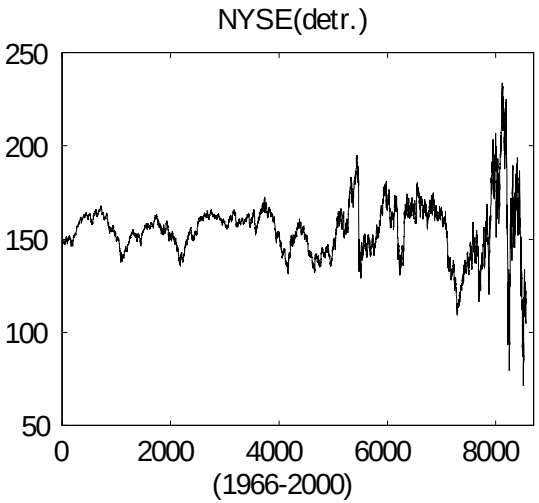
Statistical analysis of time series data

Detrended data $p(t) - q(t)$



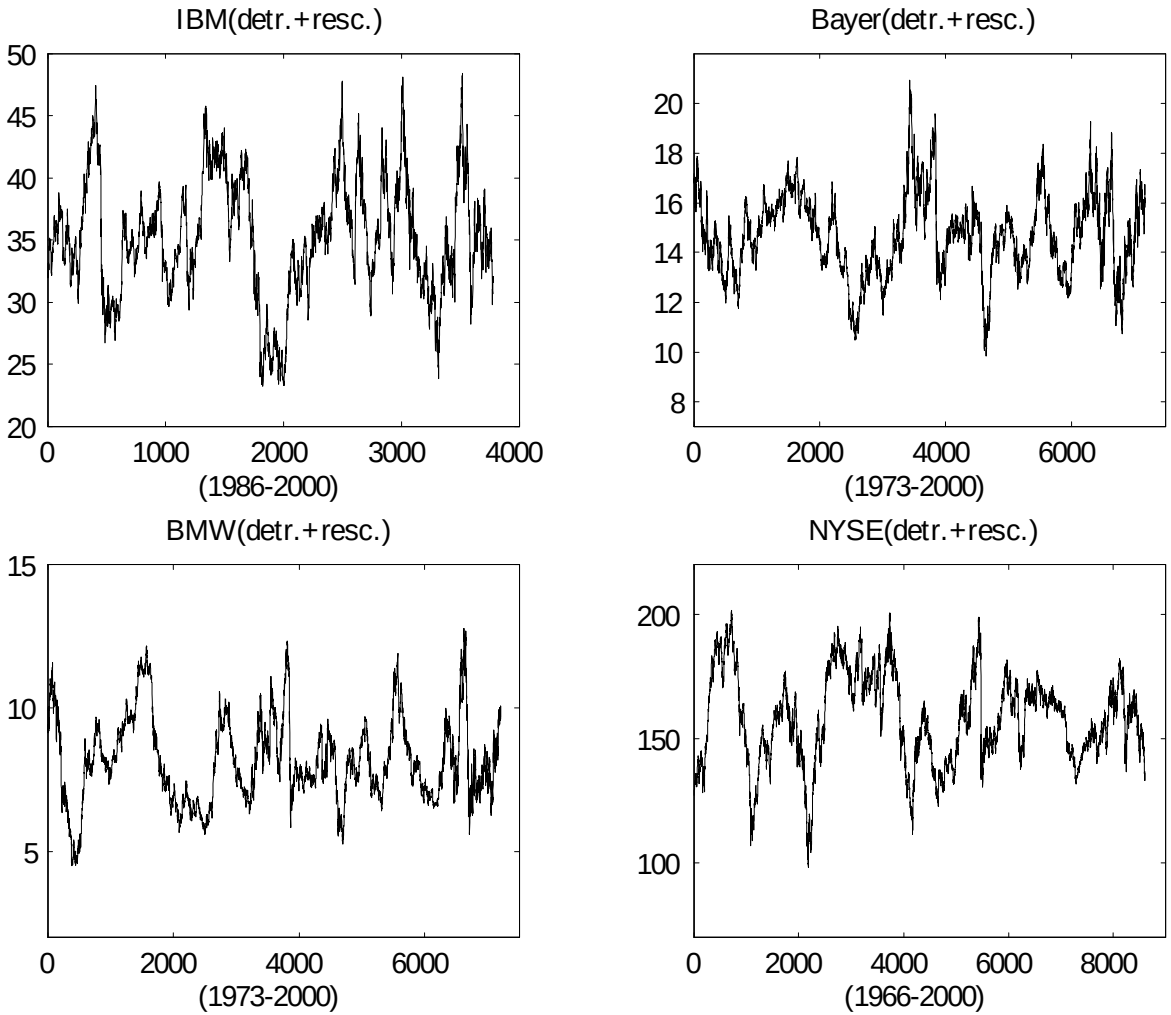






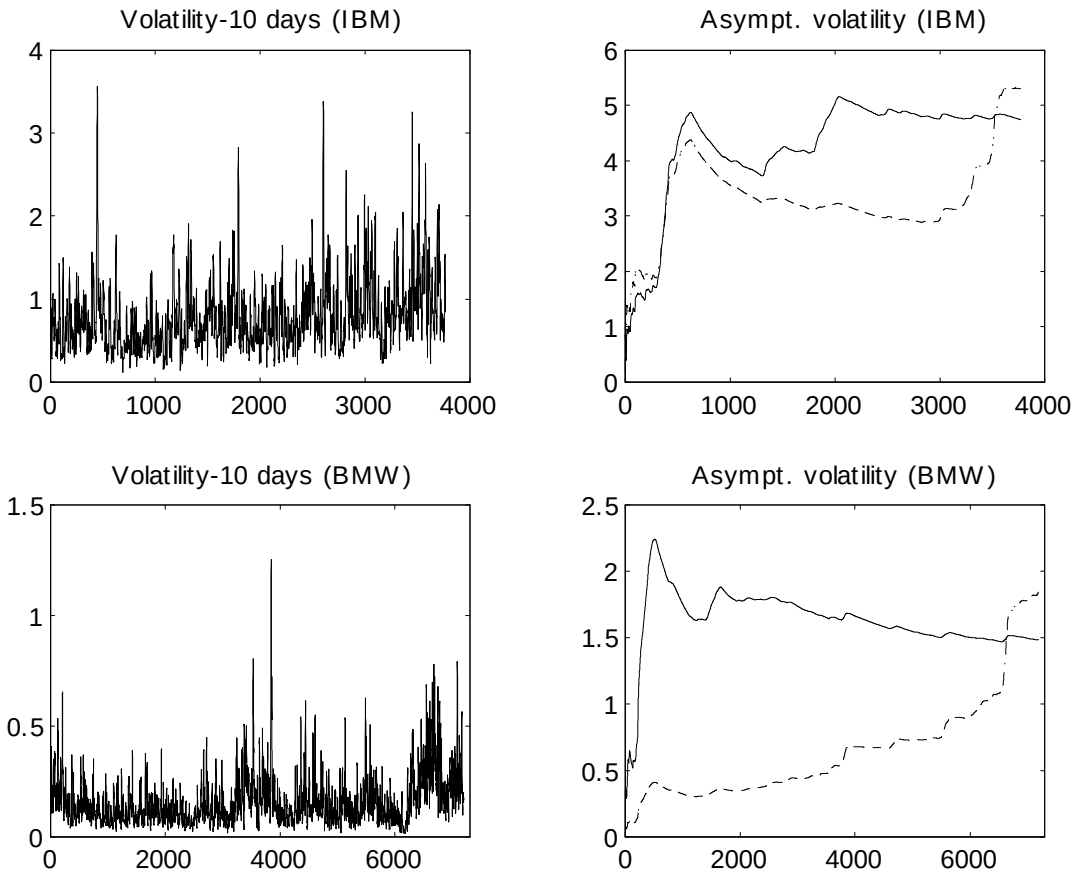
Statistical analysis of time series data

Detrended and rescaled data $x(t) = (p(t) - q(t)) \frac{<p(t)>}{q(t)}$



Statistical analysis of time series data

Ten-days window volatility and comparison of asymptotic volatility for the rescaled and non-rescaled data (IBM and BMW)



Local non-stationarity versus asymptotic stationarity

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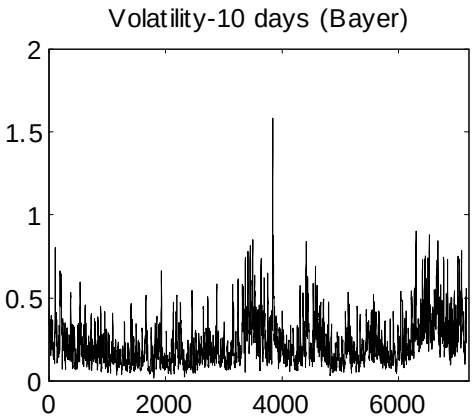

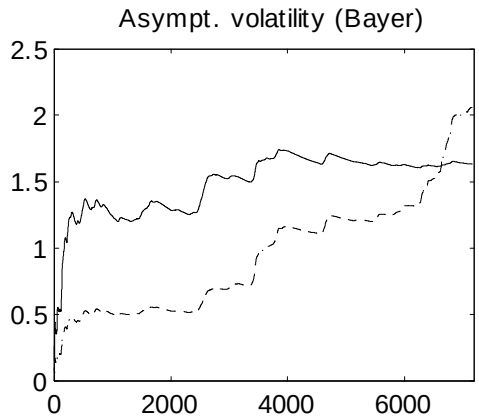

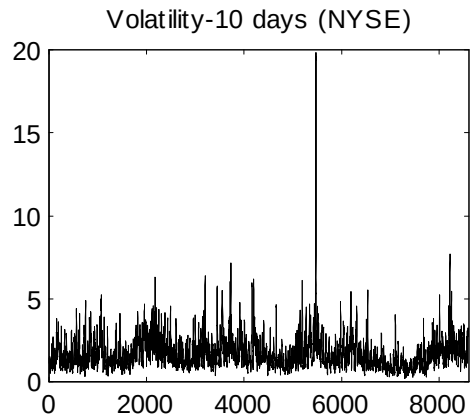

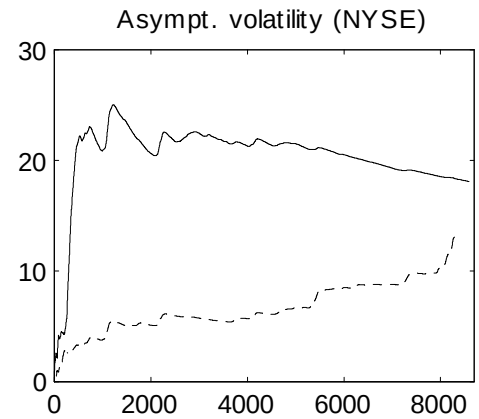

Also direct test of stationarity computing the entropies of multi-symbol words in the first and second half of the samples

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- $n-$days return

$$r(t,n) = \log p(t+n) - \log p(t)$$

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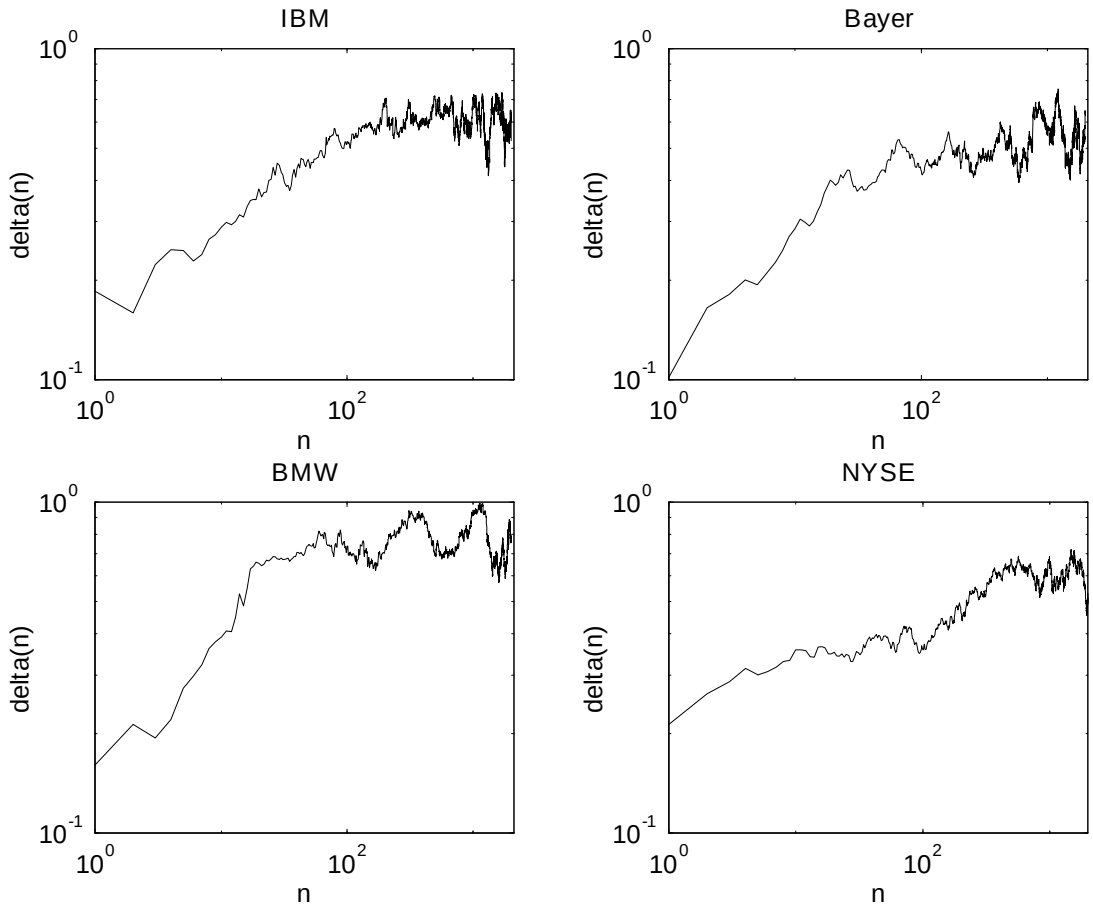
- (iii) If in some range ($n = 2$ to $n = 60$)

$$S_q(n) \sim n^{\chi(q)}$$

$\chi(q)$ is the *scaling exponent*

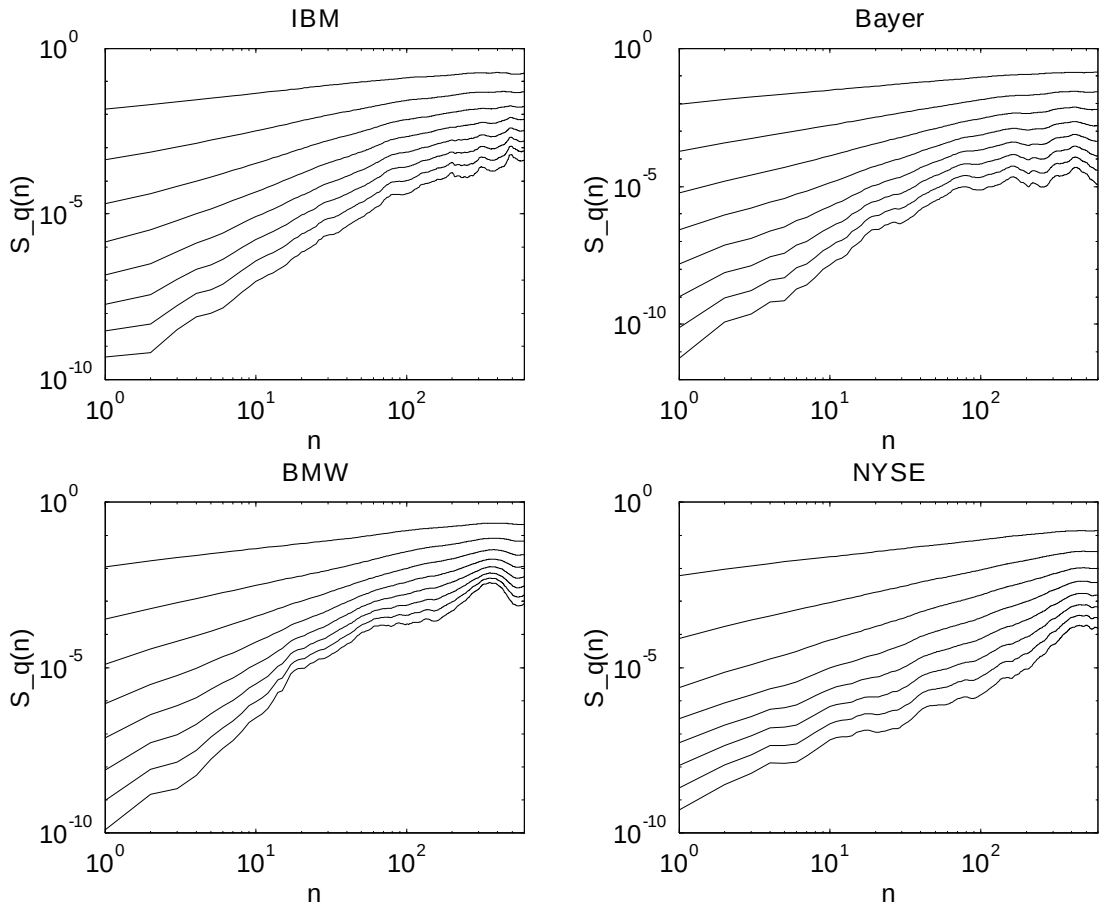
Statistical indicators

Maximum $\delta(n)$ of log-prices differences



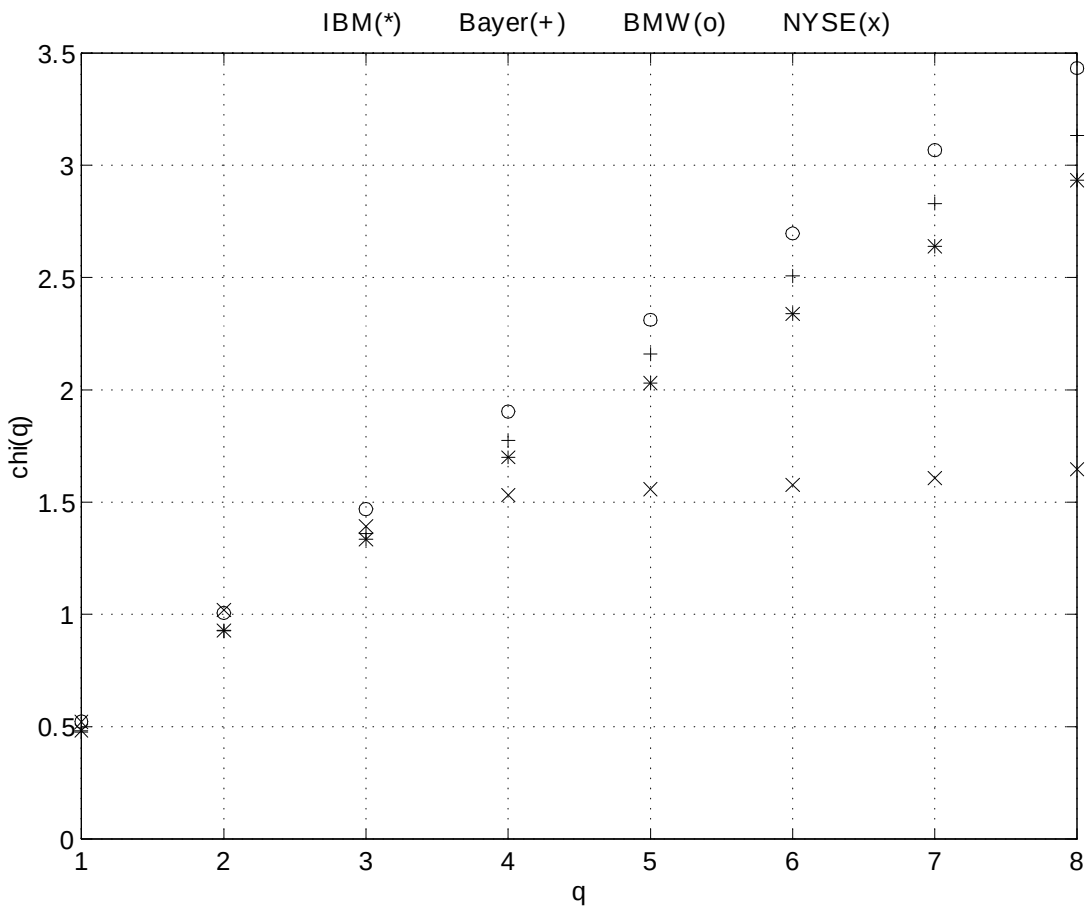
Statistical indicators

Moments of the $|r(t,n)|$ distribution



Statistical indicators

Scaling exponent $\chi(q)$



Statistical indicators

- Main conclusions:
  (a) $\delta(n)$ is log-concave and probably asymptotically constant for large r
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  (c) The scaling law $\chi(q)$ is an increasing concave function of q
  (d) $\chi(1)$ in the scaling region ($n = 2$ to $n = 60$) is close to 0.5
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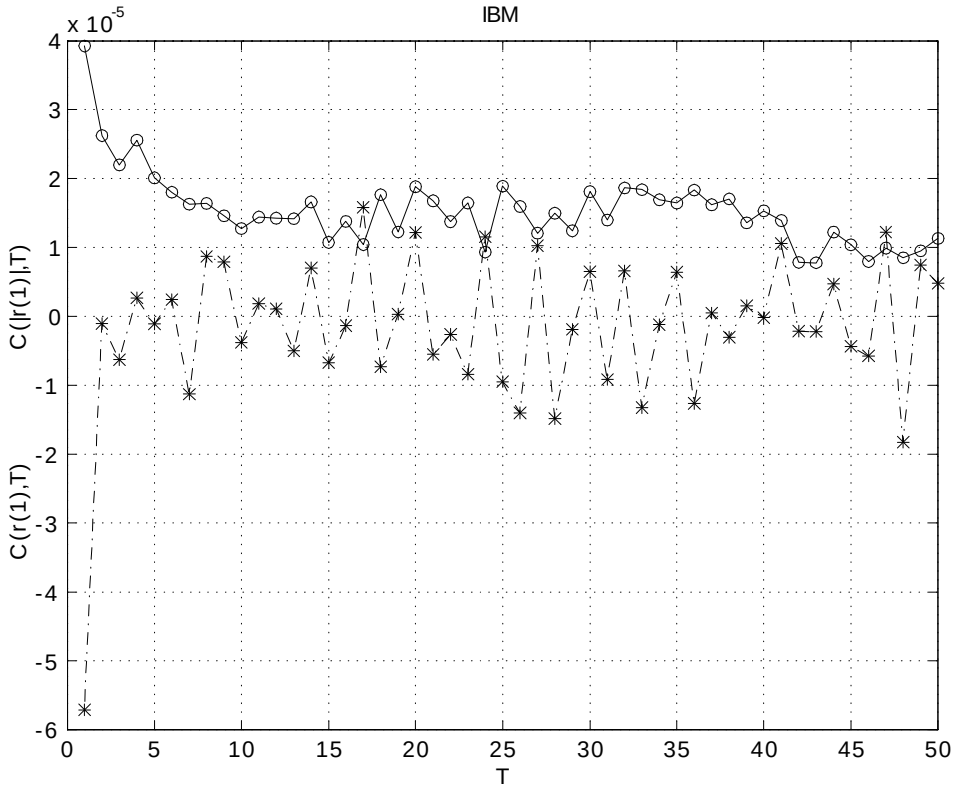
- The behavior of the statistical indicators $\delta(n)$, $S_q(n)$ and $\chi(q) \Longrightarrow$ If the process is a topological Markov chain the transitions allowed by the transition matrix T must lie inside a strictly convex domain around the diagonal of T

Statistical indicators

Correlation function of one-day returns ($\star$) and its absolute value (o)

$$C\left(r(1), T\right) = \left\langle r\left(t+T, 1\right) r\left(t, 1\right)\right\rangle$$

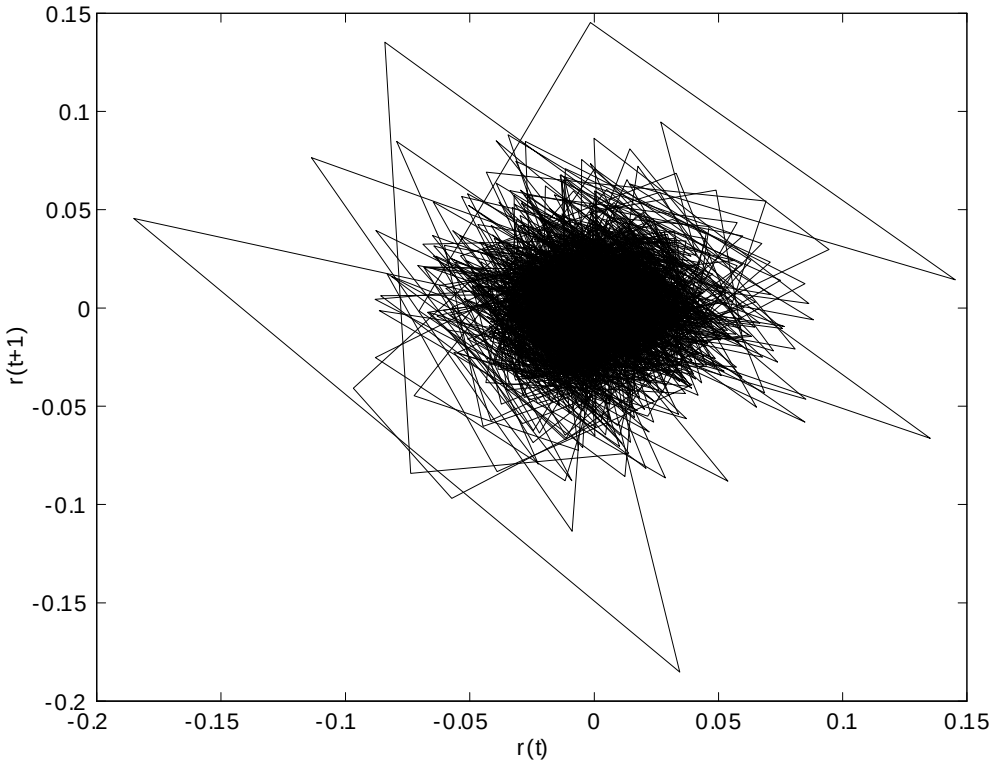
$$C\left(\left|r(1)\right|, T\right) = \left\langle \left|r\left(t+T, 1\right)\right| \left|r\left(t, 1\right)\right|\right\rangle$$



Statistical indicators

Dynamics of one-day returns

$$r(t,1) \longrightarrow r(t+1,1)$$



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Why Gibbs measures are "natural"

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- Maximum entropy principle (to maximize the uncertainty about what is not known) = use the most unbiased estimate

$$S = -\sum_i p_i \log p_i + \lambda_0 \sum_i p_i + \sum_k \lambda_k \sum_i p_i F_k(X_i)$$

$$\frac{\partial S}{\partial p_i} = 0 \qquad \Longrightarrow \qquad -\log p_i - 1 + \lambda_0 + \textstyle\sum_k \lambda_k F_k(X_i) = 0$$

$$p_i = \exp\left(-1 + \lambda_0 + \sum_k \lambda_k F_k(X_i)\right)$$

with $\lambda_0, \lambda_1, \cdots$ obtained from the constraints

Looking for a Gibbs measure

- Coding by a finite alphabet Σ
  Space Ω of orbits $\omega = i_1 i_2 \cdots i_k \cdots$, $i_k \in \Sigma$
  Dynamical law: a shift σ

$$\sigma\omega = i_2 \cdots i_k \cdots$$

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- **Gibbs measure**

$$c_1 \leq \frac{\mu\left([i_1(\omega) i_2(\omega) \cdots i_n(\omega)]\right)}{\exp\left(-nP + (S_n\phi)(\omega)\right)} \leq c_2$$

  $(S_n\phi)(\omega) = \sum_{k=0}^{n-1} \phi\left(\sigma^k\omega\right)$, ϕ being Hölder continuous function on Ω (the *potential)*
  $P(\phi, G)$: a function depending on potential and grammar (the *pressure* of ϕ)

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- Relation to the entropy

$$h(\mu) = \lim_{n\to\infty} \frac{H_n}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{i_1 \cdots i_n} \mu\left([i_1 i_2 \cdots i_n]\right) \log \mu\left([i_1 i_2 \cdots i_n]\right)$$

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- Variational principle: for each potential and grammar, $\sup_\eta \left\{ h(\eta) + \int \phi d\eta \right\}$ over all $\sigma-$invariant measures η is reached only for the Gibbs measure μ and equals the pressure

$$P(\phi, G) = h(\mu) + \int \phi d\mu$$

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- Potential may be chosen such that $P = 0$ (*normalized potential*). Then

$$\phi(\omega) = \lim_{n\to\infty} \log \frac{\mu\left([i_1(\omega) \cdots i_n(\omega)]\right)}{\mu\left([i_2(\omega) \cdots i_n(\omega)]\right)}$$

(Practical use hindered by poor statistics of large blocks)

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- Property of range$-r$ potentials: for all values $i_1 i_2 \cdots i_n$ with $n \geq r$

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$\Longrightarrow$

$$h(\mu) = -\sum_{i_1 \cdots i_k} \mu\left([i_1 \cdots i_k]\right) \log \frac{\mu\left([i_1 \cdots i_k]\right)}{\mu\left([i_1 \cdots i_{k-1}]\right)} = H_k - H_{k-1}$$

for all $k \geq r$ if $r > 1$. If $r = 1$
$h(\mu) = H_1 H_k = -\sum_{i_1 \cdots i_k} \mu\left([i_1 \cdots i_k]\right) \log \mu\left([i_1 \cdots i_k]\right)$

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- $\Longrightarrow$ **criterium to find the range of the potential**: range of the potential found when $H_k - H_{k-1}$ tends to a constant.
  Once the range is found, the potential may be constructed from the empirical weights $\widetilde{\mu}\left(\left[i_1 \cdots i_k\right]\right)$.

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- Another consequence of (1) is that for $k > r$

$$\mu\left([i_1 \cdots i_{k+1}]\right) = \frac{\mu\left([i_1 \cdots i_k]\right)\mu\left([i_2 \cdots i_{k+1}]\right)}{\mu\left([i_2 \cdots i_k]\right)}$$

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- **Application to the market fluctuations**:
  Five-symbols code $\Sigma = \{-2, -1, 0, 1, 2\}$ for

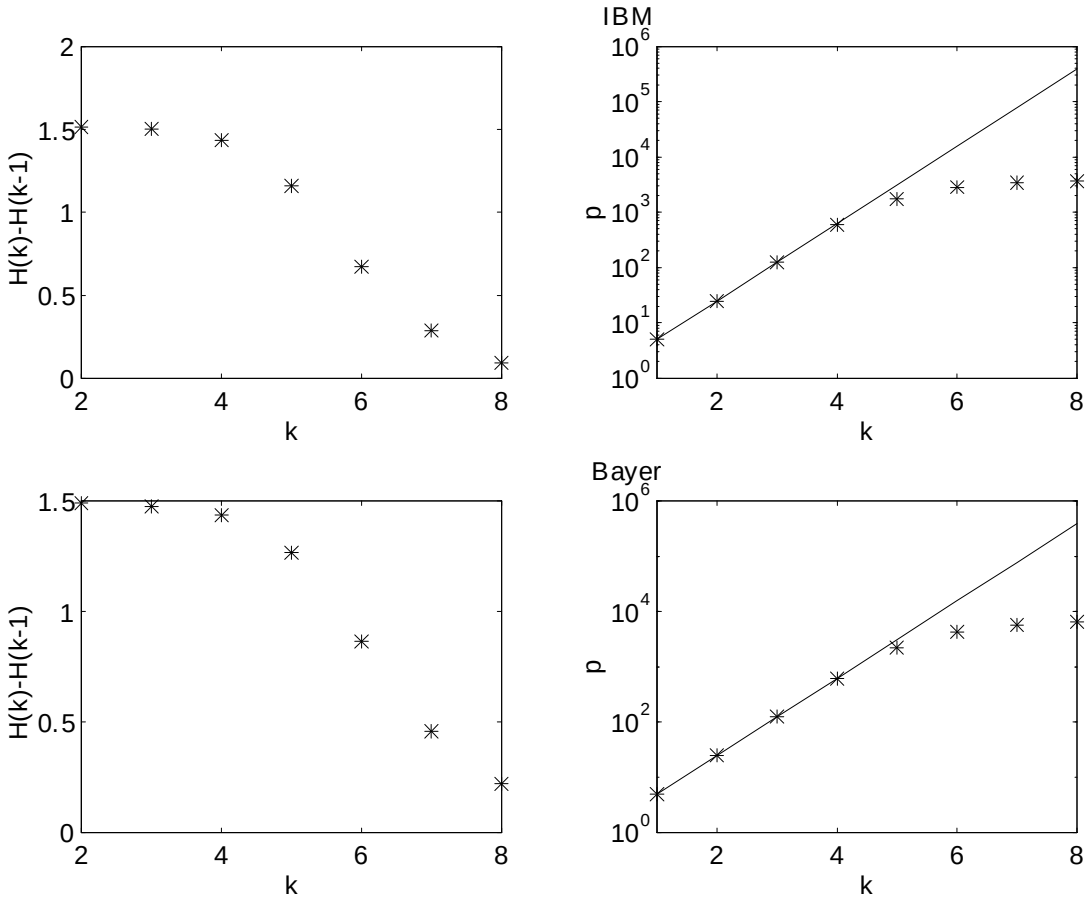
$$r(t) = \log p(t+1) - \log p(t)$$

Average $\overline{r(t)}$ and standard deviation $s = \sqrt{\overline{\left(r^2(t) - \overline{r(t)}^2\right)}}$

$$\begin{aligned}
\left(r(t) - \overline{r(t)}\right) > s &\iff 2 \\
s \geq \left(r(t) - \overline{r(t)}\right) > \tfrac{s}{3} &\iff 1 \\
\tfrac{s}{3} \geq \left(r(t) - \overline{r(t)}\right) > -\tfrac{s}{3} &\iff 0 \\
-\tfrac{s}{3} \geq \sigma\left(r(t) - \overline{r(t)}\right) > -s &\iff -1 \\
-s \geq \left(r(t) - \overline{r(t)}\right) &\iff -2
\end{aligned}$$

Looking for a Gibbs measure

$H_k - H_{k-1}$ and the number of occurring blocks of size k (IBM and Bayer)



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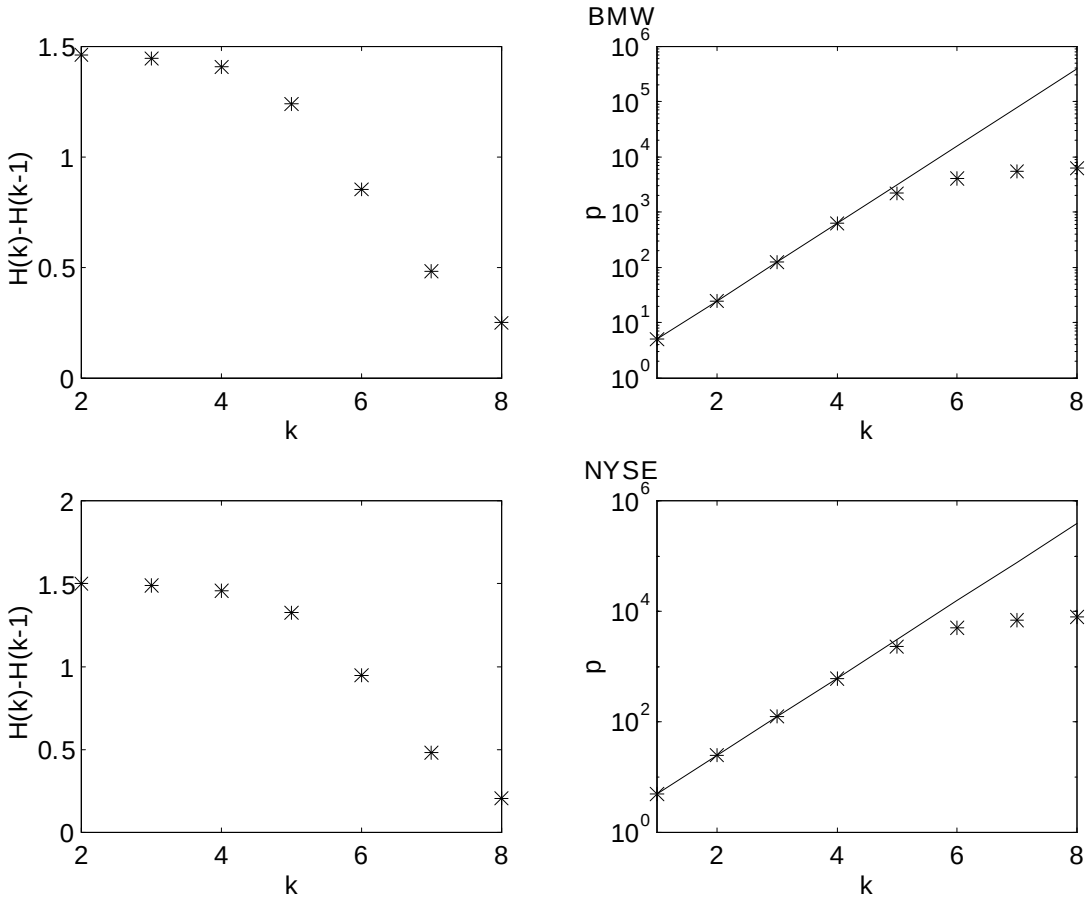

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- Behavior of $H_k - H_{k-1}$ quite different from hydrodynamic turbulence data

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  then compare with the empirically observed $\widetilde{\mu}\left(\left[i_1 \cdots i_{k+1}\right]\right)$
- Standard deviation of the relative positive errors

$$\varepsilon_k = \max\left(0, \frac{\widetilde{\mu}\left(\left[i_1 \cdots i_{k+1}\right]\right) - \mu_e\left(\left[i_1 \cdots i_{k+1}\right]\right)}{\frac{1}{2}\left(\widetilde{\mu}\left(\left[i_1 \cdots i_{k+1}\right]\right) + \mu_e\left(\left[i_1 \cdots i_{k+1}\right]\right)\right)}\right)$$

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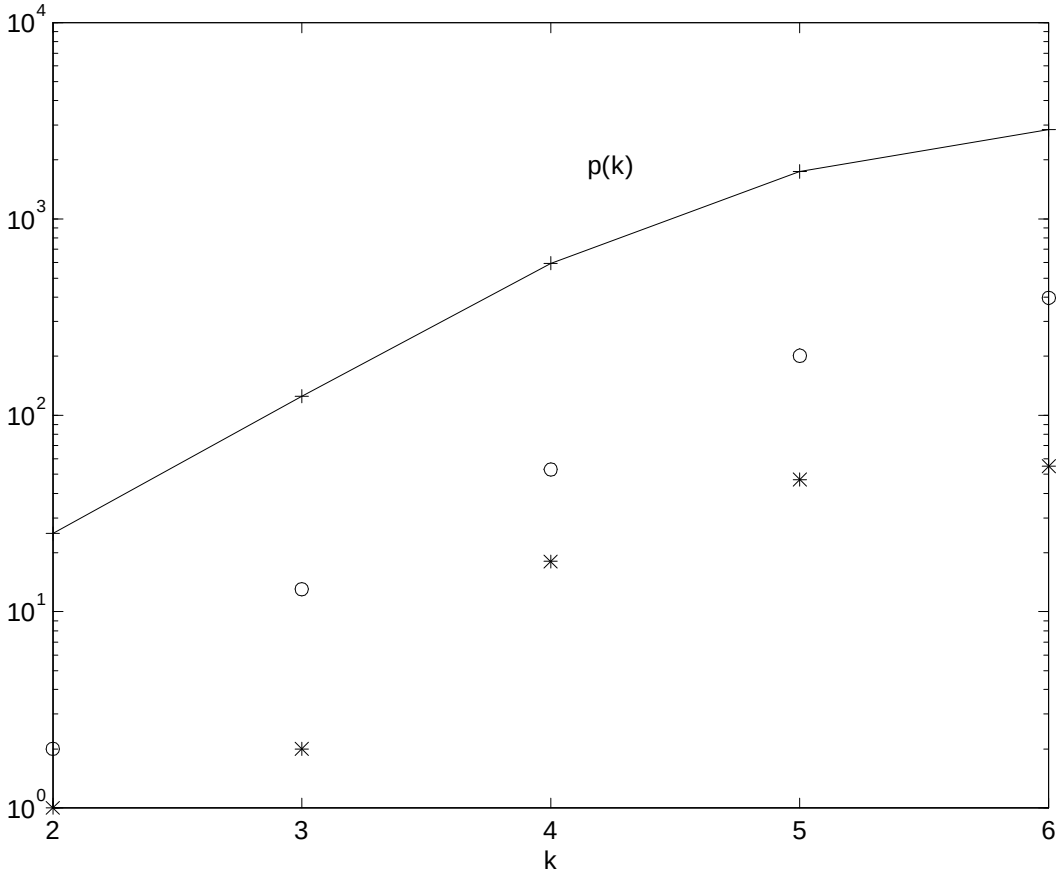
Underestimation errors one (o) and two (*) standard deviations away from the mean and the total number $p(k)$ of observed blocks



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- In any case it needs an approach suited to deal with long-memory processes

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exists $\forall a_i, j \leq -1$
3. There is a sequence $\left(\gamma_m\right)_{m \geq 1}$ with $\lim_{m \to \infty} \gamma_m = 0$, such that for all $\left\{a_j, b_j \in \Sigma, j \leq -1\right\}$ with $a_j = b_j$ for $-m \leq j \leq -1$
$$\left| \left(\frac{P\left(X_0 = a_0 | X_j = a_j, j \leq -1\right)}{P\left(X_0 = a_0 | X_j = b_j, j \leq -1\right)} - 1 \right) \right| \leq \gamma_m$$

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- Conditions 1. and 2. implicitly assumed for the pre-processed data Decays γ_m estimated from a typical sample of the process. From the empirical probabilities

$$P\left(a_0 \middle| a_1 \cdots a_m A\right) = \frac{P\left(a_0 a_1 \cdots a_m A\right)}{P\left(a_1 \cdots a_m A\right)}$$

A a block of arbitrary length

$$g\left(a_0 a_1 \cdots a_m\right) = \left(\frac{\max_A P\left(a_0 \middle| a_1 \cdots a_m A\right)}{\min_A P\left(a_0 \middle| a_1 \cdots a_m A\right)} - 1\right)$$

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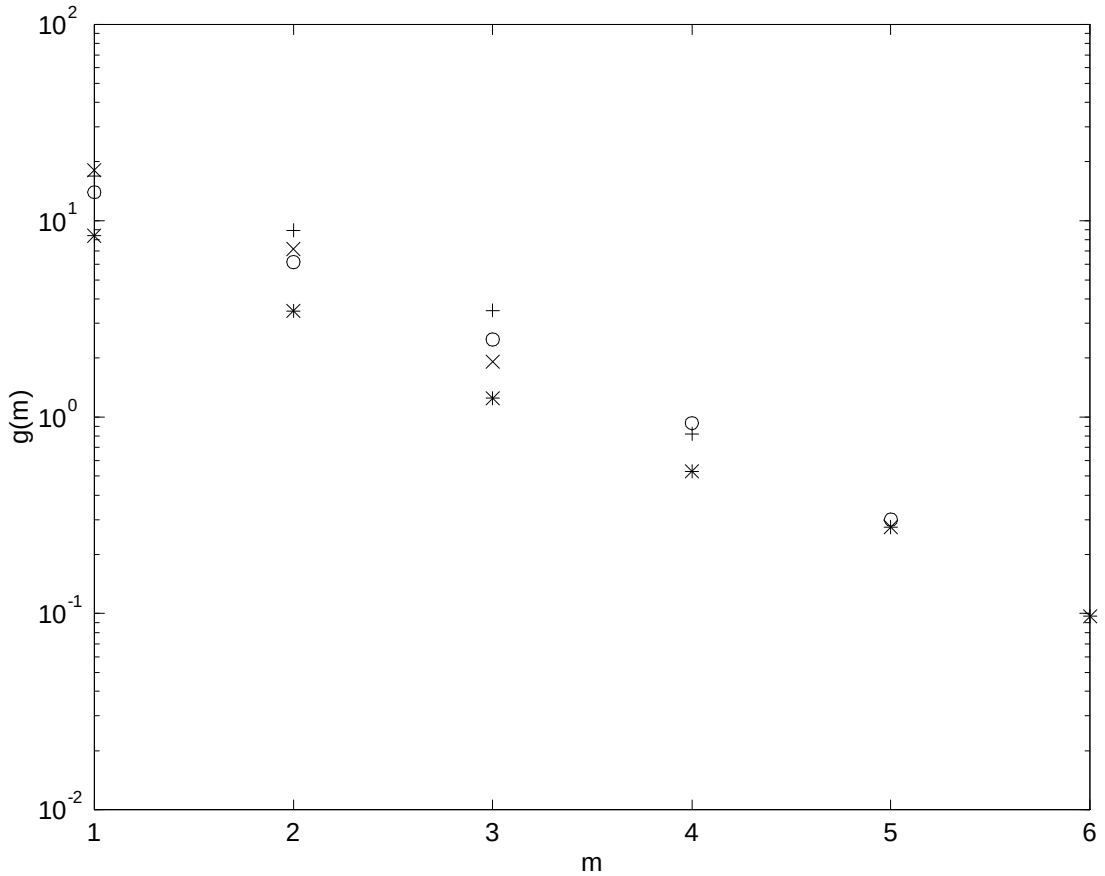
- Better estimate of the decay behavior with $\overline{g(m)} = \overline{g\left(a_0 a_1 \cdots a_m\right)}$, the average being taken over all sets $a_0 a_1 \cdots a_m$ of size m.

Chains with complete connections

$g(m)$ computed using A blocks of length 5 to 8 $(\times, +, o, *)$



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- *$\overline{d}-$distance*
  Coupling between two processes $X = \{X_n\}$ and $Y = \{Y_n\}$ is another process $\left\{\widetilde{X}_n, \widetilde{Y}_n\right\}$ over $\Sigma \times \Sigma$ such that the marginal probabilities of $\widetilde{X}$ and $\widetilde{Y}$ coincide with those of X and Y

$$\overline{d}(X, Y) = \inf\left\{P\left(\widetilde{X}_0 \neq \widetilde{Y}_0\right) : \left\{\widetilde{X}_n, \widetilde{Y}_n\right\} \quad \begin{array}{l}\text{is a stationary} \\ \text{coupling of X and Y}\end{array}\right\}$$

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- $\overline{d}-$distance tending to zero does mean that the processes will coincide after a certain time
- **Perfect simulation** always understood in the $\overline{d}-$distance sense. It does not mean **perfect prediction** (Means that a process is constructed with the same conditional probabilities of the original one)

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- **Simulation scheme by the sequence of canonical Markov approximations of finite order k ($k-$CMA)**
  $k-$CMA of a process X is a Markov chain $Y^{(k)}$ of order k with conditional probabilities $P^{(k)}$

$$P^{(k)}\left(a_0 \middle| a_1 \cdots a_k\right) = P\left(a_0 \middle| a_1 \cdots a_k\right) = \sum_A P\left(a_0 \middle| a_1 \cdots a_k A\right)$$

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- For a CCC X with summable decays

$$\overline{d}\left(X, Y^{(k)}\right) \leq C\gamma_k \tag{2}$$

The property of the Markov approximation, essential for the approximation result (2), is

$$\inf_A P\left(a_0|a_1\cdots a_k A\right) \leq P^{(k)}\left(a_0|a_1\cdots a_k\right) \leq \sup_A P\left(a_0|a_1\cdots a_k A\right) \tag{3}$$

meaning that for Markov approximation schemes, other than the canonical one, Eq.(2) holds provided (3) is satisfied

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- **For the market fluctuation data**:
  $\leq k-$Markov approximation:
  i) Empirical transition probabilities $\widetilde{P}\left(a_0|a_1\cdots a_m\right)$ inferred from the probability of blocks of order $m+1$. up to m_{Max}
  ii) $\leq k-$Simulation: look at the current block $\left(a_1\cdots a_k\right)$ and use the $k-$empirical probability to infer the next state a_0. If that block has not appeared in the trainig data, use the $k-1$ sized block $a_2\cdots a_k$ and the $k-1$ order empirical probabilities

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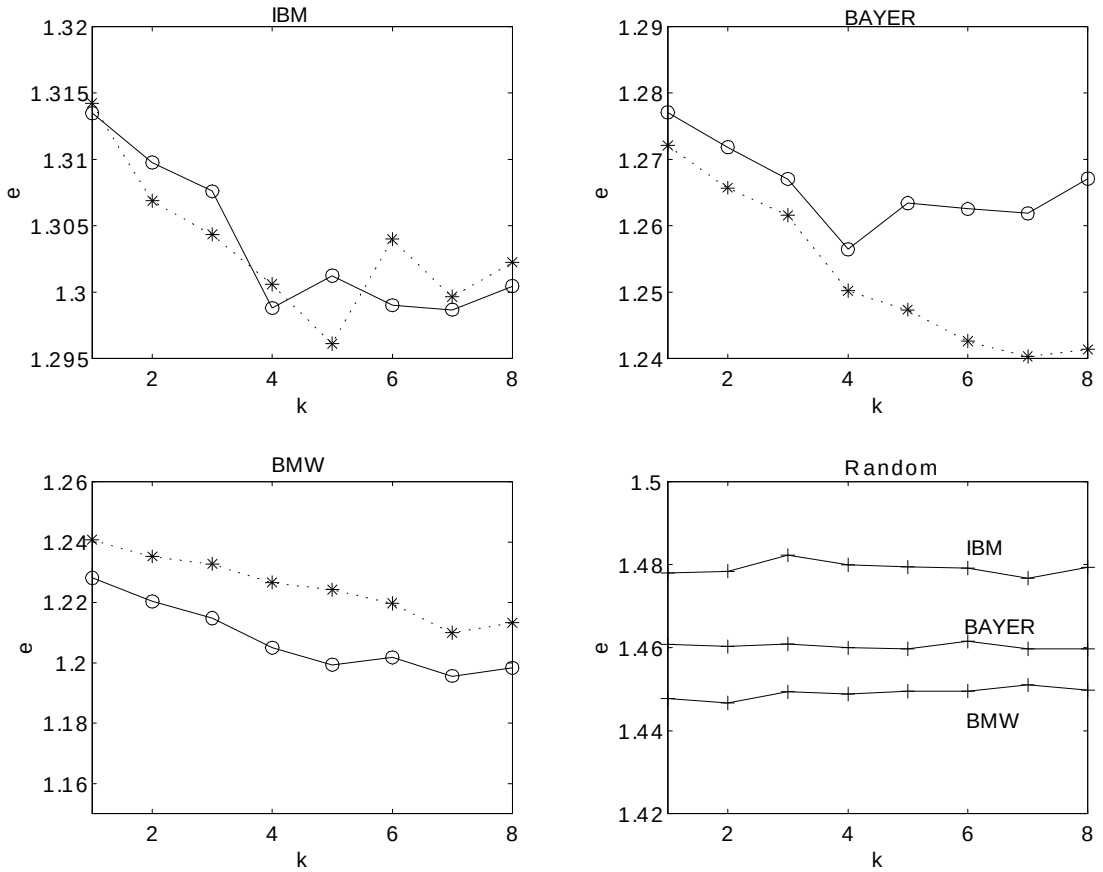
- Averaged squared error

$$e^2=\left\langle\left(\widetilde{a}_0-a_0\right)^2\right\rangle$$

Chains with complete connections

The past predicting the future (o) and the future predicting the past (*), compared to random choice



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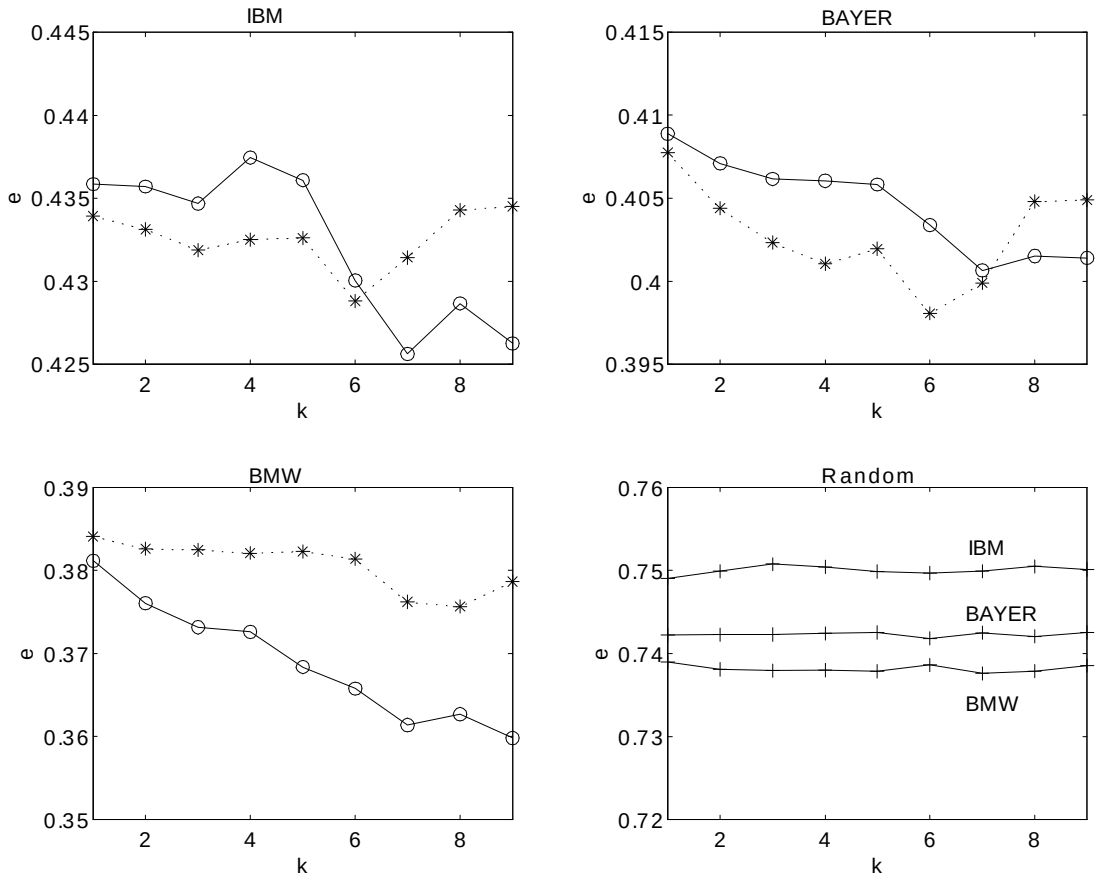

Chains with complete connections

Main conclusions:

- Average prediction better than random choice.
- Main improvement results from correct accounting of the two-symbol probabilities ($k = 1$)
- Small (but consistent) improvement using past information up $k = 4$ or 5. No significant improvement using higher order approximations
- Bulk of data represented by a short-memory process. Nevertheless there is evidence for a small long-memory component that is captured by the higher-order Markov approximations
- There is a maximum $k = k_m$ that should be used for the simulation process
- Inter-companies prediction: improvement coming from the one-symbol probabilities (as compared to random choice) is obtained. For the long-memory component the behavior is company-dependent.

Statistical analysis of market data

General conclusions

1. Bulk of the market fluctuation process is a short-memory process. In addition it has a small long-memory component associated with the large fluctuations of the returns.
2. Existence of the long-memory component suggests the *chains with complete connections and summable decays* as a framework
3. Although the decays may be exponentially converging, the lack of accurate data for long blocks prevents an accurate description by a finite range Gibbs potential.
4. The sequence of empirical based $\leq k-$Markov approximations seems the most unbiased simulation of the process. Eventual convergence in the $\overline{d}-$distance sense, because of summable decays.
5. For the future: high frequency market data. Beware of the possibly multi-scale and multi-component nature of the processes.

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