Notes on fractional Brownian motion and stochastic integration

R. Vilela Mendes

Selfsimilar processes and fractional Brownian motion (fBm)

- A process $\{X(t), t \geq 0\}$ is *selfsimilar* if for any a there is b such that

$$\{X(at)\} \stackrel{d}{=} \{bX(t)\}$$

$b = a^H$, process $H-$selfsimilar (or $H-$ss) - (Hurst exponent)

Selfsimilar processes and fractional Brownian motion (fBm)

- A process $\{X(t), t \geq 0\}$ is *selfsimilar* if for any a there is b such that

$$\{X(at)\} \stackrel{d}{=} \{bX(t)\}$$

$b = a^H$, process $H-$selfsimilar (or $H-$ss) - (Hurst exponent)

- A process has *stationary increments (si)* if any distribution of

$$\{X(t+h) - X(t), t \geq 0\}$$

is independent of $t \geq 0$

Selfsimilar processes and fractional Brownian motion (fBm)

- A process $\{X(t), t \geq 0\}$ is *selfsimilar* if for any a there is b such that

$$\{X(at)\} \stackrel{d}{=} \{bX(t)\}$$

$b = a^H$, process $H-$selfsimilar (or $H-$ss) - (Hurst exponent)

- A process has *stationary increments (si)* if any distribution of

$$\{X(t+h) - X(t), t \geq 0\}$$

is independent of $t \geq 0$

- *Theorem: If* $\{X(t), t \geq 0\}$ *is real-valued, H-ss with stationary increments and* $\mathbb{E}\left[X(1)^2\right] < \infty$, *then*

$$\mathbb{E}[X(t)X(s)] = \frac{1}{2}\left\{t^{2H} + s^{2H} - |t-s|^{2H}\right\}\mathbb{E}\left[X(1)^2\right]$$

The simplest such process is a Gaussian process called fractional Brownian motion (fBm), $B_H(t)$, defined to have $\mathbb{E}[B_H(t)] = 0$. fBm is the unique Gaussian H-ss process with stationary increments

Fractional Gaussian noise and long-range dependence

- The process

$$Y_t = B_H(t+1) - B_H(t)$$

  is called *fractional Gaussian noise*

Fractional Gaussian noise and long-range dependence

- The process

$$Y_t = B_H(t+1) - B_H(t)$$

  is called *fractional Gaussian noise*
- Renormalization group

$$T_N : Y_t \rightarrow (T_N Y)_t = \frac{1}{N^H} \sum_{i=t}^{t+N-1} Y_i$$

Fractional Gaussian noise and long-range dependence

- The process
$$Y_t = B_H(t+1) - B_H(t)$$
is called *fractional Gaussian noise*
- Renormalization group
$$T_N : Y_t \to (T_N Y)_t = \frac{1}{N^H} \sum_{i=t}^{t+N-1} Y_i$$
- *Theorem: Within the class of stationary sequences, fractional Gaussian noise is the only Gaussian fixed point of the renormalization group*

Fractional Gaussian noise and long-range dependence

- The process
$$Y_t = B_H(t+1) - B_H(t)$$
is called *fractional Gaussian noise*
- Renormalization group
$$T_N : Y_t \to (T_N Y)_t = \frac{1}{N^H} \sum_{i=t}^{t+N-1} Y_i$$
- *Theorem: Within the class of stationary sequences, fractional Gaussian noise is the only Gaussian fixed point of the renormalization group*
- *Long-range dependence:* Let $\{X(t), t \geq 0\}$ be H-ss, si, 0<H<1 with $E\left[X(1)^2\right] < \infty$ and define
$$\xi(n) = X(n+1) - X(n)$$
$$r(n) = \mathbb{E}[\xi(0)\xi(n)] = \frac{1}{2}\left\{(n+1)^{2H} - 2n^{2H} + (n-1)^{2H}\right\}\mathbb{E}\left[X(1)^2\right]$$

Long-range dependence

- Then

$$
\begin{aligned}
r(n) \quad &\underset{n\to\infty}{\sim} \quad H(2H-1)\, n^{2H-2}\mathbb{E}\left[X(1)^2\right] \quad , \quad H \neq \tfrac{1}{2} \\
r(n) \quad &= \quad 0 \qquad\qquad\qquad\qquad\qquad\qquad\quad H = \tfrac{1}{2}
\end{aligned}
$$

and

$$
\begin{array}{ccc}
0 < H < \frac{1}{2} & , & \sum_{n=0}^{\infty} |r(n)| < \infty \\
H = \frac{1}{2} & , & \text{uncorrelated} \\
\frac{1}{2} < H < 1 & , & \sum_{n=0}^{\infty} |r(n)| = \infty \quad , \quad \text{long-range dependence}
\end{array}
$$

Long-range dependence

- Then

$$
\begin{aligned}
r(n) &\underset{n\to\infty}{\sim} H(2H-1)\, n^{2H-2}\mathbb{E}\left[X(1)^2\right] \quad , \quad H \neq \tfrac{1}{2} \\
r(n) &= 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad H = \tfrac{1}{2}
\end{aligned}
$$

  and

$$
\begin{array}{ccccc}
0 < H < \frac{1}{2} & , & \sum_{n=0}^{\infty} |r(n)| < \infty & & \\
H = \frac{1}{2} & , & \text{uncorrelated} & & \\
\frac{1}{2} < H < 1 & , & \sum_{n=0}^{\infty} |r(n)| = \infty & , & \text{long-range dependence}
\end{array}
$$

- If $0 < H < \frac{1}{2}$, $r(n) < 0$ for $n \geq 1$ (negative correlation, anti-persistent process),

Long-range dependence

- Then

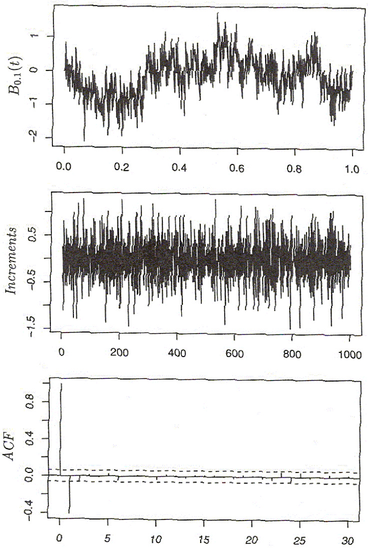
$$
\begin{aligned}
r(n) & \underset{n\to\infty}{\sim} H(2H-1)\, n^{2H-2}\mathbb{E}\left[X(1)^2\right] \quad , \quad H \neq \tfrac{1}{2} \\
r(n) & = 0 \qquad\qquad\qquad\qquad\qquad\qquad H = \tfrac{1}{2}
\end{aligned}
$$

and

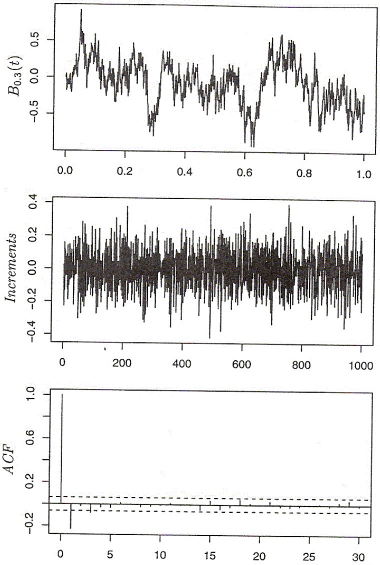
$$
\begin{array}{ccccc}
0 < H < \frac{1}{2} & , & \sum_{n=0}^{\infty} |r(n)| < \infty & & \\
H = \frac{1}{2} & , & \text{uncorrelated} & & \\
\frac{1}{2} < H < 1 & , & \sum_{n=0}^{\infty} |r(n)| = \infty & , & \text{long-range dependence}
\end{array}
$$

- If $0 < H < \frac{1}{2}$, $r(n) < 0$ for $n \geq 1$ (negative correlation, anti-persistent process),
- If $\frac{1}{2} < H < 1$, $r(n) > 0$ for $n \geq 1$ (positive correlation, persistent process).

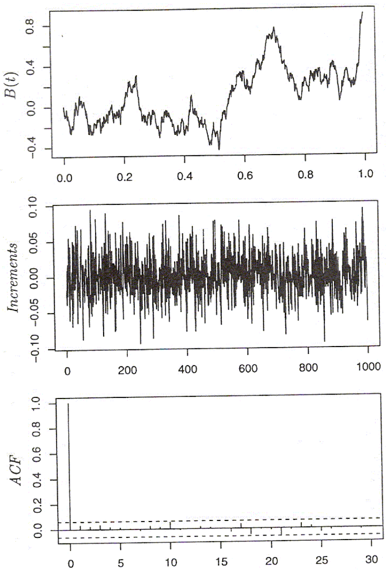
Plots of fractional Brownian motion (H=0.1)



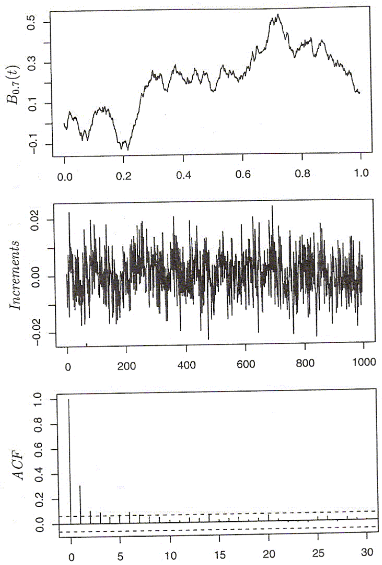
Plots of fractional Brownian motion (H=0.3)



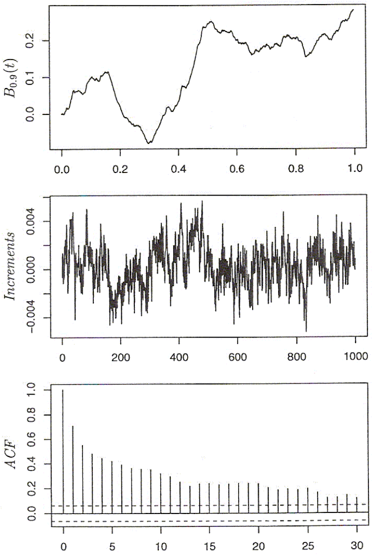
Plots of fractional Brownian motion (H=0.5)



Plots of fractional Brownian motion (H=0.7)



Plots of fractional Brownian motion (H=0.9)



Sample path properties

- Kolmogorov criterium for the existence of a continuous version of $X(t)$: $\exists \alpha \geq 1, \beta > 0, k > 0$ such that

$$\mathbb{E}\left[|X(t) - X(s)|^{\alpha}\right] \leq k\,|t-s|^{1+\beta}$$

Sample path properties

- Kolmogorov criterium for the existence of a continuous version of $X(t)$: $\exists \alpha \geq 1, \beta > 0, k > 0$ such that

$$\mathbb{E}\left[|X(t) - X(s)|^{\alpha}\right] \leq k\,|t - s|^{1+\beta}$$

- Hölder continuity of order γ

$$P\left\{\omega \in \Omega, \sup \frac{|X(t,\omega) - X(s,\omega)|}{|t-s|^{\gamma}} \leq \delta\right\} = 1$$

Sample path properties

- Kolmogorov criterium for the existence of a continuous version of $X(t)$: $\exists \alpha \geq 1, \beta > 0, k > 0$ such that

$$\mathbb{E}\left[\left|X(t) - X(s)\right|^{\alpha}\right] \leq k\left|t - s\right|^{1+\beta}$$

- Hölder continuity of order γ

$$P\left\{\omega \in \Omega, \sup \frac{\left|X(t,\omega) - X(s,\omega)\right|}{\left|t - s\right|^{\gamma}} \leq \delta\right\} = 1$$

- From

$$\mathbb{E}\left[\left|B_H(t) - B_H(s)\right|^{\alpha}\right] = \mathbb{E}\left[\left|B_H(1)\right|^{\alpha}\right]\left|t - s\right|^{\alpha H}$$

  it follows that fBm $\{B_H(t)\}$ has a continuous version $(P\{X(t) = B_H(t)\} = 1)$ the sample paths of which are Hölder continuous of order $\beta \in [0, H)$ and are almost surely nowhere locally Hölder continuous of order $\gamma > H$.

Sample path properties

- Kolmogorov criterium for the existence of a continuous version of $X(t)$: $\exists \alpha \geq 1, \beta > 0, k > 0$ such that
$$\mathbb{E}\left[|X(t) - X(s)|^{\alpha}\right] \leq k\,|t-s|^{1+\beta}$$
- Hölder continuity of order γ
$$P\left\{\omega \in \Omega, \sup \frac{|X(t,\omega) - X(s,\omega)|}{|t-s|^{\gamma}} \leq \delta\right\} = 1$$
- From
$$\mathbb{E}\left[|B_H(t) - B_H(s)|^{\alpha}\right] = \mathbb{E}\left[|B_H(1)|^{\alpha}\right]|t-s|^{\alpha H}$$
it follows that fBm $\{B_H(t)\}$ has a continuous version $(P\{X(t) = B_H(t)\} = 1)$ the sample paths of which are Hölder continuous of order $\beta \in [0, H)$ and are almost surely nowhere locally Hölder continuous of order $\gamma > H$.
- Sample paths of fBm have nowhere bounded variation and are not differentiable.

FBm for H not 1/2 is not a semimartingale

- *p - Variation of a process $X(t)$*

$$V_p(0, T) = \sup_{partitions} \sum_{k=1}^{n} |X(t_k) - X(t_{k-1})|^p$$

FBm for H not 1/2 is not a semimartingale

- *p - Variation of a process* $X(t)$

$$V_p(0,T) = \sup_{partitions} \sum_{k=1}^{n} |X(t_k) - X(t_{k-1})|^p$$

- Index of the p-variation

$$I(X, |0,T]) = \inf\{p > 0; V_p(0,T) < \infty\}$$

FBm for H not 1/2 is not a semimartingale

- *p - Variation of a process* $X(t)$

$$V_p(0,T) = \sup_{partitions} \sum_{k=1}^{n} |X(t_k) - X(t_{k-1})|^p$$

- Index of the p-variation

$$I(X, |0,T]) = \inf\{p > 0; V_p(0,T) < \infty\}$$

- From $\mathbb{E}\left[|B_H(t) - B_H(s)|^\alpha\right] = \mathbb{E}\left[|B_H(1)|^\alpha\right] |t-s|^{\alpha H}$ it is easy to conclude that

$$I(B_H, |0,T]) = \frac{1}{H}$$

The index of a semimartingale must belong to $[0,1] \cup \{2\}$. Therefore $B_H(t)$ cannot be a semimartingale unless $H = \frac{1}{2}$

Representation of fractional Brownian motion by Wiener integrals

- "Time" representation

$$B_H(t) \overset{d}{=} \frac{1}{\Gamma\left(H+\frac{1}{2}\right)}\left\{\int_{-\infty}^{0}\left((t-u)^{H-\frac{1}{2}}-(-u)^{H-\frac{1}{2}}\right)dB(u)+\int_{0}^{t}(t-u)^{H-\frac{1}{2}}\,dB(u)\right\}$$

Representation of fractional Brownian motion by Wiener integrals

- "Time" representation

$$B_H(t) \overset{d}{=} \frac{1}{\Gamma\left(H+\frac{1}{2}\right)} \left\{ \int_{-\infty}^{0} \left((t-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}} \right) dB(u) + \int_0^t (t-u)^{H-\frac{1}{2}} dB(u) \right\}$$

- Finite interval representation

$$B_H(t) \overset{d}{=} C \int_0^t K(t,u)\, dB(u)$$

$$K(t,u) = \left\{ \left(\frac{t}{u}\right)^{H-\frac{1}{2}} (t-u)^{H-\frac{1}{2}} - \left(H-\frac{1}{2}\right) u^{\frac{1}{2}-H} \int_u^t x^{H-\frac{3}{2}} (x-u)^{H-\frac{1}{2}} dx \right\}$$

Representation of fractional Brownian motion by Wiener integrals

- "Time" representation

$$B_H(t) \stackrel{d}{=} \frac{1}{\Gamma\left(H+\frac{1}{2}\right)} \left\{ \int_{-\infty}^{0} \left((t-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}} \right) dB(u) + \int_0^t (t-u)^{H-\frac{1}{2}} dB(u) \right\}$$

- Finite interval representation

$$B_H(t) \stackrel{d}{=} C \int_0^t K(t,u)\, dB(u)$$

$$K(t,u) = \left\{ \left(\frac{t}{u}\right)^{H-\frac{1}{2}} (t-u)^{H-\frac{1}{2}} - \left(H-\frac{1}{2}\right) u^{\frac{1}{2}-H} \int_u^t x^{H-\frac{3}{2}} (x-u)^{H-\frac{1}{2}} dx \right\}$$

- "Spectral" representation

$$B_H(t) \stackrel{d}{=} \frac{\Gamma\left(H+\frac{1}{2}\right)}{(2\pi)^{\frac{1}{2}} c(H)} \int_{\mathbb{R}} \frac{e^{ixt}-1}{ix} d\tilde{B}(x)$$

$$\tilde{B}(x) = B_1 + iB_2,\ B_1(x) = B_1(-x),\ B_2(x) = -B_2(-x)$$

Representation of fractional Brownian motion

- (Paley-Wigner-type) series representation

$$B_H(t) = \sum_{n \in \mathbb{Z}} \frac{e^{2i\omega_n t/T} - 1}{2i\omega_n t/T} Z_n$$

convergent in $t \in [0, T]$.
The $\omega_n' s$ are the real zeros of the Bessel function J_{1-H} and the Z_n are independent complex-valued Gaussian random variables with mean zero and variance

$$\mathbb{E}|Z_n|^2 = \begin{cases} (2-2H)^{-1}\Gamma^{-2}(1-H)\left(\frac{\omega_n}{2}\right)^{-2H} J_{-H}^{-2}(\omega_n) V_T^{-1} & \omega_n \neq 0 \\ V_T^{-1} & \omega_n = 0 \end{cases}$$

$$V_T = \frac{\Gamma(3/2-H)}{2H\Gamma(H+1/2)\Gamma(3-2H)} T^{2-2H}$$

Stochastic integration for fractional Brownian motion

- The non semimartingale property for $H \neq \frac{1}{2}$ makes the standard construction not applicable.

Stochastic integration for fractional Brownian motion

- The non semimartingale property for $H \neq \frac{1}{2}$ makes the standard construction not applicable.
- If $1/2 < H < 1$ the zero quadratic variation allows for a pathwise construction of the integral

$$\int_a^b f(t,\omega)\,\delta B_H(t) := \lim_{\Delta t_k \to 0} \sum_k f(t_k,\omega)\left(B_H(t_{k+1}) - B_H(t_k)\right)$$

However with the pathwise definition one ends up, in general with

$$\mathbb{E}\left[\int f(t, B_H(t))\,\delta B_H(t)\right] \neq 0$$

leading, for example, to arbitrage in financial applications.

Stochastic integration for fractional Brownian motion

- The non semimartingale property for $H \neq \frac{1}{2}$ makes the standard construction not applicable.
- If $1/2 < H < 1$ the zero quadratic variation allows for a pathwise construction of the integral

$$\int_a^b f(t,\omega)\,\delta B_H(t) := \lim_{\Delta t_k \to 0} \sum_k f(t_k,\omega)\left(B_H(t_{k+1}) - B_H(t_k)\right)$$

However with the pathwise definition one ends up, in general with

$$\mathbb{E}\left[\int f(t, B_H(t))\,\delta B_H(t)\right] \neq 0$$

leading, for example, to arbitrage in financial applications.
- An alternative

$$\int_a^b f(t,\omega)\,dB_H(t) := \lim_{\Delta t_k \to 0} \sum_k f(t_k,\omega) \diamond \left(B_H(t_{k+1}) - B_H(t_k)\right)$$

$\diamond$ is the Wick product. This integral reduces to Ito's for $H = \frac{1}{2}$.

A reminder of the white noise analysis formulation of Itô integration

- White noise measure: prob. measure defined on the Borelian $\sigma-$algebra $\mathcal{F} := \mathcal{B}\left(S'(\mathbb{R})\right)$ by

$$\int \exp\left(i < \omega, f >\right) d\mu = \exp\left(-\frac{1}{2}\left\|f\right\|^2_{L^2(\mathbb{R})}\right)$$

A reminder of the white noise analysis formulation of Itô integration

- White noise measure: prob. measure defined on the Borelian $\sigma-$algebra $\mathcal{F} := \mathcal{B}\left(S'(\mathbb{R})\right)$ by

$$\int \exp\left(i < \omega, f >\right) d\mu = \exp\left(-\frac{1}{2}\left\|f\right\|^2_{L^2(\mathbb{R})}\right)$$

- Brownian motion

$$B(t) = < \omega, \chi_{[0,t]} >$$

Approximating functions f by a sequence of step functions, define

A reminder of the white noise analysis formulation of Itô integration

- White noise measure: prob. measure defined on the Borelian $\sigma-$algebra $\mathcal{F} := \mathcal{B}\left(S'(\mathbb{R})\right)$ by

$$\int \exp\left(i < \omega, f >\right) d\mu = \exp\left(-\frac{1}{2}\left\|f\right\|^2_{L^2(\mathbb{R})}\right)$$

- Brownian motion

$$B(t) =< \omega, \chi_{[0,t]} >$$

Approximating functions f by a sequence of step functions, define

- Itô integral for deterministic functions

$$< \omega, f >:= \int f(t)\, dB(t)$$

A reminder of the white noise analysis formulation of Itô integration

- White noise measure: prob. measure defined on the Borelian $\sigma-$algebra $\mathcal{F} := \mathcal{B}\left(S^{'}(\mathbb{R})\right)$ by

$$\int \exp\left(i < \omega, f >\right) d\mu = \exp\left(-\frac{1}{2}\left\|f\right\|^2_{L^2(\mathbb{R})}\right)$$

- Brownian motion

$$B\left(t\right) = < \omega, \chi_{[0,t]} >$$

Approximating functions f by a sequence of step functions, define

- Itô integral for deterministic functions

$$< \omega, f > := \int f\left(t\right) dB\left(t\right)$$

- Iterated Itô integral for $f \in \hat{L}\left(\mathbb{R}^n\right)$

$$I_n\left(f\right) := n! \int_{\mathbb{R}} \int_{-\infty}^{t_n} \cdots \int_{-\infty}^{t_2} f\left(t_1, t_2, \cdots, t_n\right) dB\left(t_1\right) \cdots dB\left(t_n\right)$$

A reminder of the white noise analysis formulation of Itô integration

- Expansion of functionals $F(\omega)$ of white noise (**Wiener-Itô chaos expansion**)

  $\exists f_n \in \hat{L}(\mathbb{R}^n)$ such that

$$F(\omega) = \sum_{n=0}^{\infty} I_n(f_n)$$

A reminder of the white noise analysis formulation of Itô integration

- Expansion of functionals $F(\omega)$ of white noise (**Wiener-Itô chaos expansion**)

  $\exists f_n \in \hat{L}(\mathbb{R}^n)$ such that

$$F(\omega) = \sum_{n=0}^{\infty} I_n(f_n)$$

- A natural basis: Hermite polynomials and Hermite function

$$h_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n}\left(e^{-x^2/2}\right)$$

$$\xi_k(x) = \pi^{-\frac{1}{4}}((n-1)!)^{-\frac{1}{2}} h_{n-1}\left(\sqrt{2}x\right) e^{-\frac{x^2}{2}}$$

  Then define

A reminder of the white noise analysis formulation of Itô integration

- Expansion of functionals $F(\omega)$ of white noise (**Wiener-Itô chaos expansion**)

  $\exists f_n \in \hat{L}(\mathbb{R}^n)$ such that

$$F(\omega) = \sum_{n=0}^{\infty} I_n(f_n)$$

- A natural basis: Hermite polynomials and Hermite function

$$h_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n}\left(e^{-x^2/2}\right)$$

$$\xi_k(x) = \pi^{-\frac{1}{4}}((n-1)!)^{-\frac{1}{2}} h_{n-1}\left(\sqrt{2}x\right) e^{-\frac{x^2}{2}}$$

  Then define

- $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$

$$\mathcal{H}_\alpha(\omega) = h_{\alpha_1}(<\omega, \xi_1>) h_{\alpha_2}(<\omega, \xi_2>) \cdots h_{\alpha_n}(<\omega, \xi_n>)$$

A reminder of the white noise analysis formulation of Itô integration

- **Chaos expansion in terms of hermite functionals**

$$F(\omega) = \sum c_\alpha \mathcal{H}_\alpha(\omega)$$

A reminder of the white noise analysis formulation of Itô integration

- **Chaos expansion in terms of hermite functionals**

$$F(\omega) = \sum c_\alpha \mathcal{H}_\alpha(\omega)$$

- **Wick product** of white noise functionals

$$(F_1 \diamond F_2) = \sum c_\alpha^{(1)} c_\beta^{(2)} \mathcal{H}_{\alpha+\beta}(\omega)$$

A reminder of the white noise analysis formulation of Itô integration

- **Chaos expansion in terms of hermite functionals**

$$F(\omega) = \sum c_\alpha \mathcal{H}_\alpha(\omega)$$

- **Wick product** of white noise functionals

$$(F_1 \diamond F_2) = \sum c_\alpha^{(1)} c_\beta^{(2)} \mathcal{H}_{\alpha+\beta}(\omega)$$

- Note

$$< \omega, f > \diamond < \omega, g > = < \omega, f > < \omega, g > - (f, g)$$

A reminder of the white noise analysis formulation of Itô integration

- **Chaos expansion in terms of hermite functionals**

$$F(\omega) = \sum c_\alpha \mathcal{H}_\alpha(\omega)$$

- **Wick product** of white noise functionals

$$(F_1 \diamond F_2) = \sum c_\alpha^{(1)} c_\beta^{(2)} \mathcal{H}_{\alpha+\beta}(\omega)$$

- Note

$$< \omega, f > \diamond < \omega, g > = < \omega, f >< \omega, g > - (f, g)$$

- Chaos expansion of Brownian motion

$$\begin{aligned} B(t) &= \left\langle \omega, \chi_{[0,t]} \right\rangle \\ &= \left\langle \omega, \sum_{k=1}^{\infty} \left(\chi_{[0,t]}, \xi_k \right)_{L^2(\mathbb{R})} \xi_k(\cdot) \right\rangle \\ &= \sum_{k=1}^{\infty} \int_0^t \xi_k(s)\, ds \left\langle \omega, \xi_k \right\rangle \end{aligned}$$

A reminder of the white noise analysis formulation of Itô integration

- White noise

$$W(t) = \sum_{k=1}^{\infty} \xi_k(t) \langle \omega, \xi_k \rangle$$

then,

$$\frac{dB(t)}{dt} = W(t)$$

A reminder of the white noise analysis formulation of Itô integration

- White noise

$$W(t) = \sum_{k=1}^{\infty} \xi_k(t) \langle \omega, \xi_k \rangle$$

then,

$$\frac{dB(t)}{dt} = W(t)$$

- Stochastic integration

$$\int_{\mathbb{R}} Y(t,\omega)\, dB(t) = \int_{\mathbb{R}} Y(t,\omega) \diamond W(t)\, d(t)$$

coincides with the classical Itô integral if $Y(t,\omega)$ is adapted to the filtration generated by $B(t)$, but is more general.

Fractional Brownian motion and white noise analysis

- Relate fBm to classical Brownian motion by an operator defined first for functions in $S(\mathbb{R})$ and then extended to $L^2(\mathbb{R})$.

$$\overset{\wedge}{Mf}(y) = |y|^{\frac{1}{2}-H}\hat{f}(y)$$

or

$$\begin{array}{rcll} Mf(x) & = & C_H \int_{\mathbb{R}} \frac{f(x-t)-f(x)}{|t|^{\frac{3}{2}-H}} dt & , \quad 0 < H < \frac{1}{2} \\ Mf(x) & = & f(x) & , \quad H = \frac{1}{2} \\ Mf(x) & = & C_H \int_{\mathbb{R}} \frac{f(t)}{|t-x|^{\frac{3}{2}-H}} dt & , \quad \frac{1}{2} < H < 1 \end{array}$$

$$C_H = \left\{2\Gamma\left(H-\tfrac{1}{2}\right)\cos\left(\tfrac{\pi}{2}\left(H-\tfrac{1}{2}\right)\right)\right\}^{-1}\left\{\Gamma(2H+1)\sin(\pi H)\right\}^{\frac{1}{2}}$$

Fractional Brownian motion and white noise analysis

- Relate fBm to classical Brownian motion by an operator defined first for functions in $S(\mathbb{R})$ and then extended to $L^2(\mathbb{R})$.

$$\overset{\wedge}{Mf}(y) = |y|^{\frac{1}{2}-H} \hat{f}(y)$$

or

$$\begin{array}{rcll} Mf(x) & = & C_H \int_{\mathbb{R}} \frac{f(x-t)-f(x)}{|t|^{\frac{3}{2}-H}} dt & , \quad 0 < H < \frac{1}{2} \\ Mf(x) & = & f(x) & , \quad H = \frac{1}{2} \\ Mf(x) & = & C_H \int_{\mathbb{R}} \frac{f(t)}{|t-x|^{\frac{3}{2}-H}} dt & , \quad \frac{1}{2} < H < 1 \end{array}$$

$$C_H = \left\{2\Gamma\left(H-\tfrac{1}{2}\right)\cos\left(\tfrac{\pi}{2}\left(H-\tfrac{1}{2}\right)\right)\right\}^{-1}\left\{\Gamma(2H+1)\sin(\pi H)\right\}^{\frac{1}{2}}$$

- Now define a space $L^2_H(\mathbb{R})$ by

$$(f, g)_{L^2_H(\mathbb{R})} = (Mf, Mg)_{L^2(\mathbb{R})}$$

and a process $\tilde{B}_H(t) := \left\langle \omega, M_{[0,t]}(\cdot) \right\rangle$ with $M_{[0,t]}(x) = M\chi_{[0,t]}(x)$

Fractional Brownian motion and white noise analysis

- Computing

$$\mathbb{E}\left[\tilde{B}_H(s)\,\tilde{B}_H(t)\right] = \frac{1}{2}\left\{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right\}$$

one concludes that the continuous version of $\tilde{B}_H(t)$ is fBm.

Fractional Brownian motion and white noise analysis

- Computing

$$\mathbb{E}\left[\tilde{B}_H(s)\,\tilde{B}_H(t)\right]=\frac{1}{2}\left\{|t|^{2H}+|s|^{2H}-|t-s|^{2H}\right\}$$

one concludes that the continuous version of $\tilde{B}_H(t)$ is fBm.

- *Integration of deterministic functions:* Let $f(x)$ be a step function

$$f(x)=\sum_j a_j\chi_{[t_j,t_{j+1}]}(x)$$

$$\begin{aligned}\langle\omega, Mf\rangle &= \sum_j a_j\left\langle\omega, M_{[t_j,t_{j+1}]}\right\rangle=\sum_j a_j\left(B_H(t_{j+1})-B_H(t_j)\right)\\ &= \int_{\mathbb{R}} f(t)\,dB_H(t)\end{aligned}$$

Extended to $L^2(\mathbb{R})$ provides a notion of integration for deterministic functions, which coincides with the pathwise definition.

Fractional Brownian motion and white noise analysis

- An orthonormal basis for $L^2_H(\mathbb{R})$

$$\left\{ e_k(x) = M^{-1}\xi_k(x), \qquad k = 1, 2, \cdots \right\}$$

$\xi_k(x)$ is an Hermite function

Fractional Brownian motion and white noise analysis

- An orthonormal basis for $L^2_H(\mathbb{R})$

$$\left\{ e_k(x) = M^{-1}\xi_k(x), \qquad k = 1, 2, \cdots \right\}$$

$\xi_k(x)$ is an Hermite function

- Chaos expansion of $B_H(t)$

$$\begin{aligned}
B_H(t) &= \left\langle \omega, M\chi_{[0,t]} \right\rangle \\
&= \left\langle M\omega, \textstyle\sum_{k=1}^{\infty} \left(\chi_{[0,t]}, e_k \right)_{L^2_H(\mathbb{R})} e_k(\cdot) \right\rangle \\
&= \left\langle M\omega, \textstyle\sum_{k=1}^{\infty} \left(M\chi_{[0,t]}, Me_k \right)_{L^2(\mathbb{R})} e_k(\cdot) \right\rangle \\
&= \textstyle\sum_{k=1}^{\infty} \left(M\chi_{[0,t]}, \xi_k \right)_{L^2(\mathbb{R})} \langle \omega, Me_k \rangle \\
&= \textstyle\sum_{k=1}^{\infty} \left(\chi_{[0,t]}, M\xi_k \right)_{L^2(\mathbb{R})} \langle \omega, Me_k \rangle \\
&= \textstyle\sum_{k=1}^{\infty} \int_0^t M\xi_k(s)\, ds \, \langle \omega, \xi_k \rangle
\end{aligned}$$

M extended to $S'(\mathbb{R})$ by $\langle M\omega, f \rangle = \langle \omega, Mf \rangle$

Fractional Brownian motion and white noise analysis

- Fractional White Noise

$$W_H(t) = \sum_{k=1}^{\infty} M\xi_k(t) \langle \omega, \xi_k \rangle$$

then,

$$\frac{dB_H(t)}{dt} = W_H(t)$$

in $(S)^*$

- Fractional White Noise

$$W_H(t) = \sum_{k=1}^{\infty} M\xi_k(t) \langle \omega, \xi_k \rangle$$

then,

$$\frac{dB_H(t)}{dt} = W_H(t)$$

in $(S)^*$

- Then the natural extension of Ito's integral to the fractional case is

$$\int_{\mathbb{R}} Y(t,\omega)\, dB_H(t) = \int_{\mathbb{R}} Y(t,\omega) \diamond W_H(t)\, d(t)$$

Example:

$$\int_0^T B_H(t)\, dB_H(t) = \frac{1}{2}(B_H(T))^2 - \frac{1}{2} T^{2H}$$

implying $\mathbb{E}\left[\int_0^T B_H(t)\, dB_H(t)\right] = 0$

Fractional Brownian motion and white noise analysis

- Directional derivative and (fractional) Malliavin derivative

$$D_\gamma^{(H)} F(\omega) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \{F(\omega + \varepsilon M\gamma) - F(\omega)\}$$

If there is a function $\Psi : \mathbb{R} \to (S)^*$ such that

$$D_\gamma^{(H)} F(\omega) = \int_{\mathbb{R}} M\Psi(t)\, M\gamma(t)\, dt$$

then

$$D_t^{(H)} F := \frac{\partial^{(H)}}{\partial \omega} F(t, \omega) = \Psi(t)$$

is the (fractional) Malliavin derivative or *stochastic gradient* of F.

Fractional Brownian motion and white noise analysis

Fractional Ito formulas

- $H \in (0,1)$

$$\begin{aligned} f\left(t, B_H\left(t\right)\right) &= f\left(0,0\right) + \int_0^t \frac{\partial f}{\partial s}\left(s, B_H\right) ds + \int_0^t \frac{\partial f}{\partial x}\left(s, B_H\right) dB_H\left(s\right) \\ &+ H \int_0^t \frac{\partial^2 f}{\partial x^2}\left(s, B_H\left(s\right)\right) s^{2H-1} ds \end{aligned}$$

Fractional Ito formulas

- $H \in (0,1)$

$$f(t, B_H(t)) = f(0,0) + \int_0^t \frac{\partial f}{\partial s}(s, B_H)\, ds + \int_0^t \frac{\partial f}{\partial x}(s, B_H)\, dB_H(s) + H \int_0^t \frac{\partial^2 f}{\partial x^2}(s, B_H(s))\, s^{2H-1} ds$$

- $H \in \left(\frac{1}{2}, 1\right)$ $\qquad dX_i(t) = \sum_{j=1}^m \sigma_{ij}(t, \omega)\, dB_H^j(t)$

$$\begin{aligned} f(t, X(t)) &= f(0, X(0)) + \int_0^t \frac{\partial f}{\partial s}(s, X(s))\, ds \\ &+ \int_0^t \sum_{i=1}^n \frac{\partial f}{\partial x^i}(s, X(s))\, dX_i(s) \\ &+ \int_0^t \left\{ \sum_{i,j=1}^m \frac{\partial^2 f}{\partial x^i \partial x^j}(s, X(s)) \sum_{k=1}^m \sigma_{ik}(s)\, D_{\phi,s}^k(X_j(s)) \right\} \end{aligned}$$

$D_{\phi,t} F = \int_{\mathbb{R}} \phi(s,t)\, D_s F ds$ and $\phi(s,t) = H(2H-1)|s-t|^{2H-2}$

References

- P. Embrechts and M. Maejima; *Selfsimilar processes*, Princeton Univ. Press, 2002.
- P. Doukhan, G. Oppenheim and M. S. Taqqu (Eds.); *Theory and Applications of long-range dependence*, Birkhäuser, Boston 2003.
- R. J. Elliot and J. Van der Hoek; *A general fractional white noise theory and applications to finance*, Mathematical Finance 13 (2003) 301-330.
- F. Biagini, B. Oksendal, A. Sulem, and N. Wallner; *An introduction to white noise theory and Malliavin Calculus for fractional Brownian motion*, Proc. Royal Soc. 460 (2004) 347-372.
- F. Biagini, Y. Hu, B. Oksendal and T. Zhang; *Stochastic calculus for fractional Brownian motion and applications*, Springer 2007.
- K. Dzhaparidze and H. van Zanten; *Krein's spectral theory and the Paley-Wiener expansion for fractional Brownian motion*, Annals of Prob. 33 (2005) 620-644.