Non-commutative tomography: A tool for data analysis and signal processing

Rui Vilela Mendes
IPFN, EURATOM-IST Association
CMAF, FCT - Lisbon
http://label2.ist.utl.pt/vilela/

Outline

- Integral transforms: linear and bilinear
- Wavelet-type, quasi-distributions and tomograms: Examples and relations
- Tomograms and the conformal group operators
- The time-frequency tomogram: Aplications:

1. Denoising and component separation
2. Plasma reflectometry

- Signal-adapted tomography

- **Integral transforms**

Tomographic data analysis. General setting

- **Integral transforms**
- *Linear transforms*: Fourier, Wavelets, Hilbert, ...

Tomographic data analysis. General setting

- **Integral transforms**
- *Linear transforms*: Fourier, Wavelets, Hilbert, ...
- *Bilinear transforms*: Wigner-Ville, Bertrand, Tomograms

Tomographic data analysis. General setting

- **Integral transforms**
- *Linear transforms*: Fourier, Wavelets, Hilbert, ...
- *Bilinear transforms*: Wigner-Ville, Bertrand, Tomograms
- **General setting**
  Consider signals $f(t)$ as vectors $\mid f\rangle \in$ dense nuclear subspace $\mathcal{N}$ of a Hilbert space $\mathcal{H}$ with dual space $\mathcal{N}^*$

Tomographic data analysis. General setting

- **Integral transforms**
- *Linear transforms*: Fourier, Wavelets, Hilbert, ...
- *Bilinear transforms*: Wigner-Ville, Bertrand, Tomograms
- **General setting**
  Consider signals $f(t)$ as vectors $\mid f\rangle \in$ dense nuclear subspace $\mathcal{N}$ of a Hilbert space $\mathcal{H}$ with dual space $\mathcal{N}^*$
- $\{U(\alpha) : \alpha \in I\}$ a family of operators defined on $\mathcal{N}^*$. (In many cases $U(\alpha)$ generates a unitary group $U(\alpha) = e^{iB(\alpha)}$)

Tomographic data analysis. General setting

- **Integral transforms**
- *Linear transforms*: Fourier, Wavelets, Hilbert, ...
- *Bilinear transforms*: Wigner-Ville, Bertrand, Tomograms
- **General setting**
  Consider signals $f(t)$ as vectors $\mid f\rangle \in$ dense nuclear subspace $\mathcal{N}$ of a Hilbert space $\mathcal{H}$ with dual space $\mathcal{N}^*$
- $\{U(\alpha) : \alpha \in I\}$ a family of operators defined on $\mathcal{N}^*$. (In many cases $U(\alpha)$ generates a unitary group $U(\alpha) = e^{iB(\alpha)}$)
- **Three types of transforms**
  Let $h \in \mathcal{N}^*$ be a reference vector such that the linear span of $\{U(\alpha)h \in \mathcal{N}^* : \alpha \in I\}$ is dense in $\mathcal{N}^*$. In the set $\{U(\alpha)h\}$, a complete set of vectors can be chosen to serve as a basis

Tomographic data analysis. General setting

- **Integral transforms**
- *Linear transforms*: Fourier, Wavelets, Hilbert, ...
- *Bilinear transforms*: Wigner-Ville, Bertrand, Tomograms
- **General setting**
  Consider signals $f(t)$ as vectors $\mid f\rangle \in$ dense nuclear subspace $\mathcal{N}$ of a Hilbert space $\mathcal{H}$ with dual space $\mathcal{N}^*$
- $\{U(\alpha) : \alpha \in I\}$ a family of operators defined on $\mathcal{N}^*$. (In many cases $U(\alpha)$ generates a unitary group $U(\alpha) = e^{iB(\alpha)}$)
- **Three types of transforms**
  Let $h \in \mathcal{N}^*$ be a reference vector such that the linear span of $\{U(\alpha)h \in \mathcal{N}^* : \alpha \in I\}$ is dense in $\mathcal{N}^*$. In the set $\{U(\alpha)h\}$, a complete set of vectors can be chosen to serve as a basis
- **1 - Wavelet-type transform**

$$W_f^{(h)}(\alpha) = \langle U(\alpha)\, h \mid f\rangle,$$

Tomographic data analysis. General setting

- **2 - Quasi-distribution**

$$Q_f(\alpha) = \langle U(\alpha) f \mid f \rangle.$$

- **2 - Quasi-distribution**

$$Q_f(\alpha) = \langle U(\alpha) f \mid f \rangle.$$

- If $U(\alpha)$ is a unitary operator there is a self-adjoint operator $B(\alpha)$

$$W_f^{(h)}(\alpha) = \langle h \mid e^{iB(\alpha)} \mid f \rangle$$

$$Q_f^{(B)}(\alpha) = \langle f \mid e^{iB(\alpha)} \mid f \rangle$$

Tomographic data analysis. General setting

- **2 - Quasi-distribution**

$$Q_f(\alpha) = \langle U(\alpha) f \mid f \rangle .$$

- If $U(\alpha)$ is a unitary operator there is a self-adjoint operator $B(\alpha)$

$$W_f^{(h)}(\alpha) = \langle h \mid e^{iB(\alpha)} \mid f \rangle$$

$$Q_f^{(B)}(\alpha) = \langle f \mid e^{iB(\alpha)} \mid f \rangle$$

- **3 - Tomographic transform or tomogram**

$$M_f^{(B)}(X) = \langle f \mid \delta(B(\alpha) - X) \mid f \rangle$$

($\delta(B(\alpha) - X) = |X\rangle\langle X| =$projector on the eigenvector of $B(\alpha)$ with eigenvalue X)

Examples for wavelet-type and quasi-distributions

- **Fourier transform**: is $W_f^{(h)}(\alpha)$ if $U(\alpha)$ is unitary generated by $B_F\left(\overrightarrow{\alpha}\right) = \alpha_1 t + i\alpha_2 \frac{d}{dt}$ and h is a (generalized) eigenvector of the time-translation operator

Examples for wavelet-type and quasi-distributions

- **Fourier transform**: is $W_f^{(h)}(\alpha)$ if $U(\alpha)$ is unitary generated by $B_F\left(\overrightarrow{\alpha}\right) = \alpha_1 t + i\alpha_2 \frac{d}{dt}$ and h is a (generalized) eigenvector of the time-translation operator
- **Ambiguity function:** $Q_f(\alpha)$ for the same $B_F\left(\overrightarrow{\alpha}\right)$

Examples for wavelet-type and quasi-distributions

- **Fourier transform**: is $W_f^{(h)}(\alpha)$ if $U(\alpha)$ is unitary generated by $B_F(\overrightarrow{\alpha}) = \alpha_1 t + i\alpha_2 \frac{d}{dt}$ and h is a (generalized) eigenvector of the time-translation operator
- **Ambiguity function:** $Q_f(\alpha)$ for the same $B_F(\overrightarrow{\alpha})$
- **Wigner–Ville transform**: $Q_f(\alpha)$ for the same $B_F(\overrightarrow{\alpha})$ plus the parity operator

$$B^{(WV)}(\alpha_1, \alpha_2) = -i2\alpha_1 \frac{d}{dt} - 2\alpha_2 t + \frac{\pi\left(t^2 - \frac{d^2}{dt^2} - 1\right)}{2}.$$

Examples for wavelet-type and quasi-distributions

- **Fourier transform**: is $W_f^{(h)}(\alpha)$ if $U(\alpha)$ is unitary generated by $B_F\left(\overrightarrow{\alpha}\right) = \alpha_1 t + i\alpha_2 \frac{d}{dt}$ and h is a (generalized) eigenvector of the time-translation operator
- **Ambiguity function:** $Q_f(\alpha)$ for the same $B_F\left(\overrightarrow{\alpha}\right)$
- **Wigner–Ville transform**: $Q_f(\alpha)$ for the same $B_F\left(\overrightarrow{\alpha}\right)$ plus the parity operator

$$B^{(WV)}(\alpha_1, \alpha_2) = -i2\alpha_1 \frac{d}{dt} - 2\alpha_2 t + \frac{\pi\left(t^2 - \frac{d^2}{dt^2} - 1\right)}{2}.$$

- **Wavelet transform**: $W_f^{(h)}(\alpha)$ for $B_W\left(\overrightarrow{\alpha}\right) = \alpha_1 D + i\alpha_2 \frac{d}{dt}$, D being the dilation operator $D = -\frac{1}{2}\left(it\frac{d}{dt} + i\frac{d}{dt}t\right)$

Examples for wavelet-type and quasi-distributions

- **Fourier transform**: is $W_f^{(h)}(\alpha)$ if $U(\alpha)$ is unitary generated by $B_F\left(\overrightarrow{\alpha}\right) = \alpha_1 t + i\alpha_2 \frac{d}{dt}$ and h is a (generalized) eigenvector of the time-translation operator
- **Ambiguity function:** $Q_f(\alpha)$ for the same $B_F\left(\overrightarrow{\alpha}\right)$
- **Wigner–Ville transform**: $Q_f(\alpha)$ for the same $B_F\left(\overrightarrow{\alpha}\right)$ plus the parity operator

$$B^{(WV)}(\alpha_1, \alpha_2) = -i2\alpha_1 \frac{d}{dt} - 2\alpha_2 t + \frac{\pi\left(t^2 - \frac{d^2}{dt^2} - 1\right)}{2}.$$

- **Wavelet transform**: $W_f^{(h)}(\alpha)$ for $B_W\left(\overrightarrow{\alpha}\right) = \alpha_1 D + i\alpha_2 \frac{d}{dt}$, D being the dilation operator $D = -\frac{1}{2}\left(it\frac{d}{dt} + i\frac{d}{dt}t\right)$
- **Bertrand transform:** $Q_f(\alpha)$ for B_W

The tomographic transform (tomogram)

- $$M_f^{(B)}(X) = \langle f \mid \delta\left(B\left(\alpha\right) - X\right) \mid f\rangle$$

The tomographic transform (tomogram)

- $$M_f^{(B)}(X) = \langle f \mid \delta\left(B\left(\alpha\right) - X\right) \mid f \rangle$$
- $M_f^{(B)}(\alpha)$ is positive and may be interpreted as a probability distribution. Benefits from the properties of the bilinear transforms, without interpretation ambiguities

The tomographic transform (tomogram)

- $$M_f^{(B)}(X) = \langle f \mid \delta\left(B\left(\alpha\right) - X\right) \mid f \rangle$$
- $M_f^{(B)}(\alpha)$ is positive and may be interpreted as a probability distribution. Benefits from the properties of the bilinear transforms, without interpretation ambiguities
- For normalized $\mid f \rangle$,
  $$\langle f \mid f \rangle = 1$$
  the tomogram is normalized
  $$\int M_f^{(B)}\left(X\right)\, dX = 1$$
  It is a probability distribution for the random variable X corresponding to the observable defined by the operator $B\left(\alpha\right)$

The tomographic transform (tomogram)

- $$M_f^{(B)}(X) = \langle f \mid \delta\left(B\left(\alpha\right) - X\right) \mid f \rangle$$
- $M_f^{(B)}(\alpha)$ is positive and may be interpreted as a probability distribution. Benefits from the properties of the bilinear transforms, without interpretation ambiguities
- For normalized $\mid f \rangle$,
  $$\langle f \mid f \rangle = 1$$
  the tomogram is normalized
  $$\int M_f^{(B)}\left(X\right) \, dX = 1$$
  It is a probability distribution for the random variable X corresponding to the observable defined by the operator $B\left(\alpha\right)$
- The tomogram is a homogeneous function
  $$M_f^{(B/p)}(X) = |p| M_f^{(B)}(pX)$$

Relations between the three types of transforms

- $$M_f^{(B)}(X) = \frac{1}{2\pi} \int Q_f^{(kB)}(\alpha)\, e^{-ikX}\, dk$$

Relations between the three types of transforms

- $$M_f^{(B)}(X) = \frac{1}{2\pi} \int Q_f^{(kB)}(\alpha)\, e^{-ikX}\, dk$$
- $$Q_f^{(B)}(\alpha) = \int M_f^{(B/p)}(X)\, e^{ipX}\, dX.$$

Relations between the three types of transforms

- $$M_f^{(B)}(X) = \frac{1}{2\pi} \int Q_f^{(kB)}(\alpha)\, e^{-ikX}\, dk$$
- $$Q_f^{(B)}(\alpha) = \int M_f^{(B/p)}(X)\, e^{ipX}\, dX.$$
- $$Q_f^{(B)}(\alpha) = W_f^{(f)}(\alpha),$$

Relations between the three types of transforms

- $$M_f^{(B)}(X) = \frac{1}{2\pi} \int Q_f^{(kB)}(\alpha)\, e^{-ikX}\, dk$$
- $$Q_f^{(B)}(\alpha) = \int M_f^{(B/p)}(X)\, e^{ipX}\, dX.$$
- $$Q_f^{(B)}(\alpha) = W_f^{(f)}(\alpha),$$
- $$W_f^{(h)}(\alpha) = \frac{1}{4} \int e^{iX} \begin{bmatrix} M_{f_1}^{(B)}(X) - iM_{f_2}^{(B)}(X) \\ -M_{f_3}^{(B)}(X) + iM_{f_4}^{(B)}(X) \end{bmatrix} dX,$$

with

$$\begin{aligned} &| \quad f_1\rangle =| h\rangle + | f\rangle; \qquad && | f_3\rangle =| h\rangle - | f\rangle; \\ &| \quad f_2\rangle =| h\rangle + i \mid f\rangle; \qquad && | f_4\rangle =| h\rangle - i \mid f\rangle. \end{aligned}$$

The conformal group

- The generators of the conformal group

$$\text{in } \mathbb{R}^d \qquad \begin{array}{l} \omega_k = i\frac{\partial}{\partial t_k} \\ D = i\left(t \bullet \nabla + \frac{d}{2}\right) \\ R_{j,k} = i\left(t_j \frac{\partial}{\partial t_k} - t_k \frac{\partial}{\partial t_j}\right) \\ K_j = i\left(t_j^2 \frac{\partial}{\partial t_j} + t_j\right) \end{array}$$

The conformal group

- The generators of the conformal group

in $\mathbb{R}^d$

$$\begin{aligned} \omega_k &= i\tfrac{\partial}{\partial t_k} \\ D &= i\left(t \bullet \nabla + \tfrac{d}{2}\right) \\ R_{j,k} &= i\left(t_j \tfrac{\partial}{\partial t_k} - t_k \tfrac{\partial}{\partial t_j}\right) \\ K_j &= i\left(t_j^2 \tfrac{\partial}{\partial t_j} + t_j\right) \end{aligned}$$

- For $d = 1$

in $\mathbb{R}$

$$\begin{aligned} \omega &= i\tfrac{d}{dt} \\ D &= i\left(t\tfrac{d}{dt} + \tfrac{1}{2}\right) \\ K &= i\left(t^2\tfrac{d}{dt} + t\right) \end{aligned}$$

Tomograms associated to the conformal group

- Time-frequency tomogram

$$B_1 = \mu t + i\nu \frac{d}{dt}$$

Tomograms associated to the conformal group

- Time-frequency tomogram

$$B_1 = \mu t + i\nu \frac{d}{dt}$$

- Time-scale

$$B_2 = \mu t + i\nu \left(t \frac{d}{dt} + \frac{1}{2} \right)$$

Tomograms associated to the conformal group

- Time-frequency tomogram

$$B_1 = \mu t + i\nu \frac{d}{dt}$$

- Time-scale

$$B_2 = \mu t + i\nu \left(t\frac{d}{dt} + \frac{1}{2} \right)$$

- Frequency-scale

$$B_3 = i\mu \frac{d}{dt} + i\nu \left(t\frac{d}{dt} + \frac{1}{2} \right)$$

Tomograms associated to the conformal group

- Time-frequency tomogram

$$B_1 = \mu t + i\nu \frac{d}{dt}$$

- Time-scale

$$B_2 = \mu t + i\nu \left(t\frac{d}{dt} + \frac{1}{2} \right)$$

- Frequency-scale

$$B_3 = i\mu \frac{d}{dt} + i\nu \left(t\frac{d}{dt} + \frac{1}{2} \right)$$

- Time-conformal

$$B_4 = \mu t + i\nu \left(t^2\frac{d}{dt} + t \right)$$

Tomograms associated to the conformal group

- General construction of the tomograms: Let

$$\int dY \left|Y\right\rangle \left\langle Y\right| = 1$$

be a decomposition of the unit, with generalized eigenvectors of the operator B. Then

$$M(\alpha, X) = \int dY \langle f \mid \delta\left(B\left(\alpha\right) - X\right) \left|Y\right\rangle \left\langle Y\right| \mid f \rangle = \left|\langle X \mid f \rangle\right|^2$$

Tomograms associated to the conformal group

- General construction of the tomograms: Let

$$\int dY \left|Y\right\rangle \left\langle Y\right| = 1$$

be a decomposition of the unit, with generalized eigenvectors of the operator B. Then

$$M(\alpha, X) = \int dY \langle f \mid \delta\left(B\left(\alpha\right) - X\right) \left|Y\right\rangle \left\langle Y\right| \mid f\rangle = \left|\langle X \mid f\rangle\right|^2$$

- Therefore the construction of the tomograms reduces to the calculation of the generalized eigenvectors of each B operator

Tomograms associated to the conformal group

- General construction of the tomograms: Let

$$\int dY \left| Y \right\rangle \left\langle Y \right| = 1$$

be a decomposition of the unit, with generalized eigenvectors of the operator B. Then

$$M(\alpha, X) = \int dY \langle f \mid \delta \left(B\left(\alpha \right) - X \right) \left| Y \right\rangle \left\langle Y \right| \mid f \rangle = \left| \langle X \mid f \rangle \right|^2$$

- Therefore the construction of the tomograms reduces to the calculation of the generalized eigenvectors of each B operator
- $B_1 \psi_1 \left(\mu, \nu, t, X \right) = X \psi_1 \left(\mu, \nu, t, X \right)$

$$\psi_1 \left(\mu, \nu, t, X \right) = \exp i \left(\frac{\mu t^2}{2\nu} - \frac{tX}{\nu} \right)$$

$$\int dt \psi_1^* \left(\mu, \nu, t, X \right) \psi_1 \left(\mu, \nu, t, X' \right) = 2\pi\nu\delta \left(X - X' \right)$$

Tomograms associated to the conformal group

- $B_2\psi_2(\mu,\nu,t,X) = X\psi_2(\mu,\nu,t,X)$

$$\psi_2(\mu,\nu,t,X) = \frac{1}{\sqrt{|t|}} \exp i\left(\frac{\mu t}{\nu} - \frac{X}{\nu}\log|t|\right)$$

$$\int dt\psi_2^*(\mu,\nu,t,X)\,\psi_2(\mu,\nu,t,X') = 4\pi\nu\delta(X-X')$$

Tomograms associated to the conformal group

- $B_2\psi_2(\mu,\nu,t,X) = X\psi_2(\mu,\nu,t,X)$

$$\psi_2(\mu,\nu,t,X) = \frac{1}{\sqrt{|t|}}\exp i\left(\frac{\mu t}{\nu} - \frac{X}{\nu}\log|t|\right)$$

$$\int dt\psi_2^*(\mu,\nu,t,X)\,\psi_2(\mu,\nu,t,X') = 4\pi\nu\delta(X-X')$$

- $B_3\psi_3(\mu,\nu,\omega,X) = X\psi_3(\mu,\nu,\omega,X)$

$$\psi_3(\mu,\nu,t,X) = \exp(-i)\left(\frac{\mu}{\nu}\omega - \frac{X}{\nu}\log|\omega|\right)$$

$$\int d\omega\psi_1^*(\mu,\nu,\omega,X)\,\psi_1(\mu,\nu,\omega,X') = 2\pi\nu\delta(X-X')$$

Tomograms associated to the conformal group

- $B_2\psi_2(\mu,\nu,t,X) = X\psi_2(\mu,\nu,t,X)$

$$\psi_2(\mu,\nu,t,X) = \frac{1}{\sqrt{|t|}} \exp i\left(\frac{\mu t}{\nu} - \frac{X}{\nu}\log|t|\right)$$

$$\int dt\psi_2^*(\mu,\nu,t,X)\,\psi_2(\mu,\nu,t,X') = 4\pi\nu\delta(X-X')$$

- $B_3\psi_3(\mu,\nu,\omega,X) = X\psi_3(\mu,\nu,\omega,X)$

$$\psi_3(\mu,\nu,t,X) = \exp(-i)\left(\frac{\mu}{\nu}\omega - \frac{X}{\nu}\log|\omega|\right)$$

$$\int d\omega\psi_1^*(\mu,\nu,\omega,X)\,\psi_1(\mu,\nu,\omega,X') = 2\pi\nu\delta(X-X')$$

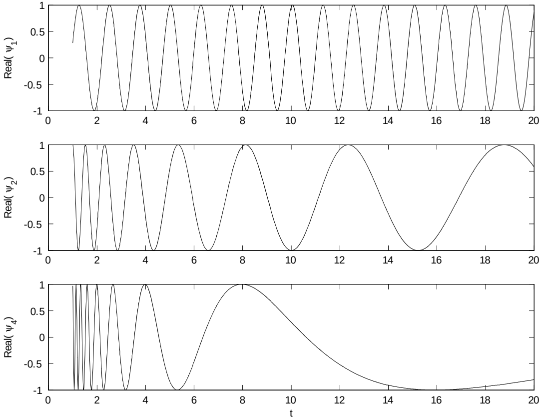
- $B_4\psi_4(\mu,\nu,t,X) = X\psi_4(\mu,\nu,t,X)$

$$\psi_4(\mu,\nu,t,X) = \frac{1}{|t|} \exp i\left(\frac{X}{\nu t} + \frac{\mu}{\nu}\log|t|\right)$$

$$\int dt\psi_4^*(\mu,\nu,t,s)\,\psi_4(\mu,\nu,t,s') = 2\pi\nu\delta(s-s')$$

Tomograms associated to the conformal group

$\mu = 0$



Tomograms associated to the conformal group

- Time-frequency tomogram

$$M_1(\mu,\nu,X)=\frac{1}{2\pi|\nu|}\left|\int \exp\left[\frac{i\mu t^2}{2\nu}-\frac{itX}{\nu}\right] f(t)\,dt\right|^2$$

Tomograms associated to the conformal group

- Time-frequency tomogram

$$M_1\left(\mu,\nu,X\right)=\frac{1}{2\,\pi|\nu|}\left|\int \exp\left[\frac{i\mu t^2}{2\,\nu}-\frac{itX}{\nu}\right]f\left(t\right)dt\right|^2$$

- Time-scale tomogram

$$M_2(\mu,\nu,X)=\frac{1}{2\pi|\nu|}\left|\int dt\,\frac{f\left(t\right)}{\sqrt{|t|}}e^{\left[i\left(\frac{\mu}{\nu}t-\frac{X}{\nu}\log|t|\right)\right]}\right|^2$$

Tomograms associated to the conformal group

- Time-frequency tomogram

$$M_1\left(\mu,\nu,X\right)=\frac{1}{2\,\pi|\nu|}\left|\int\exp\left[\frac{i\mu t^2}{2\,\nu}-\frac{itX}{\nu}\right]f\left(t\right)dt\right|^2$$

- Time-scale tomogram

$$M_2(\mu,\nu,X)=\frac{1}{2\pi|\nu|}\left|\int dt\,\frac{f(t)}{\sqrt{|t|}}e^{\left[i\left(\frac{\mu}{\nu}t-\frac{X}{\nu}\log|t|\right)\right]}\right|^2$$

- Frequency-scale tomogram

$$M_3(\mu,\nu,X)=\frac{1}{2\pi|\nu|}\left|\int d\omega\,\frac{f(\omega)}{\sqrt{|\omega|}}e^{\left[-i\left(\frac{\mu}{\nu}\omega-\frac{X}{\nu}\log|\omega|\right)\right]}\right|^2$$

$f(\omega)=$ Fourier transform of $f(t)$

Tomograms associated to the conformal group

- Time-frequency tomogram

$$M_1\left(\mu,\nu,X\right)=\frac{1}{2\,\pi|\nu|}\left|\int \exp\left[\frac{i\mu t^2}{2\,\nu}-\frac{itX}{\nu}\right]f\left(t\right)dt\right|^2$$

- Time-scale tomogram

$$M_2(\mu,\nu,X)=\frac{1}{2\pi|\nu|}\left|\int dt\,\frac{f\left(t\right)}{\sqrt{|t|}}e^{\left[i\left(\frac{\mu}{\nu}t-\frac{X}{\nu}\log|t|\right)\right]}\right|^2$$

- Frequency-scale tomogram

$$M_3(\mu,\nu,X)=\frac{1}{2\pi|\nu|}\left|\int d\omega\,\frac{f\left(\omega\right)}{\sqrt{|\omega|}}e^{\left[-i\left(\frac{\mu}{\nu}\omega-\frac{X}{\nu}\log|\omega|\right)\right]}\right|^2$$

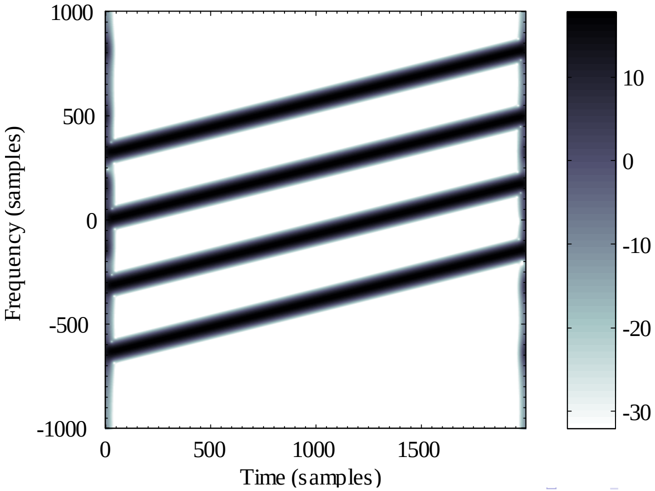
$f\left(\omega\right)=$ Fourier transform of $f\left(t\right)$

- Time-conformal tomogram

$$M_4(\mu,\nu,X)=\frac{1}{2\pi|\nu|}\left|\int dt\,\frac{f\left(t\right)}{|t|}e^{\left[i\left(\frac{X}{\nu t}+\frac{\mu}{\nu}\log|t|\right)\right]}\right|^2$$

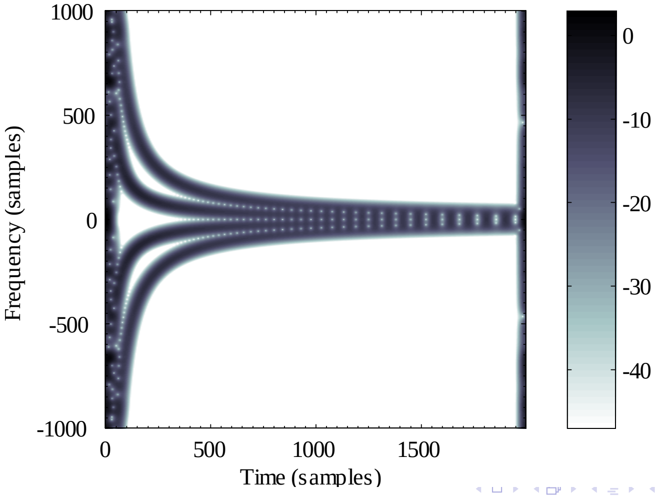
Basis functions of the tomograms in the time-frequency plane

Time-frequency



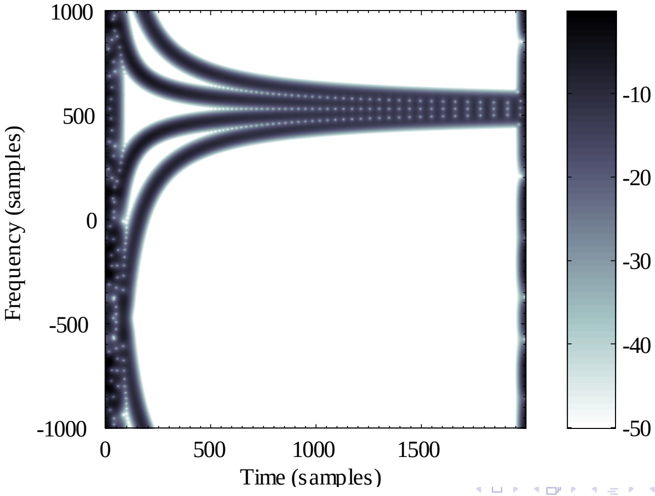
Basis functions of the tomograms in the time-frequency plane

Time-scale



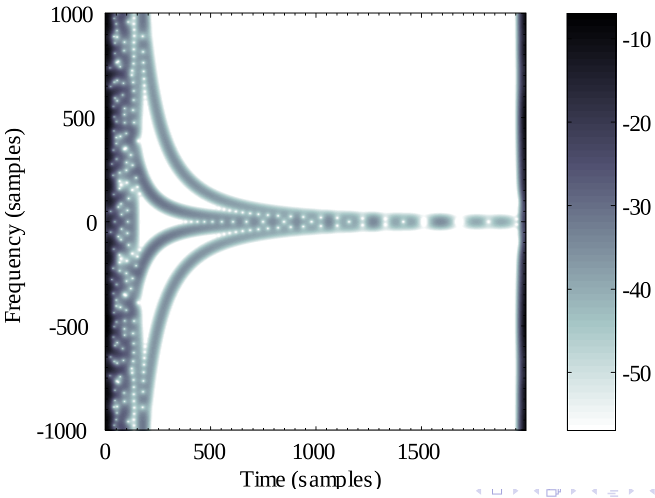
Basis functions of the tomograms in the time-frequency plane

Time-scale



Basis functions of the tomograms in the time-frequency plane

Time-conformal



Applications: Component decomposition

- Most natural and man-made signals are nonstationary and have a multicomponent structure.
  Examples: Bat echolocation, whale sounds, radar, sonar, etc.

Applications: Component decomposition

- Most natural and man-made signals are nonstationary and have a multicomponent structure.
  Examples: Bat echolocation, whale sounds, radar, sonar, etc.
- The concept of signal component is not uniquely defined. The notion of *component* depends as much on the observer as on the observed object. When we speak about a component of a signal we are in fact referring to a particular feature of the signal that we want to emphasize.

Applications: Component decomposition

- Most natural and man-made signals are nonstationary and have a multicomponent structure.
  Examples: Bat echolocation, whale sounds, radar, sonar, etc.
- The concept of signal component is not uniquely defined. The notion of *component* depends as much on the observer as on the observed object. When we speak about a component of a signal we are in fact referring to a particular feature of the signal that we want to emphasize.
- One possibility: Separation of components using its behavior in the time-frequency plane. Consider the finite-time tomogram

$$M(\theta, X) = \left| \int f(t) \psi_{\theta,X}(t) \, dt \right|^2 = |< f, \psi >|^2$$

with

$$\psi_{\theta,X}(t) = \frac{1}{\sqrt{T}} \exp\left(\frac{-i\cos\theta}{2\sin\theta} t^2 + \frac{iX}{\sin\theta} t \right)$$

$\mu = \cos\theta, \nu = \sin\theta$.

Component decomposition

- θ is a parameter that interpolates between the time and the frequency operators, running from 0 to $\pi/2$ whereas X is allowed to be any real number.

Component decomposition

- θ is a parameter that interpolates between the time and the frequency operators, running from 0 to $\pi/2$ whereas X is allowed to be any real number.
- For all different θ's the $U(\theta)$ are unitarily equivalent operators, hence all the tomograms share the same information. The component separation technique is based on the search for an intermediate value of θ where a good compromise might be found between time localization and frequency information.

Component decomposition

- θ is a parameter that interpolates between the time and the frequency operators, running from 0 to $\pi/2$ whereas X is allowed to be any real number.
- For all different θ's the $U(\theta)$ are unitarily equivalent operators, hence all the tomograms share the same information. The component separation technique is based on the search for an intermediate value of θ where a good compromise might be found between time localization and frequency information.
- First select a subset X_n in such a way that the corresponding family $\left\{\psi_{\theta,X_n}(t)\right\}$ is orthogonal and normalized,

$$< \psi_{\theta,X_n}\psi_{\theta,X_m} >= \delta_{m,n}$$

  This is possible by taking the sequence

$$X_n = X_0 + \frac{2n\pi}{T}\sin\theta$$

  where X_0 is freely chosen (in general we take $X_0 = 0$)

Component decomposition and denoising

- We then consider the projections of the signal $f(t)$

$$c_{X_n}^{\theta}(f) = < f, \psi_{\theta, X_n} >$$

which are used for the signal processing.

Component decomposition and denoising

- We then consider the projections of the signal $f(t)$

$$c_{X_n}^{\theta}(f) =< f, \psi_{\theta,X_n} >$$

which are used for the signal processing.

- Denoising consists in eliminating the $c_{X_n}^{\theta}(f)$ such that

$$\left| c_{X_n}^{\theta}(f) \right|^2 \leq \epsilon$$

for some threshold ϵ

Component decomposition and denoising

- We then consider the projections of the signal $f(t)$

$$c_{X_n}^{\theta}(f) = < f, \psi_{\theta, X_n} >$$

which are used for the signal processing.

- Denoising consists in eliminating the $c_{X_n}^{\theta}(f)$ such that

$$\left| c_{X_n}^{\theta}(f) \right|^2 \leq \epsilon$$

for some threshold ϵ

- Multi-component analysis is done by selecting subsets $\mathcal{F}_k$ of the X_n and reconstructing partial signals (k-components) by restricting the sum to

$$f_k(t) = \sum_{n \in \mathcal{F}_k} c_{X_n}^{\theta}(f) \psi_{\theta, X_n}(t)$$

for each k.

Component decomposition. Examples

- $$y(t) = y_1(t) + y_2(t) + y_3(t) + b(t)$$

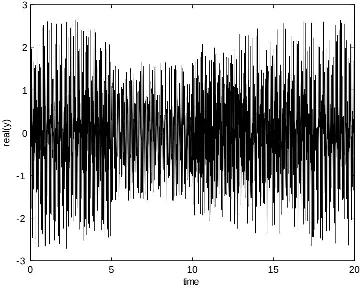
$$\begin{aligned} y_1(t) &= \exp(i25t), t \in [0, 20] \\ y_2(t) &= \exp(i75t), t \in [0, 5] \\ y_3(t) &= \exp(i75t), t \in [10, 20] \end{aligned}$$

Component decomposition. Examples

- $$y(t) = y_1(t) + y_2(t) + y_3(t) + b(t)$$

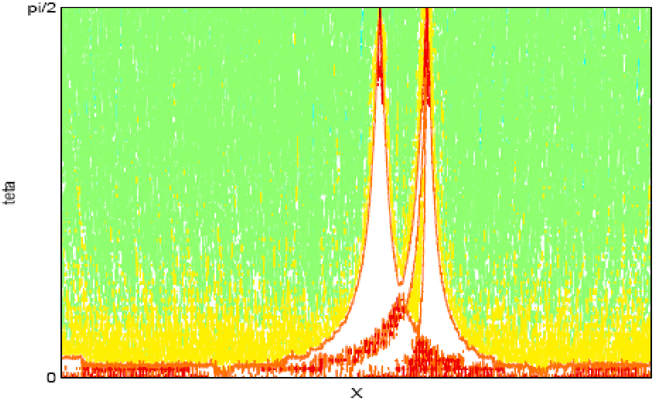
$$\begin{aligned} y_1(t) &= \exp(i25t), t \in [0, 20] \\ y_2(t) &= \exp(i75t), t \in [0, 5] \\ y_3(t) &= \exp(i75t), t \in [10, 20] \end{aligned}$$

- Real part of the time signal



Component decomposition

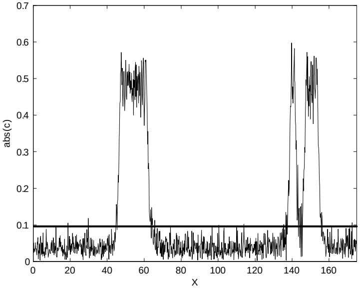
The tomogram



Component decomposition. Examples

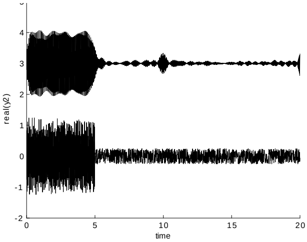
- Separation at $\theta = \frac{\pi}{5}$

Component decomposition. Examples

- Separation at $\theta = \frac{\pi}{5}$
- 

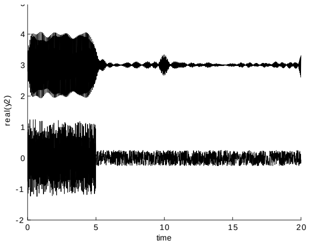
Component decomposition. Examples

- Reconstrution of the $y_2\left(t\right)$

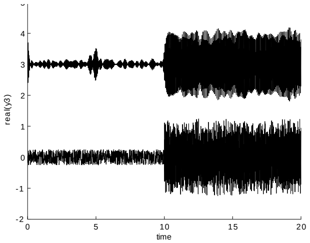


Component decomposition. Examples

- Reconstrution of the $y_2\left(t\right)$



- and $y_3\left(t\right)$ components



Component decomposition. Examples

- Sum $y(t) = y_0(t) + y_R(t) + b(t)$ of an "incident" $y_0(t)$ and a "deformed reflected" chirp $y_R(t)$ delayed by $3s$ with white noise added.

$$y_0(t) = e^{i\Phi_0(t)} \qquad\qquad y_R(t) = e^{i\Phi_R(t)}$$

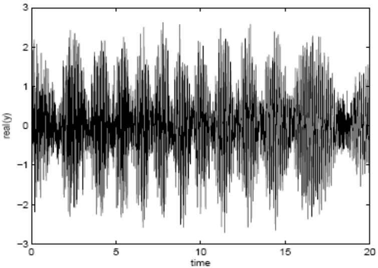
$\Phi_0(t) = a_0 t^2 + b_0 t$ and
$\Phi_R(t) = a_R(t - t_R)^2 + b_R(t - t_R) + 10(t - t_R)^{\frac{3}{2}}$.

Component decomposition. Examples

- Sum $y(t) = y_0(t) + y_R(t) + b(t)$ of an "incident" $y_0(t)$ and a "deformed reflected" chirp $y_R(t)$ delayed by $3s$ with white noise added.

  $$y_0(t) = e^{i\Phi_0(t)} \qquad y_R(t) = e^{i\Phi_R(t)}$$

  $\Phi_0(t) = a_0 t^2 + b_0 t$ and
  $\Phi_R(t) = a_R(t - t_R)^2 + b_R(t - t_R) + 10(t - t_R)^{\frac{3}{2}}$.



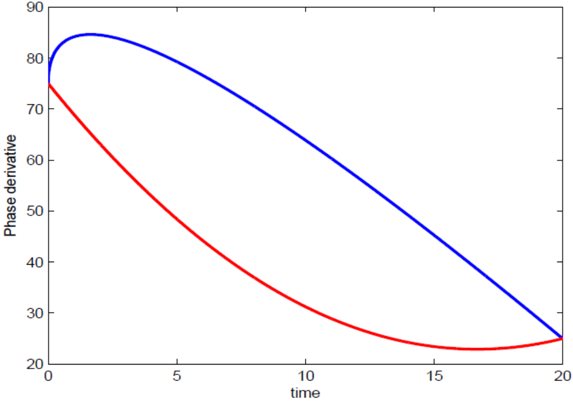
-

Component decomposition. Examples

- Comparison of the phase derivatives $\frac{d}{dt}\Phi_0(t)$ and $\frac{d}{dt}\Phi_R(t)$. Except for the three first seconds, the spectrum of the signals $y_0(t)$ and $y_R(t)$ is almost the same

Component decomposition. Examples

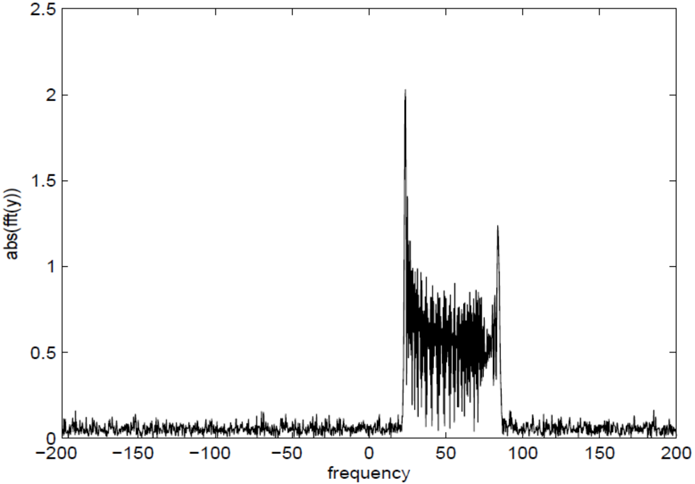
- Comparison of the phase derivatives $\frac{d}{dt}\Phi_0(t)$ and $\frac{d}{dt}\Phi_R(t)$. Except for the three first seconds, the spectrum of the signals $y_0(t)$ and $y_R(t)$ is almost the same



-

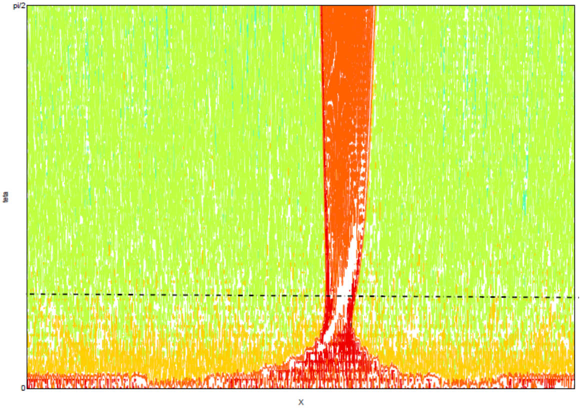
Component decomposition. Examples

- Frequency representation

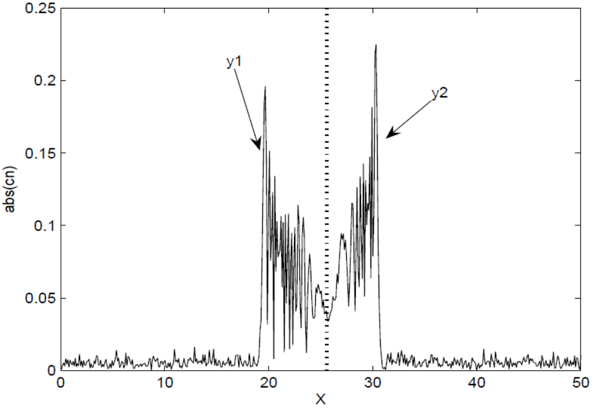


Component decomposition. Examples

- Tomogram of the chirps signal

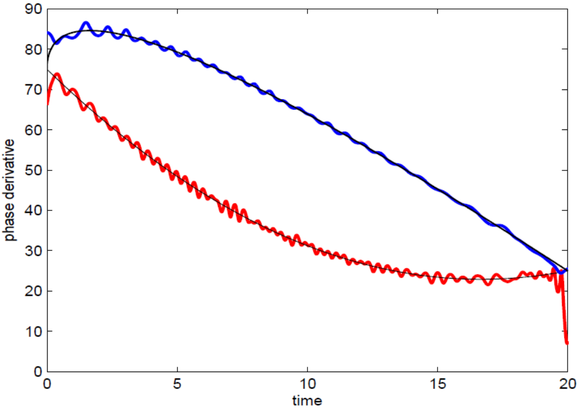


Separable spectrum at $\theta = \frac{\pi}{5}$

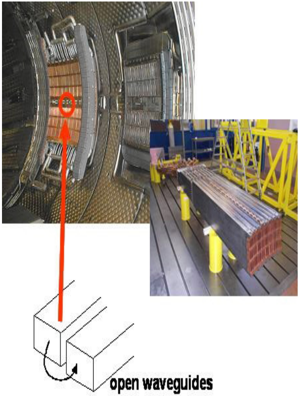


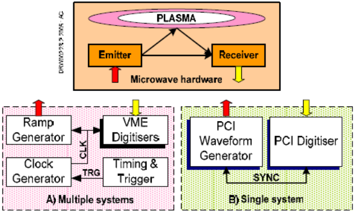
Component decomposition. Examples

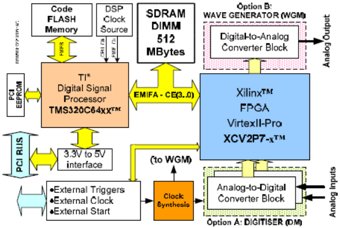
The phase derivative



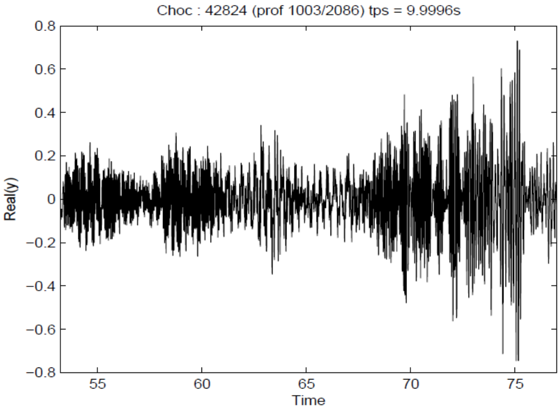
Component decomposition. Reflectometry



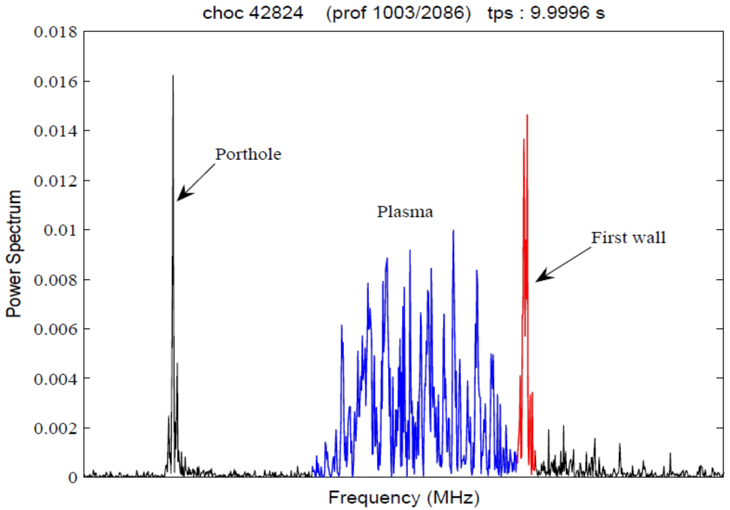




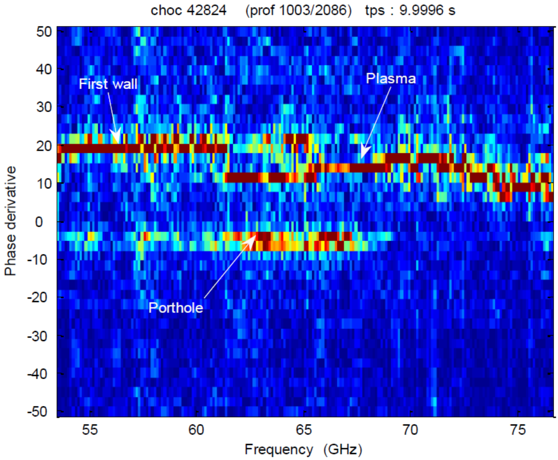
- Reflectometry signal



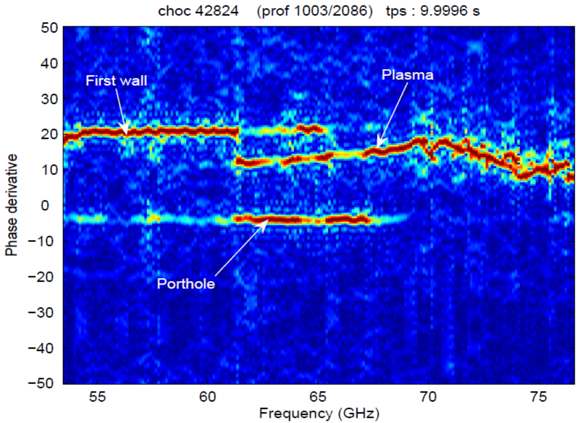
Component decomposition. Examples



Spectrogram

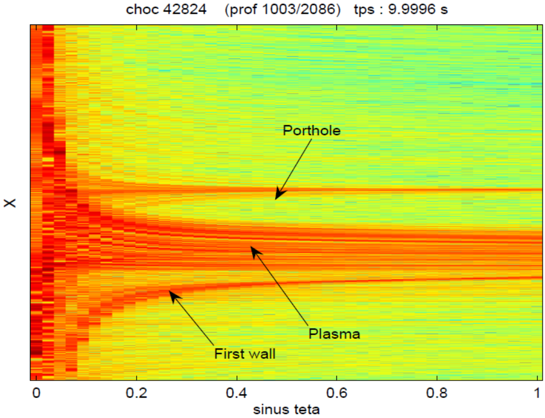


Oversampled spectrogram



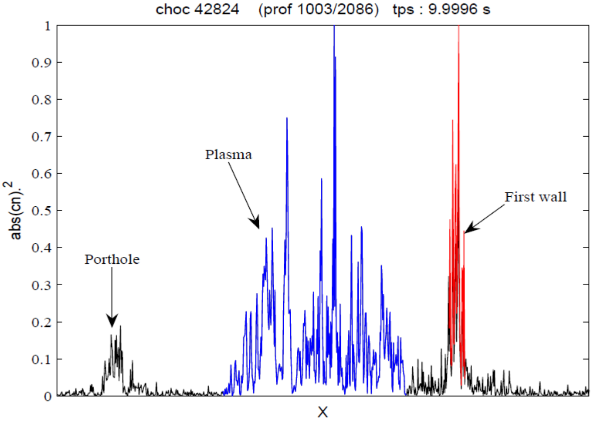
Component decomposition. Examples

- Tomogram of the reflectometry signal

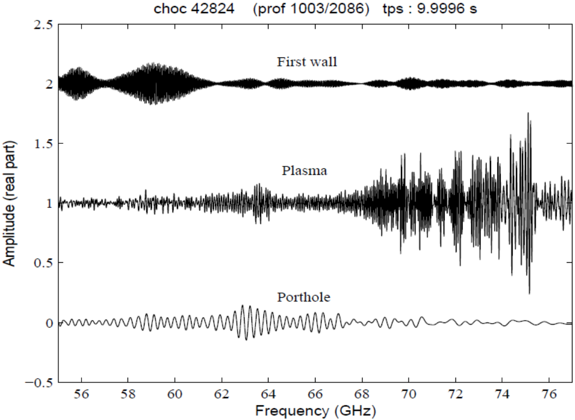


Component decomposition. Examples

- "Spectrum" at $\theta = \pi - \frac{\pi}{5}$

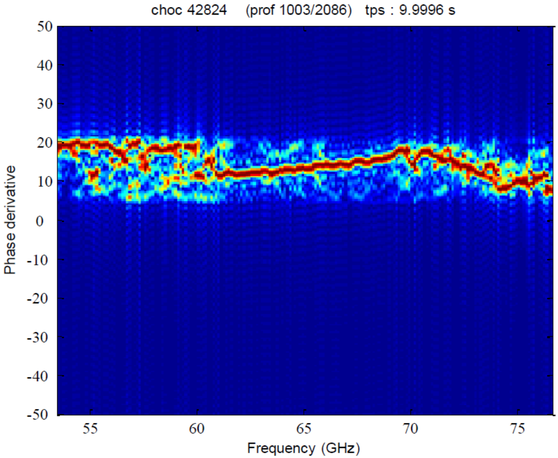


Component decomposition. Examples



Component decomposition. Examples

Spectrogram of the plasma component



Tomography with an adapted operator pair

- In $B(\mu,\nu) = \mu t + \nu S$, choose an operator S, specially tuned to the features of the signal that one wants to extract.
- At particular values of (μ,ν) noise effects may cancel out. Separates the information of very small signals from large noise and obtain reliable information on the temporal structure of the signal. **A signal-adapted filtering technique**.
- To construct S consider a set of $N-$dimensional time sequences $\{\overrightarrow{x_1}, \cdots, \overrightarrow{x_k}\}$, typical of the signal one wants to detect. (May be considered as the code words that later one wishes to detect in a noisy signal).
- Form the $k \times N$ matrix $U \in \mathcal{M}_{k\times N}$.

$$U = \begin{pmatrix} x_1(1\Delta t) & x_1(2\Delta t) & \dots & x_1(N\Delta t) \\ \vdots & \vdots & & \vdots \\ x_k(1\Delta t) & x_k(2\Delta t) & \dots & x_k(N\Delta t) \end{pmatrix}$$

with $k < N$ typically.

Signal-adapted tomography

- Construct the square matrices $A = U^T U \in \mathcal{M}_{N\times N}$ and $B = UU^T \in \mathcal{M}_{k\times k}$.
- Diagonalization of A provides k non-zero eigenvalues $(\alpha_1, \cdots, \alpha_j)$ and its corresponding orthogonal $N-$dimensional eigenvectors $(\Phi_1, \cdots, \Phi_k)$, $\Phi_j \in \mathbb{R}^N$. Correspondingly, the diagonalization of B would provide the same k eigenvalues and eigenvectors $(\Psi_1, \cdots, \Psi_k)$ with $\Psi_j \in \mathbb{R}^k$. If needed one may obtain, by the Gram-Schmidt method, the remaining $N-k$ eigenvectors to span $\mathbb{R}^N$, which in this context are associated to the eigenvalue zero.
- The linear operator S constructed from the set of typical signals is

$$S = \sum_{i=1}^{k} \alpha_i \Phi_i \Phi_i^t$$

where $S \in \mathcal{M}_{N\times N}$.

Signal-adapted tomography

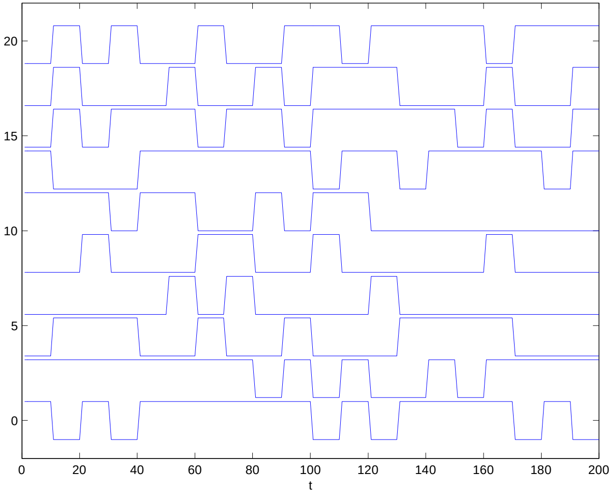
- For the tomogram consider an operator $B(\mu,\nu)$ of the form

$$B(\mu,\nu)=\mu t+\nu S=\mu\begin{pmatrix}1\Delta t & & & \\ & 2\Delta t & & \\ & & \ddots & \\ & & & N\Delta t\end{pmatrix}+\nu\sum_{i=1}^{k}\alpha_i\Phi_i\Phi_i^t$$

  where $B\in\mathcal{M}_{N\times N}$.
- The eigenvectors of each $B(\mu,\nu)$ are the columns of the matrix that diagonalizes it. From the projections of the signal on these eigenvectors one constructs a tomogram adapted to the operator pair (t,S).

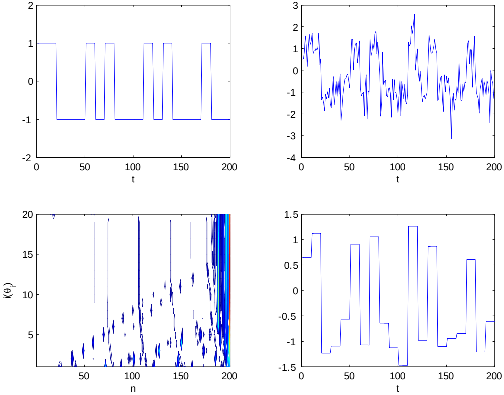
Tomography with an adapted operator pair: An example

Typical data: a set of 40 random signals with pulses of duration $\Delta t = 10$ and intensities $+1$ or -1. The total length of the signal is 200 time units.



Tomography with an adapted operator pair: An example

Eigenbasis of $B(\theta) = t\cos\theta + S\sin\theta$ is used to project the signal.
Tomogram for 20 different values of θ at intervals $\Delta\theta = \pi/40$



Right lower plot is projection on eigenvectors 185 to 200 at $\theta_{19} = 19\pi/40$.

References

- "*Noncommutative time–frequency tomography" (*V. I. Man'ko and RVM), Phys. Lett. A 263 (1999) 53–59
- "*Tomograms and other transforms: a unified view"* (M. A. Man'ko, V. I. Man'ko and RVM), J. Phys. A: Math. Gen. 34 (2001) 8321-8332
- "*A tomographic analysis of reflectometry data I: Component factorization*" (F. Briolle, R. Lima, V. I. Man'ko and RVM), Meas. Sci. Technol. 20 (2009) 105501.
- "*A tomographic analysis of reflectometry data II: The phase derivative*, (F. Briolle, R. Lima and RVM), Meas. Sci. Technol. 20 (2009) 105502.
- *"Analysis and separation of time-frequency components in signals with chaotic behavior" (*B. Ricaud, F. Briolle and F. Clairet), arXiv: 1003.0734
- *"Non-commutative tomography: A tool for data analysis and signal processing" (*F. Briolle, V. I. Man'ko, B. Ricaud and RVM), Jour. Russ. Laser Research 33 (2012) 103-121.

References

- C. Aguirre, P. Pascual, D. Campos and E.Serrano; *Single neuron transient activity detection by means of tomography*, BMC Neuroscience 2011, 12(Suppl 1):P297
- C. Aguirre and R. Vilela Mendes; *Signal recognition and adapted filtering by non-commutative tomography*, forthcoming