Correlations and synchronization in collective dynamics

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Coupled oscillators and natural systems

- Many natural systems can be described as networks of oscillators coupled to each other:
  Earthquakes
  Ecosystems
  Neurons
  Cardiac pacemaker cells
  Animal behavior
- Coupled oscillators may display synchronized behavior, i.e. follow a common dynamical evolution. Famous examples include:
  The synchronization of circadian rhythms
  Crickets that chirp in unison
  Flashing fireflies
  Market herd behavior
  Fashion trends
- Synchronization properties are dependent on the coupling pattern among the oscillators, represented as an interaction network

Coupled oscillators and collective behavior

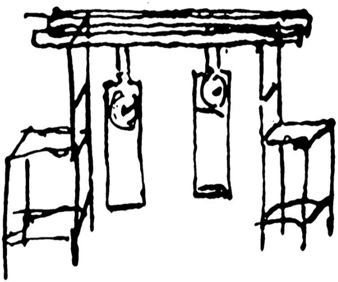
- The central question concerns the emergence of coherent behavior: synchronization or other types of fixed correlation.
- This occurs both for systems with regular behavior as well as for systems which have chaotic dynamics (lasers, neural networks, physiological processes, etc.)
- Chaotic systems are characterized by a very strong sensitivity to initial conditions, and two identical uncoupled chaotic systems will become uncorrelated at large times even if they start from very similar (but not identical) states. Nevertheless, the coupling of such systems can lead them to follow the same chaotic trajectories

Historical examples

Christiaan Huygens (1629-1695) observed synchronization of two pendulum clocks

A very small coupling of two clocks

"... the motions of each pendulum in opposite swings were so much in agreement that they never receded the least bit from each other and the sound of each was always heard simultaneously. ······ if this agreement was disturbed by some interference, it reestablished itself in a short time. ······, after a careful examination finally found that the cause of this is due to the motion of the beam, even though this is hardly perceptible."



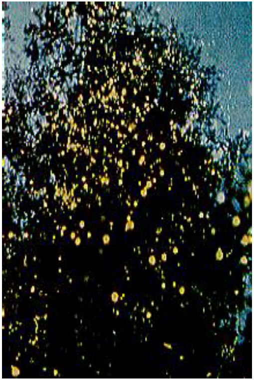
Lord Rayleigh described synchronization in acoustical systems:
"When two organ-pipes of the same pitch stand side by side, complications ensue which not unfrequently give trouble in practice. In extreme cases the pipes may almost reduce one another to silence. Even when the mutual influence is more moderate, it may still go so far as to cause the pipes to speak in absolute unison, in spite of inevitable small differences."

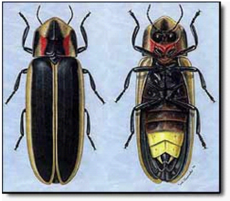
Historical examples

- **W. H. Eccles and J. H. Vincent** applied for a British Patent confirming their discovery of the synchronization property of a triode generator
- **Jean-Jacques Dortous de Mairan** reported in 1729 on his experiments with the haricot bean and found a circadian rhythm (24-hours-rhythm): motion of leaves continues even without variations of the illuminance
- **Engelbert Kaempfer** wrote after his voyage to Siam in 1680: *"The glowworms . . . represent another shew, which settle on some Trees, like a fiery cloud, with this surprising circumstance, that a whole swarm of these insects, having taken possession of one Tree, and spread themselves over its branches, sometimes hide their Light all at once, and a moment after make it appear again with the utmost regularity and exactness . . ."*.

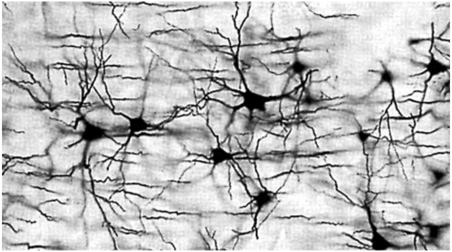
Historical examples

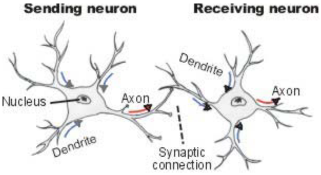
Kuala Selangor firefly park (Malaysia), http://www.fireflypark.com





Synchronization in neuronal ensembles





Synchronization in neuronal ensembles is believed to be the reason for emergence of pathological rhythms in the Parkinson disease and in the Epilepsy.

Synchronization of oscillators. Models

- **The Kuramoto model**

$$\frac{dx_i}{dt} = \omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin\left(x_j - x_i\right); \qquad x_i \in S^1$$

- **A simpler model with the same qualitative behavior (Syncnet)**

$$x_i\left(t+1\right) = x_i\left(t\right) + \omega_i + \frac{K}{N-1}\sum_{j=1}^{N} f^{(n)}\left(x_j - x_i\right)$$

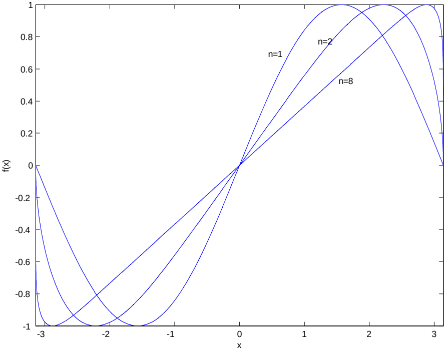
with $x_i \in \left[-\pi, \pi\right)$ and $f^{(n)}$ is a deformed version of the Kuramoto interaction

$$f^{(n)}\left(x\right) = \mathrm{sign}\left(x\right)\left(\sin\left(\frac{\left|x\right|^n}{\pi^{n-1}}\right)\right)^{1/n}$$

For $n = 1$ $f^{(1)} = \sin\left(x\right)$ and when $n \to \infty$ it becomes

$$f^{(\infty)}\left(x_j - x_i\right) = \left(x_j - x_i\right) \pmod{\pi}$$

Synchronization of oscillators. Models



The Kuramoto model

$$\frac{dx_i}{dt} = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(x_j - x_i) = \omega_i + KR \sin(\theta - x_i)$$

defining a complex order parameter

$$R\ e^{i\theta} = \frac{1}{N} \sum_{i=1}^{N} e^{ix_i}$$

Let us look for stationary solutions $(R = const)$. In the rotating frame $\theta = 0$, hence

$$\frac{dx_i}{dt} = \omega_i - KR \sin(x_i)$$

The Kuramoto model

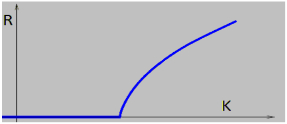
Oscillators with frequencies $|\omega_i| < KR$ become locked in the rotating frame at some angle x_i such that $\omega_i - KR\sin(x_i) = 0$. The others drift around the circle and do not contribute to the collective variable R.
Self-consistent condition

$$R = \langle e^{ix} \rangle = \int g(\omega)\cos(x)\,d\omega = \int g(KR\sin(x))\cos^2(x)\,KRdx$$

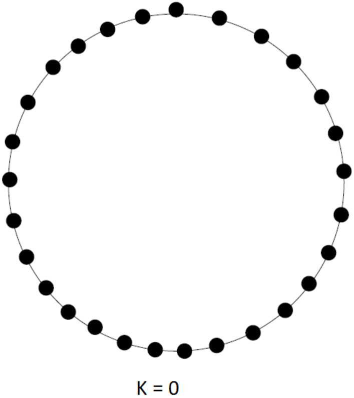
$$1 = K\int g(KR\sin(x))\cos^2(x)\,dx$$

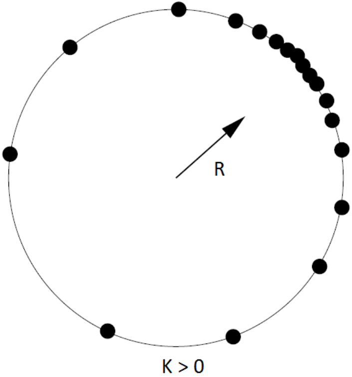
Critical K

$$K_c = \frac{2}{\pi g(0)}$$

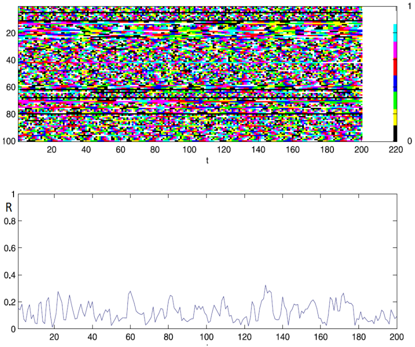


The Kuramoto model

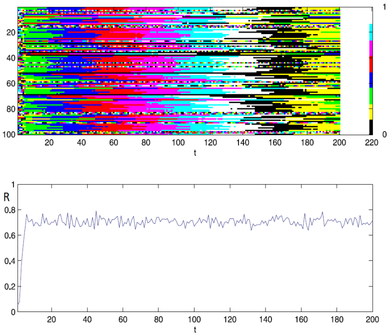




$K < K_c$

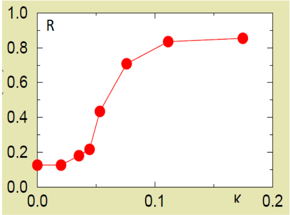


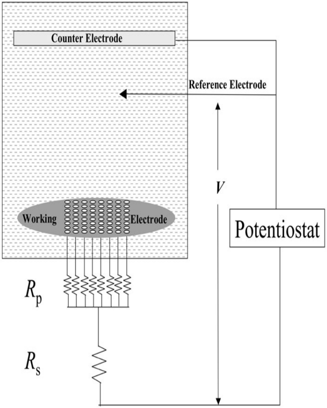
$K > K_c$



The Kuramoto model

An experiment with an array of 64 nickel electrodes in sulfuric acid, current proportional to the rate of metal dissolution. K controlled through the use of external series and parallel resistors (Science 296 (2002) 1676-1678)





The Kuramoto model with additional phase shift

$$\frac{dx_i}{dt} = \omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin\left(x_j - x_i - \beta\right) = \omega_i + \frac{KR}{N} \sin\left(\theta - x_i - \beta\right)$$

- If all frequencies are similar
  $-\frac{\pi}{2} < \beta < \frac{\pi}{2} \implies$ equal phases and $R \simeq 1$
  $-\pi < \beta < -\frac{\pi}{2}$ or $\frac{\pi}{2} < \beta < \pi \implies$ asynchronous phases and $R \simeq 0$
- Suggests that R is not a parameter characterizing all types of correlations that might exist

Synchronization: An "ergodically solvable" model

$$x_i\left(t+1\right) = x_i\left(t\right) + \omega_i + \frac{K}{N-1}\sum_{j=1}^{N} f^{(\infty)}\left(x_j - x_i\right)$$

with $x_i \in \left[-\pi, \pi\right)$ and

$$f^{(\infty)}\left(x_j - x_i\right) = \left(x_j - x_i\right) \pmod{\pi}$$

- The $\omega_i' s$ are distributed according to the Cauchy distribution

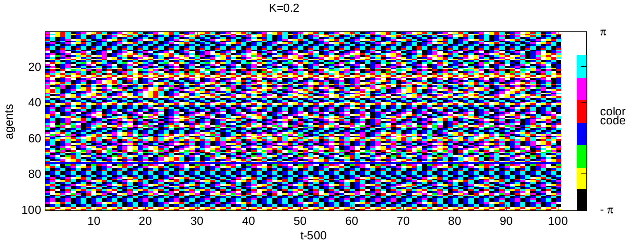
$$p\left(\omega\right) = \frac{\gamma}{\pi\left[\gamma^2 + \left(\omega - \omega_0\right)^2\right]}$$

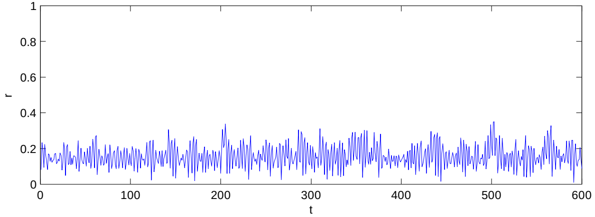
- An order parameter for synchronization

$$r\left(t\right) = \left|\frac{1}{N}\sum_{j=1}^{N} e^{i2\pi x_j(t)}\right|$$

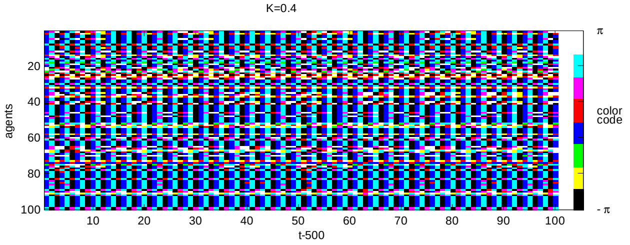
- Notice that this is a crude measure of synchronization. A better one is the entropy of the phase distribution (F. Rodriguez)

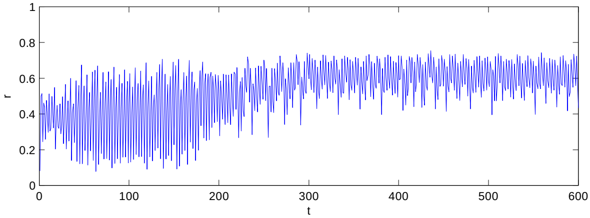
The SyncNet model



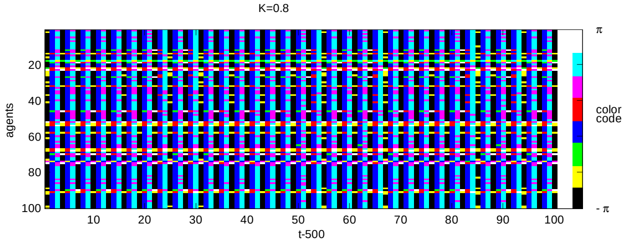


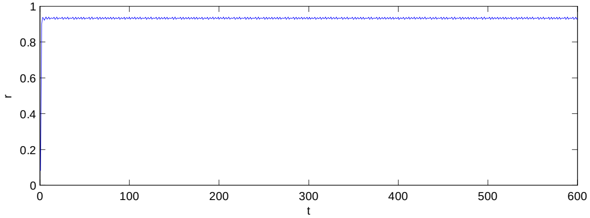
The SyncNet model





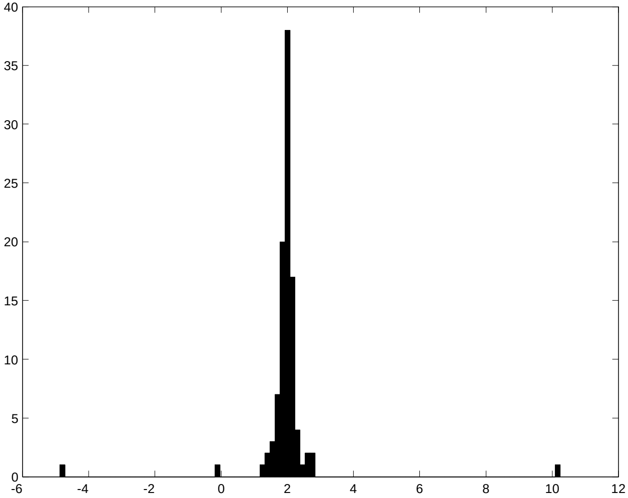
The SyncNet model





The SyncNet model

A typical distribution of the individual oscillator frequencies



The SyncNet model

- SyncNet is qualitatively identical to Kuramoto
- Is there anything more or is it synchronization all there is in the dynamics of interacting oscillators?
- An hint is obtained by computing the Lyapunov spectrum of model
  The Lyapunov spectrum is composed of one isolated zero and $\log\left(1 - \frac{N}{N-1}K\right)$ $(N-1)$-times. Therefore although it is only for sufficiently large K that synchronization effects occur, there are, for all small $K > 0$, $N-1$ contracting directions.
  Therefore, even before synchronization one expects strong correlations (dimension reduction in the dynamical space).
  Dynamical dimension $= 1$.
- Some methods to go beyond synchronization in the study the dynamics:
  - *The geometry of the dynamics*
  - *Dynamical communities*
  - *Conditional exponents*

Multidimensional scaling

- MDS begins with a $N \times N$ distance matrix $D = \{d_{ij}\}$. The objective is to find a configuration of points in $p-$dimensional space such that the coordinates of the points yield a Euclidean distance matrix whose elements are as close as possible to the elements of the given distance matrix.
- **The Classical Solution**
  When is a distance matrix Euclidean?
  A distance matrix is Euclidean if for $x_1 \cdots x_p \in \mathbb{R}^p$

$$d_{ij}^2 = (x_i - x_j)(x_i - x_j)^T$$

- **THEOREM** Define $A = \{a_{ij}\}$ with $a_{ij} = -\frac{1}{2} d_{ij}^2$ and H the centering matrix. Then D is Euclidean if and only if $B = HAH$ is positive semidefinite.

Multidimensional scaling

Recovery of coordinates

Imagine that the distances d_{ij} are obtained from a $N \times p$ coordinate matrix

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & \cdots & x_{2p} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{N1} & x_{N2} & \cdots & \cdots & x_{Np} \end{pmatrix}$$

Define a $N \times N$ matrix B

$$B = XX^T$$

a assume a decomposiition of the squared distance matrix into

$$d_{ij}^2 = \left| \overrightarrow{x_i} - \overrightarrow{x_j} \right|^2 = b_{ii} + b_{jj} - 2b_{ij}$$

Now obtaining the $b_{ij}'s$ from the $d_{ij}'s$ one can derive X by factoring B. By a simple translation of the origin in $\mathbb{R}^p$, $\sum_{i=1}^{N} x_{ik} = 0$ for all k.

Multidimensional scaling

Then

$$b_{ij} = -\frac{1}{2}\left\{d_{ij}^2 - \frac{1}{N}\left(\sum_{j=1}^{N} d_{ij}^2 + \sum_{i=1}^{N} d_{ij}^2 - \frac{1}{N}\sum_{i,j=1}^{N} d_{ij}^2\right)\right\}$$

Diagonalizing B

$$B = V\Lambda V^T$$

where $\Lambda = (\lambda_1 \cdots \lambda_N)$,$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$ is the diagonal matrix of eigenvalues and $V = [V_1, \cdots, V_N]$ the matrix of normalized eigenvectors. When D arises from an $N \times p$ data matrix, the rank of B is p (with the last $N - p$ eigenvalues being zero). Then one may write

$$B = V^* \Lambda^* V^{*T}$$

where V^* contains the first p eigenvectors and Λ^* the first p eigenvalues. Thus, a solution for X is $X = V^* \Lambda^{*1/2}$.

Multidimensional scaling

When the observed proximity matrix is not Euclidean, the matrix B is not positive-definite. In such case, some of the eigenvalues of B will be negative, correspondingly some coordinate values will be complex numbers. If B has only a small number of small negative eigenvalues, it's still possible to use the eigenvectors associated with the p largest positive eigenvalues.

Adequacy of the resulting solution might be assessed using

$$AD = \frac{\sum_{i=1}^{p} |\lambda_i|}{\sum_{i=1}^{N} |\lambda_i|}$$

If instead of a distance matrix one has a similarity matrix $S = [s_{ij}]$ one may obtain a distance (dissimilarity) matrix by, for example

$$d_{ij} = constant - s_{ij}; \quad d_{ij} = \frac{1}{s_{ij}} - constant; \quad d_{ij} = (s_{ii} + s_{jj} - 2s_{ij})^{1/2}$$

In Matlab $[V^*, \Lambda]$=cmdscale(D).

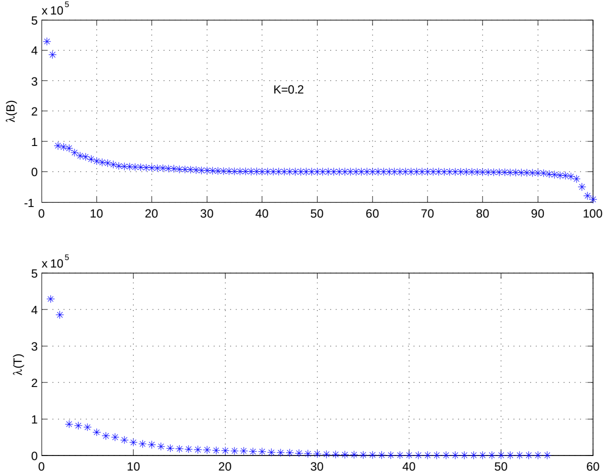
The SyncNet model. Geometrical analysis

- *Dynamical distance* = the sum of the circle distances over the last 100 time steps.
- Embed as points in Euclidean space using MDscaling or geometrical embedding. $\lambda(B)$ are the eigenvalues of the B matrix
- Reduce coordinates to the center of mass and compute the inertial tensor

$$T_{ij} = \sum_{k=1}^{N} y_i(k) y_j(k)$$

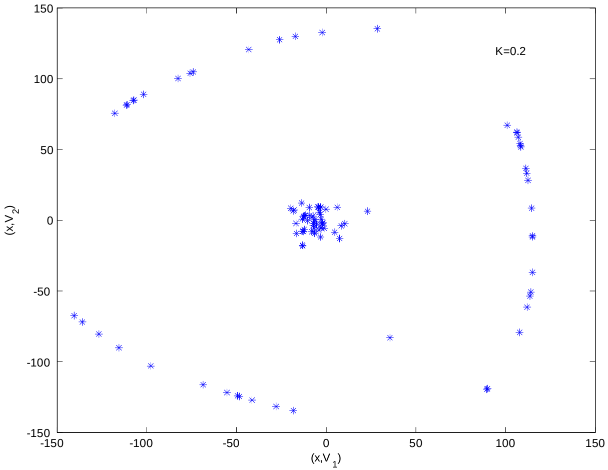
  $\lambda(T)$ are the eigenvalues of T
- Once found the eigenvalues $\{\lambda_k\}$ and eigenvectors $\{V_k\}$ of T, relevant quantities are the projections (x_i, V_k) of the coordinate vectors on the eigenvectors, in particular on those associated to the largest eigenvalues.

K=0.2

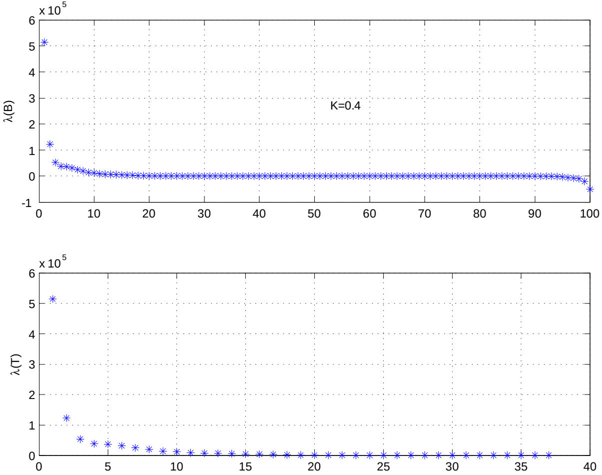


The SyncNet model. Geometrical analysis

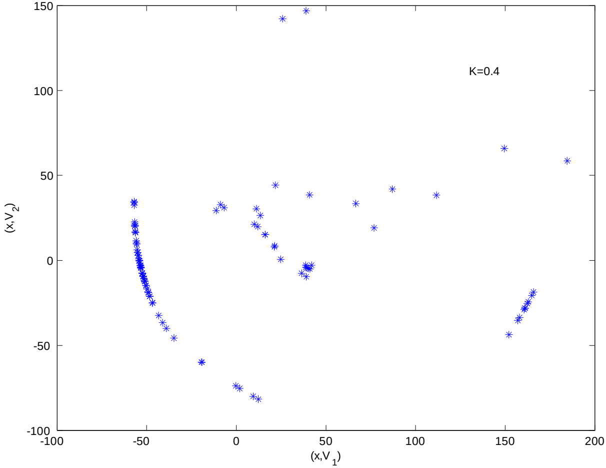
K=0.2 (Projection of the dynamics on the first and second eigenvectors)



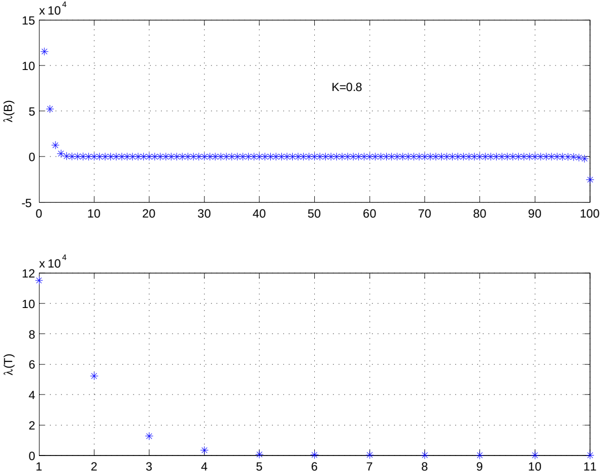
K=0.4


x 10^5
6
5
4
3
2
1
0
-1
λ(B)
K=0.4
0
10
20
30
40
50
60
70
80
90
100
x 10^5
6
5
4
3
2
1
0
λ(T)
0
5
10
15
20
25
30
35
40

K=0.4 (Projection of the dynamics on the first and second eigenvectors)



K=0.8


x 10^4
15
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0
-5
λ(B)
K=0.8
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80
90
100
x 10^4
12
10
8
6
4
2
0
λ(T)
1
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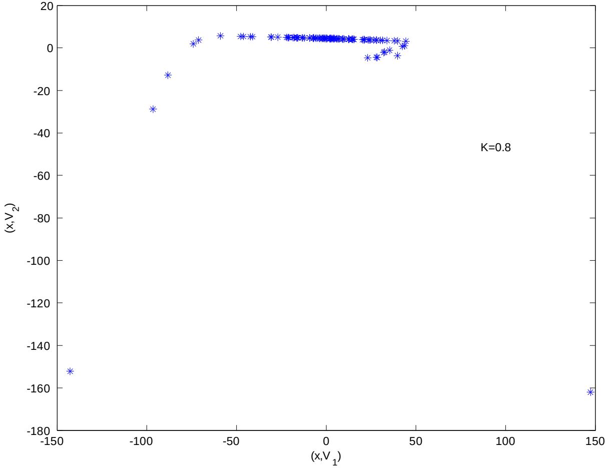
11

The SyncNet model. Geometrical analysis

K=0.8 (Projection of the dynamics on the first and second eigenvectors)



The SyncNet model. Geometrical analysis and order parameters

The projection of the embedded coordinates $\{x_i\}$ on the eigenvectors V_k associated to the largest eigenvalues of T may be considered as new order parameters, in particular $P_k = \sum_{i=1}^{N} |(x_i, V_k)|$

The dynamical communities approach

- Construct for each agent

$$\Delta_i(t) = x_i(t) - x_i(t-1)$$

and use this to find the dynamical distance of the agents

$$d_{ij} = \sqrt{\sum_{t=1}^{T} \left|\Delta_i(t) - \Delta_j(t)\right|^2}$$

- From the distances find an adjacency matrix

$$A_{ij} = \exp\left(-\beta(d_{ij} - d_{\min})\right)$$

- The degree matrix is

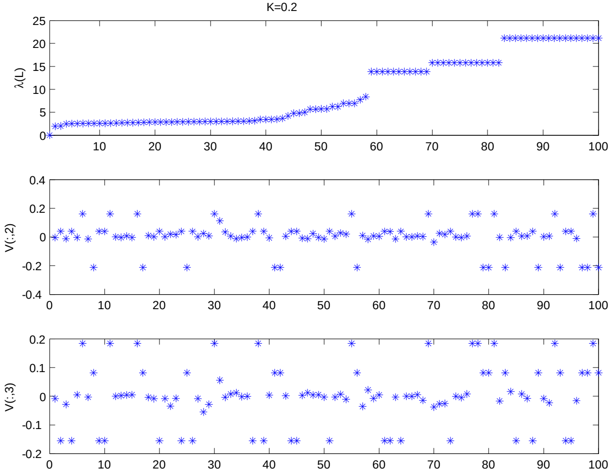
$$(G)_{ii} = \sum_{j \neq i} A_{ij}$$

- and the Laplacian matrix

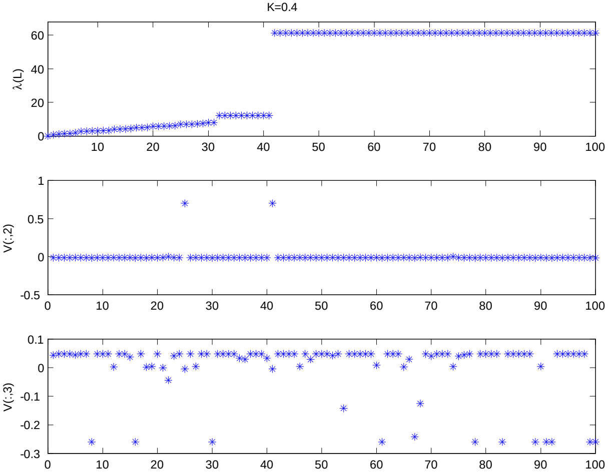
$$L = G - A$$

- From the spectrum of L identify dynamical communities

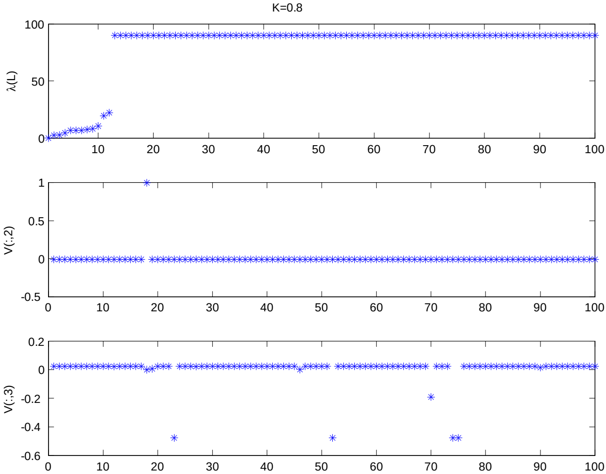
The dynamical communities approach (K=0.2)



The dynamical communities approach (K=0.4)



The dynamical communities approach (K=0.8)



3. Structural characterization of the dynamics in agent-based models

Ergodic tools. Exponents and entropies

- *Invariant measures and ergodic parameters*

$$I_F(\mu) = \lim_{T\to\infty} \frac{1}{T} \sum_{n=1}^{T} F\left(f^n x_0\right)$$

- *Lyapunov and conditional exponents*

From the k x k and (m-k)x(m-k) blocks of the Jacobian , obtain the conditional exponents as the eigenvalues of the limits

$$\lim_{n\to\infty} \left(D_k f^{n*}(x)\, D_k f^n(x)\right)^{\frac{1}{2n}}$$

$$\lim_{n\to\infty} \left(D_{m-k} f^{n*}(x)\, D_{m-k} f^n(x)\right)^{\frac{1}{2n}}$$

or

$$\lim_{n\to\infty} \frac{1}{n} \log \left\| D_k f^n(x) u \right\| = \xi_i^{(k)}$$

$$0 \neq u \in E_x^i / E_x^{i+1}$$

E_x^i is the subspace spanned by the eigenstates corresponding to eigenvalues $\leq \exp(\xi_i^{(k)})$

Structures and self-organization

- Structure index

$$S = \frac{1}{N}\sum_{i=1}^{N_+}\left(\frac{\lambda_0}{\lambda_i} - 1\right)$$

diverges whenever a Lyapunov exponent approaches zero from above (points where long time correlations develop)

- Self-organization (partitions $\Sigma_k = R^k \times R^{m-k}$)

$$I_\Sigma(\mu) = \sum_{k=1}^{N}\{h_k(\mu) + h_{m-k}(\mu) - h(\mu)\}$$

$$h_k(\mu) = \sum_{\xi_i^{(k)}>0} \xi_i^{(k)};\, h_{m-k}(\mu) = \sum_{\xi_i^{(m-k)}>0} \xi_i^{(m-k)};\, h(\mu) = \sum_{\lambda_i>0} \lambda_i$$

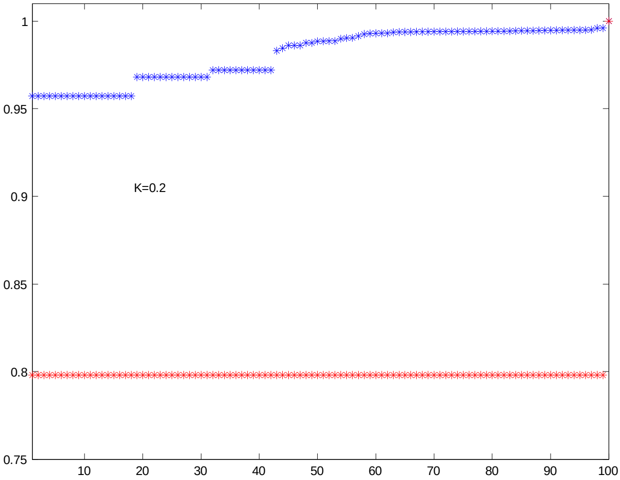
Characterizing the dynamical correlations by ergodic invariants

- **Conditional exponents:** The view that each agent has of its dependence on the dynamics of the other agents.
- Introduced by Pecora and Carroll in their study of synchronization of chaotic systems.
- They are good ergodic invariants playing an important role as self-organization parameters.
- Computed from

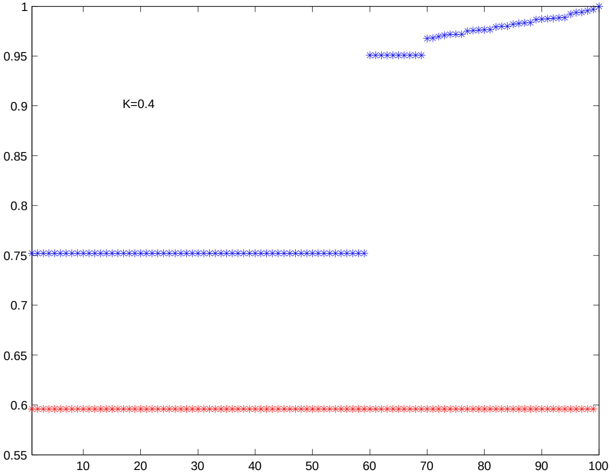
$$x_i\left(t+1\right)=x_i\left(t\right)+\omega_i+\frac{K}{N-1}\sum_{j=1}^{N}A_{ij}f^{(n)}\left(x_j-x_i\right)$$

- The adjacency matrix is the same that was derived in the previous section from the dynamical similarities of the agents' time evolution.

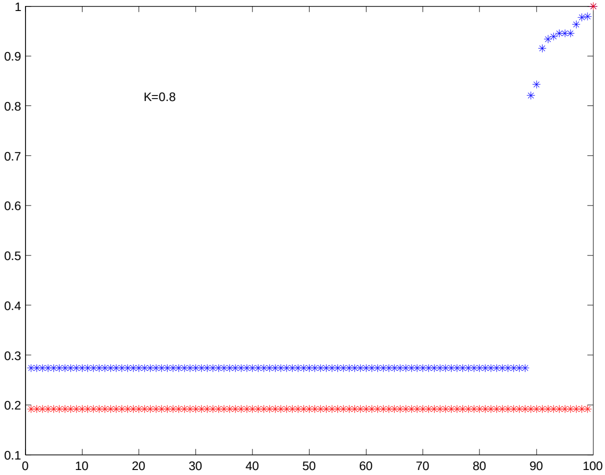
Characterizing the dynamical correlations by ergodic invariants



Characterizing the dynamical correlations by ergodic invariants



Characterizing the dynamical correlations by ergodic invariants



Synchronization and the network structure

Consider a set of identical oscillators with dynamics

$$\begin{aligned} \frac{dx_i}{dt} &= \Omega\left(x_i\right) + \beta \sum_{j \in N(i)} \left\{ f\left(x_i\right) - f\left(x_j\right) \right\} \\ &= \Omega\left(x_i\right) + \beta \sum_{j=1}^{N} L_{ij} f\left(x_j\right) \end{aligned}$$

L being the Laplacian matrix.
Consider a synchronized state and small perturbations around this state

$$x_i\left(t\right) = s\left(t\right) + \xi_i\left(t\right)$$

In the neighborhood of the synchronized state

$$\begin{aligned} \Omega\left(x_i\right) &\simeq \Omega\left(s\right) + \Omega'\left(s\right)\xi_i \\ f\left(x_i\right) &\simeq f\left(s\right) + f'\left(s\right)\xi_i \end{aligned}$$

Then

$$\frac{d\xi_i}{dt} = \Omega'(s)\,\xi_i + \beta\sum_{j=1}^{N} L_{ij} f'(s)\,\xi_j$$

The eigenvalues of L are $0 = \lambda_1 \leq \lambda_2 \leq \cdots$. Decouple the equation into eigenmodes $\{\sigma_i\}$

$$\frac{d\sigma_i}{dt} = \left[\Omega'(s) + \beta\lambda_i f'(s)\right]\sigma_i$$

with solution

$$\sigma_i(t) = \sigma_i(0)\, e^{\left[\Omega'(s)+\beta\lambda_i f'(s)\right]t}$$

for $i = 1$ it is $\sigma_i(t) = \sigma_i(0)\, e^{\Omega'(s)t}$, that is, the dynamics (stable or chaotic) of the synchronized state. Stability of this state depends on

$$\left[\Omega'(s) + \beta\lambda_i f'(s)\right] \leq 0$$

for all $i \geq 1$ along the trajectory of the synchronized state. If for example $\max\left[\Omega'(s)\right] > 0$, stability depends on $\beta\lambda_i f'(s)$ being sufficiently negative, in particular for $\beta < 0$ on the size of the spectral gap λ_2.

Synchronization and the network structure

Therefore the structure of the network connections, through the spectral gap on the Laplacian spectrum, has a bearing on synchronization.
There is in the literature a wide (mostly phenomenological) exploration of synchronization in many different types of networks (regular, small-world, scale-free, etc. - see References)

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