Structure-generating mechanisms in agent-based models

Rui Vilela Mendes

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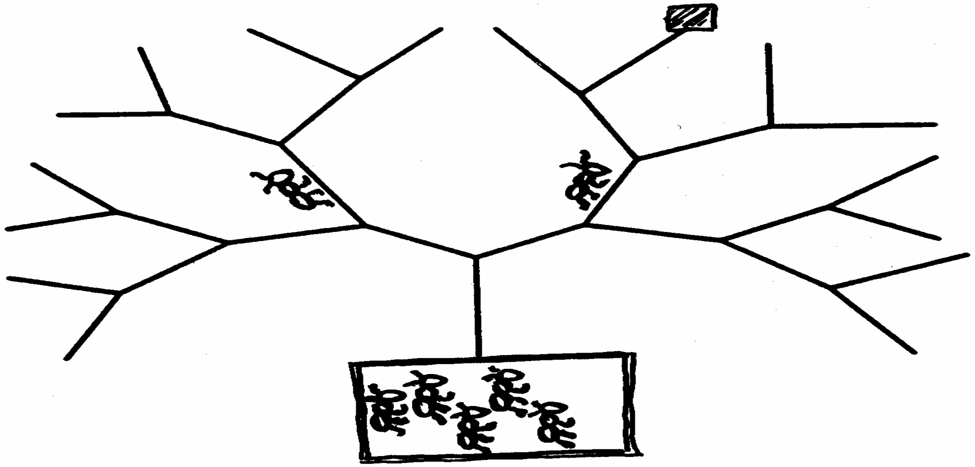

No.	SEQUENCE OF TURNS TO SYRUP	MEAN TIME SEC.	SAMPLE STANDARD DEVIATION	NUMBER OF TESTS
1	LLL	72	8	18
2	RRR	75	5	15
3	LLLLLL	84	6	9
4	RRRRR	78	8	10
5	LLLLL	90	9	8
6	RRRRRR	88	9	5
7	LRLRLR	130	11	4
8	RLRLRL	135	9	8
9	LLR	69	4	12
10	LRLL	100	11	10
11	RLLLR	120	9	6
12	RRLRL	150	16	8
13	RLRRRL	180	20	6
14	RRLRRR	220	15	7
15	LRLLRL	200	18	5

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- **Mechanisms**
  (1) Sensitive-dependence and convex coupling
  (2) Sensitive-dependence and extremal dynamics
  (3) Interaction through a collectively generated field. (Multistability and evolution)

Structure-generation through density-dependent coupling

- Bernoulli agents on circle with nearest-neighbour convex coupling

$$x_i(t+1) = (1-c)f(x_i(t)) + \frac{c}{2}\left(f(x_{i+1}(t)) + f(x_{i-1}(t))\right) \qquad (1)$$

$f(x) = 2x \pmod{1}$ and periodic boundary conditions

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- Coupling is also dynamical by a *reproduction and death* mechanism
  - After each R time cycles, agents with $x_i > 0.5$ are coded 1 and those with $x_i \leq 0.5$ are coded 0.
  - Configurations 0110 :candidates for reproduction with probability p_r
  - Configurations 0000 :candidates for death with probability p_m
  - Reproduction : transition $0110 \rightarrow 0X110$, the state of the new agent X chosen at random in the interval $(0,1)$
  - Death : transition $0000 \rightarrow 000$

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- The Lyapunov exponents for the dynamical system in (1) are

$$\lambda_k = \log \left\{ 2\left(1-c\right) + 2c\cos\left(\frac{2\pi}{n}k\right) \right\} \qquad k = 0, \cdots, N-1$$

All positive for $c < 0.5$ but above this value structures are created when each Lyapunov exponent crosses zero. Collective modes have different probabilities, a new collective mode being frozen each time a Lyapunov exponent reaches the zero value.

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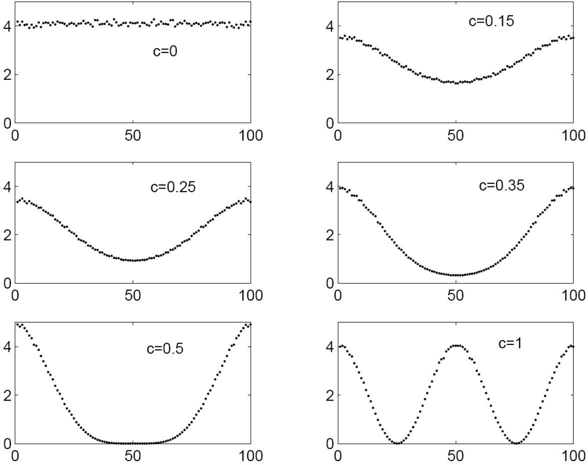
- The eigenvectors corresponding to each exponent are $\left\{e^{in\theta_k}\right\}$ with $\theta_k = \frac{2\pi}{N}k$, $k=0,\cdots,N-1$. Therefore $y_k = \frac{1}{N}\sum_{n=1}^{N}\cos\left(\frac{2\pi}{N}kn\right)$ are the coordinates of the collective eigenmodes

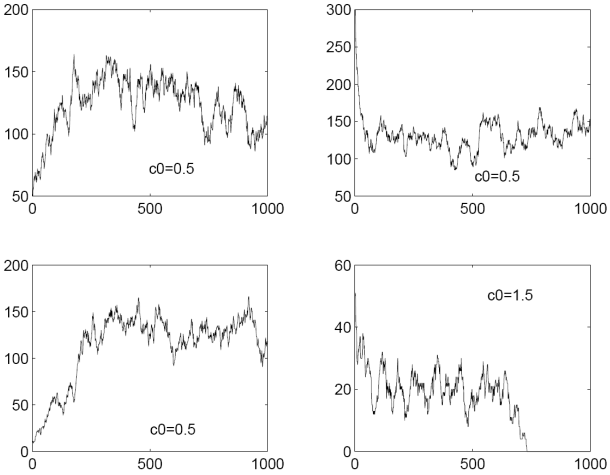
Structure-generation through density-dependent coupling

Average energy $E_k = \langle y_k^2 \rangle$ of the collective modes
(mode $k = 0$ not shown)



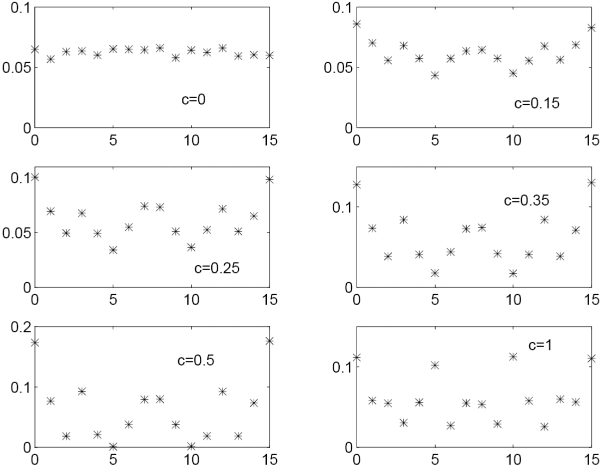
Structure-generation through density-dependent coupling

Evolution of the population plotted against the reproduction-death cycle number ($p_r = 1$ and $p_m = 0.5$). Population controlled by collective structures



Structure-generation through density-dependent coupling

Relative probability of each one of the 16 different configurations of four neighbours $(x_1 x_2 x_3 x_4)$, labelled by $x_1 + 2 \times x_2 + 4 \times x_2 + 8 \times x_2$



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- Multistability and evolution :
  In some cases, the essential mechanism, self-organizing the system, is the evolution (a slow dynamics), the fast dynamics only provides the multi-attractor background which is selected by the slow evolution.

Coupled map minority model

(Inspired on Brian Arthur's El Farol model)

- *Continuous version*:
  Fix a number c (*the cut*) to divide the interval [0,1]
  Each agent chooses a value x_i between 0 and 1
  The *average* $x_m = \frac{1}{N} \sum_i x_i$ is computed

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- *Winning agents:* those for which x_i lie on the side opposite to x_m
  Payoff of agent i at time t :
  $P_i(t) = \frac{1}{2}\left(1 - \text{sign}\left\{(x_m(t) - c)\,(x_i(t) - c)\right\}\right)$

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- *Agents' strategy*

$$x_i(t+1) = f_i(x_m(t), \alpha_i)$$

Here f_i is either a shifted tent map

$$f_i(x) = 2 + 2x\text{sign}\left(\frac{1}{2} - (x + \alpha_i)\right) \pmod 1$$

or a shifted p-ary multiplication

$$f_i(x) = p(x + \alpha_i) \pmod 1$$

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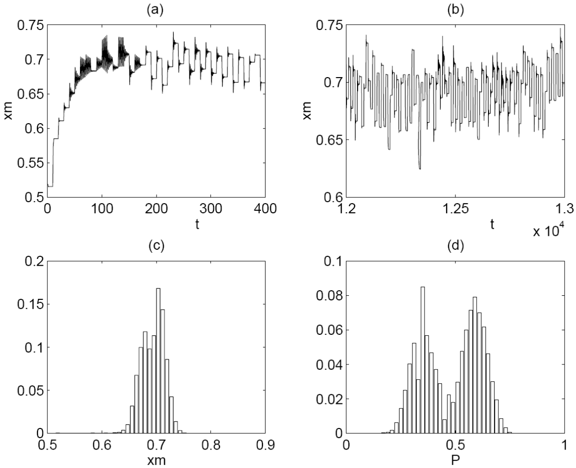
- P =fraction of winning agents

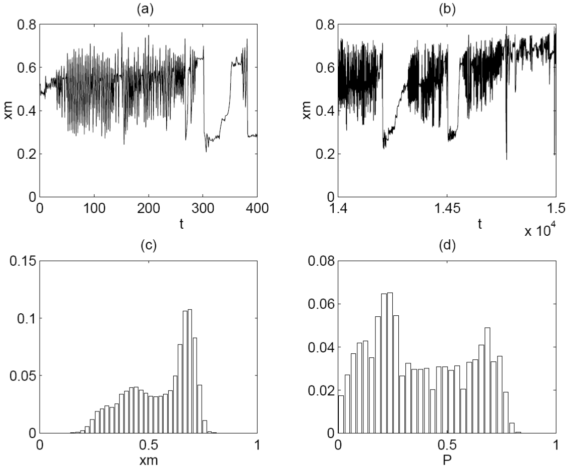
$$P = \frac{1}{N} \sum_i P_i$$

Shifted tent map
($N = 100$, $k = r = 10$ and $k^{'} = 3$)



Coupled map minority model

2-ary multiplication



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- The dynamics of x_m is

$$x_m(t+1) = \frac{1}{N}\sum_i f_i\left(x_m(t) + \alpha_i\right)$$

the Lyapunov exponent being

$$\lambda = \lim_{k\to\infty}\frac{1}{k}\log \left(\frac{1}{N}\left|\sum_i f_i'(x_m(t)+\alpha_i)\right| \cdots \frac{1}{N}\left|\sum_i f_i'(x_m(t+k)+\alpha_i)\right|\right)$$

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- *For the tent map*: for a large number agents, uniform distribution of the α's, $\frac{1}{N}\left|\sum_i f_i'\right|$ of order $\frac{1}{\sqrt{N}}$, $\Longrightarrow \lambda$ negative of order $-\frac{1}{2}\log N$

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$$DT = \begin{pmatrix} \frac{1}{N} f_1' & \frac{1}{N} f_1' & \cdots & \frac{1}{N} f_1' \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \frac{1}{N} f_N' & \frac{1}{N} f_N' & \cdots & \frac{1}{N} f_N' \end{pmatrix}$$

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- Eigenvalues of $\left(DT^k\right)^T\left(DT^k\right)$ are $N-1$ zeros and one equal to

$$N\left(\frac{1}{N^2}\sum_i f_i'^2\right)\left(\frac{1}{N}\sum_i f_i'\right)^2\cdots\left(\frac{1}{N}\sum_i f_i'\right)^2$$

One non-trivial Lyapunov exponent identical to the Lyapunov exponent of the x_m dynamics.

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- *Features*:
  1 - the evolution dynamics organizes the system (x_m around the cut)
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  3 - behavior of the collective variable around the average value is quite irregular. Compatible with the fast contraction of negative Lyapunov exponents because of sensitivity of the attractor to small changes of parameters

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- Non-periodic attractors.
- In conclusion: *self-organization is driven by the slow (evolution) dynamics on the attractor background supplied by the (fast) agent dynamics*

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- The change in the log price is

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- *Objective* : to increase the total wealth $m_t + p_t \times s_t$ at the expense of the rest of the market.

A market-like game

- [*Comparison with the minority model*
  $z_t \leftrightarrow x_m$
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- **Agent strategies**

  Two types of informations are taken into account:
  The *misprice*

$$zv_t - z_t = \log(v_t) - \log(p_t)$$

A market-like game

- and the *price trend*

$$z_t - z_{t-1} = \log(p_t) - \log(p_{t-1})$$

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- # Non-decreasing function $f(x)$ such that $f(-\infty) = 0$ and $f(\infty) = 1$
  Two examples

$$\begin{aligned} f_1(x) &= \theta(x) \\ f_2(x) &= \frac{1}{1+\exp(-\beta x)} \end{aligned}$$

  Four-component vector γ

$$\gamma_t = \begin{pmatrix} f(zv_t - z_t) f(z_t - z_{t-1}) \\ f(zv_t - z_t)\,(1 - f(z_t - z_{t-1})) \\ (1 - f(zv_t - z_t))\, f(z_t - z_{t-1}) \\ (1 - f(zv_t - z_t))\,(1 - f(z_t - z_{t-1})) \end{pmatrix}$$

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$$\gamma_t = \begin{pmatrix} f(zv_t - z_t) f(z_t - z_{t-1}) \\ f(zv_t - z_t)\,(1 - f(z_t - z_{t-1})) \\ (1 - f(zv_t - z_t))\, f(z_t - z_{t-1}) \\ (1 - f(zv_t - z_t))\,(1 - f(z_t - z_{t-1})) \end{pmatrix}$$

- *Strategy of each investor :*
  four-component vector $\alpha^{(i)}$ with entries -1, 0, or 1
  Investment of agent i : $\alpha^{(i)} \cdot \gamma$

A market-like game

- Examples:
  Fundamental (value-investing strategy)
  $\alpha^{(i)} = (1, 1, -1, -1)$
  Pure trend-following (technical trading)
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A market-like game

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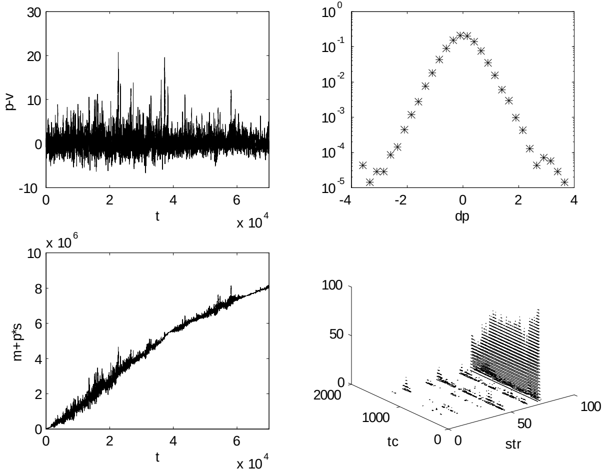
  (Fundamental strategy = no.72 and pure trend-following = no.60)
- *Evolution dynamics :*
  - After r time steps s agents copy the strategy of the s best performers plus
  - Mutation probability
  (In the figures: $r = 50$, $s = 10$, $\lambda_0 = 10000$), $45 = (0, 1, -1, -1)$, $18 = (-1, 1, -1, -1)$, $73 = (1, 1, -1, 0)$, $75 = (1, 1, 0, -1)$

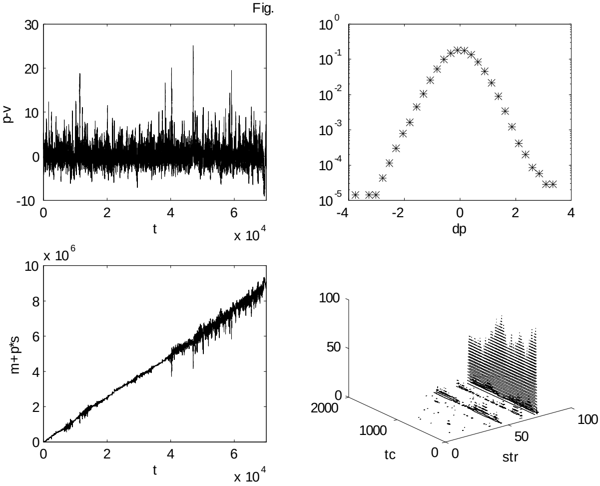
A market-like game

Market game simulation with evolution. Initial condition: all traders in the fundamental strategy



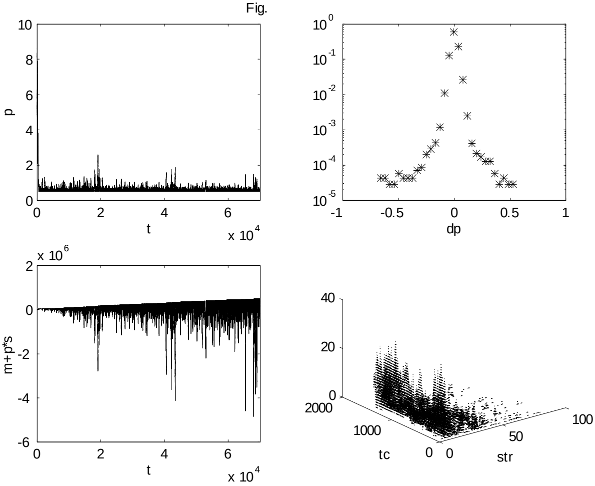
A market-like game

Market game simulation with evolution. Initial condition: 50% fundamental and 50% trend-following



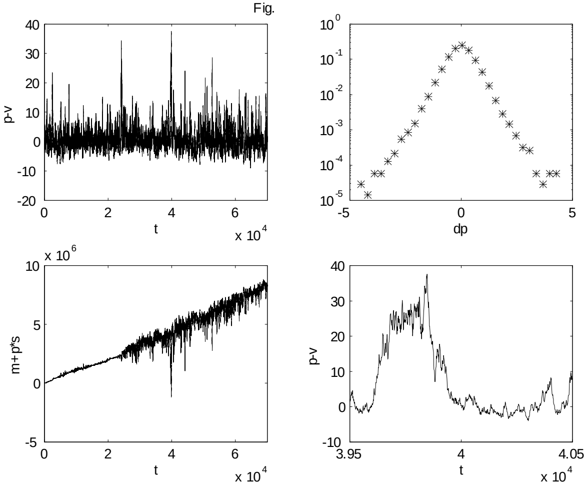
A market-like game

Market game simulation with evolution. Initial condition: random strategies



A market-like game

Market game simulation without evolution. 50% of fundamental strategies and 50% trend-following



A market-like game

- Lyapunov exponents for the log-price (z_t) dynamics. The Jacobian of

$$\begin{pmatrix} z_t \\ z_{t-1} \end{pmatrix} \to \begin{pmatrix} z_{t+1} \\ z_t \end{pmatrix}$$

is

$$M_t = \begin{pmatrix} 1 + \frac{\partial}{\partial z_t} \frac{\sum_i \omega^{(i)}}{\lambda_0 + \lambda_1 \left| \sum_i \omega^{(i)} \right|} & \frac{\partial}{\partial z_{t-1}} \frac{\sum_i \omega^{(i)}}{\lambda_0 + \lambda_1 \left| \sum_i \omega^{(i)} \right|} \\ 1 & 0 \end{pmatrix}$$

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$$\lim_{N\to\infty} \left| M_{t+N-1}^T \cdots M_t^T M_t \cdots M_{t+N-1} \right|^{1/2N}$$

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- Lyapunov exponents computed for $f = f_2$ for several values of β. and a $50 - 50$ admixture of fundamental and trend-following strategies Typically one Lyapunov number equal to zero and the other smaller than but very close to one