Structure-generating mechanisms in agent-based models

Rui Vilela Mendes
Organization, structure and patterns

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- Pattern detected $\rightarrow$ (♦ obtain a compressed description ♦ predict the outcome)

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- Measures of complexity for patterns
- Computational mechanics
Why is the dynamical behavior of a composite system qualitatively different from the dynamics of the components in isolation?
Dynamical mechanisms leading to collective structures

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- Collective structure formation in systems composed of many agents in interaction, each one of which has *simple* dynamics. *(simple to describe in law, but not necessarily with simple orbits. Small logic depth, but capable of generating orbits with high Kolmogorov complexity)*

Example:

\[ x_{n+1} = px_n \pmod{.1} \]

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- Positive Lyapunov exponents and Kolmogorov entropy,
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**Mechanisms**

1. Sensitive-dependence and convex coupling
2. Sensitive-dependence and extremal dynamics
3. Interaction through a collectively generated field. (Multistability and evolution)
Bernoulli agents on circle with nearest-neighbour convex coupling

\[ x_i(t + 1) = (1 - c) f(x_i(t)) + \frac{c}{2} (f(x_{i+1}(t)) + f(x_{i-1}(t))) \]  \hspace{1cm} (1)

\[ f(x) = 2x \mod 1 \] and periodic boundary conditions
Structure-generation through density-dependent coupling

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- Agents assumed to live in a limited space with the intensity of the coupling a function of the total number of agents \( N \), for example
  \[ c = c_m \left( 1 - e^{-\alpha N} \right) \]
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  (2)

- Coupling is also dynamical by a reproduction and death mechanism
  - After each \(R\) time cycles, agents with \(x_i > 0.5\) are coded 1 and those with \(x_i \leq 0.5\) are coded 0.
  - Configurations 0110 : candidates for reproduction with probability \(p_r\)
  - Configurations 0000 : candidates for death with probability \(p_m\)
  - Reproduction : transition 0110 \(\rightarrow\) 0X110 , the state of the new agent \(X\) chosen at random in the interval (0, 1)
  - Death : transition 0000 \(\rightarrow\) 000
Without coupling 0110 and 0000 appear, on average, the same number of times. The population density depends only on the relative values of $p_r$ and $p_m$. 

\[ \lambda_k = \log_2 \left( \frac{1}{c} \right) + \frac{2}{c} \cos \frac{2\pi k}{N} \]

All positive for $c < 0.5$ but above this value structures are created when each Lyapunov exponent crosses zero. Collective modes have different probabilities, a new collective mode being frozen each time a Lyapunov exponent reaches the zero value. The eigenvectors corresponding to each exponent are $\theta_k$ with $\theta_k = \frac{2\pi n}{k}$, $k = 0, 1, \ldots, N - 1$. Therefore $y_k = \frac{1}{N} \sum_{n=1}^{N} \cos \frac{2\pi kn}{N}$ are the coordinates of the collective eigenmodes.
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The Lyapunov exponents for the dynamical system in (1) are

\[
\lambda_k = \log \left\{ 2 (1 - c) + 2c \cos \left( \frac{2\pi}{n} k \right) \right\} \quad k = 0, \cdots, N - 1
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Average energy $E_k = \langle y_k^2 \rangle$ of the collective modes (mode $k = 0$ not shown)
Structure-generation through density-dependent coupling

Evolution of the population plotted against the reproduction-death cycle number \((p_r = 1\) and \(p_m = 0.5\)). Population controlled by collective structures.
Structure-generation through density-dependent coupling

Relative probability of each one of the 16 different configurations of four neighbours \((x_1 x_2 x_3 x_4)\), labelled by \(x_1 + 2 \times x_2 + 4 \times x_2 + 8 \times x_2\)

- \(c=0\)
- \(c=0.15\)
- \(c=0.25\)
- \(c=0.35\)
- \(c=0.5\)
- \(c=1\)
Interaction through collective variables

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- Multistability and evolution:
  In some cases, the essential mechanism, self-organizing the system, is the evolution (a slow dynamics), the fast dynamics only provides the multi-attractor background which is selected by the slow evolution.
Coupled map minority model

(Inspired on Brian Arthur’s El Farol model)

- Continuous version:
  - Fix a number $c$ (the cut) to divide the interval $[0,1]$
  - Each agent chooses a value $x_i$ between 0 and 1
  - The average $x_m = \frac{1}{N} \sum_i x_i$ is computed
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- **Winning agents:** those for which $x_i$ lie on the side opposite to $x_m$

Payoff of agent $i$ at time $t$:

$P_i(t) = \frac{1}{2} \left( 1 - \text{sign}\left\{ (x_m(t) - c) (x_i(t) - c) \right\} \right)$
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- **Agents’ strategy**
  \[ x_i(t+1) = f_i(x_m(t), \alpha_i) \]

  Here $f_i$ is either a shifted tent map
  \[ f_i(x) = 2 + 2x \text{sign} \left( \frac{1}{2} - (x + \alpha_i) \right) \pmod{1} \]

  or a shifted $p$-ary multiplication
  \[ f_i(x) = p(x + \alpha_i) \pmod{1} \]
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- $\alpha_i$ is a number between zero and one and at $t = 0$ the strategies (the $\alpha_i$’s) are random
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- **Evolution**: Each $r$ time steps $k$ agents have their strategies modified
  # The $k'$ agents with less earnings in that period have new (random) $\alpha$’s
  # The remaining $k - k'$ copy the $\alpha$’s of the $k - k'$ best performers with a small error

The behavior of the model:
- Approach to a regime where the average value $x_m$ oscillates around the value of the cut $c$ (even when $c$ is very different from the random value $0.5$).
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- $P =$ fraction of winning agents

$$P = \frac{1}{N} \sum_i P_i$$
Coupled map minority model

Shifted tent map
\((N = 100, k = r = 10 \text{ and } k' = 3)\)
Coupled map minority model

2-ary multiplication

(a) (b) (c) (d)
Coupled map minority model

- $x_m = 0.694$ and $\sigma(x_m) = 0.02$
- ($x_m = 0.5$ and $\sigma(x_m) = 0.288$ for random choice)
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- The Lyapunov exponents for the dynamics of $x_m$ and for the dynamics of the agents control the fluctuations
- The dynamics of $x_m$ is

$$x_m(t + 1) = \frac{1}{N} \sum_i f_i (x_m(t) + \alpha_i)$$

the Lyapunov exponent being

$$\lambda = \lim_{k \to \infty} \frac{1}{k} \log \left( \frac{1}{N} \left| \sum f_i'(x_m(t) + \alpha_i) \right| \cdots \frac{1}{N} \left| \sum f_i'(x_m(t + k) + \alpha_i) \right| \right)$$
Coupled map minority model

- *For the tent map*: for a large number agents, uniform distribution of the \( \alpha \)'s, \( \frac{1}{N} |\sum_i f'_i| \) of order \( \frac{1}{\sqrt{N}} \), \( \implies \lambda \) negative of order \( -\frac{1}{2} \log N \)
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$$DT = \begin{pmatrix} \frac{1}{N} f'_1 & \frac{1}{N} f'_1 & \cdots & \frac{1}{N} f'_1 \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N} f'_N & \frac{1}{N} f'_N & \cdots & \frac{1}{N} f'_N \end{pmatrix}$$
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\end{pmatrix}$$

- Eigenvalues of $(DT^k)^T (DT^k)$ are $N - 1$ zeros and one equal to

$$N \left( \frac{1}{N^2} \sum_i f_i'^2 \right) \left( \frac{1}{N} \sum_i f_i' \right)^2 \cdots \left( \frac{1}{N} \sum_i f_i' \right)^2$$

One non-trivial Lyapunov exponent identical to the Lyapunov exponent of the $x_m$ dynamics.
Coupled map minority model

**Features:**

1. the evolution dynamics organizes the system ($x_m$ around the cut)
2. the fast dynamics controls the nature of the fluctuations around this value
3. behavior of the collective variable around the average value is quite irregular. Compatible with the fast contraction of negative Lyapunov exponents because of sensitivity of the attractor to small changes of parameters
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  Large fluctuations around the mean collective value
  \[ \bar{x}_m = 0.554, \sigma(x_m) = 0.145, \bar{P} = 0.378 \text{ and } \sigma(P) = 0.223 \]
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- In conclusion: *self-organization is driven by the slow (evolution) dynamics on the attractor background supplied by the (fast) agent dynamics*
A market-like game

- **Collective variable**: stock prices (which the investors themselves influence through their investments)
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- **Evolution**: investors adjust their strategies in order to maximize profits
  - Set of investors playing against the market (effect on an existing market that is also influenced by other factors)
  - The rest of the market impact is represented by a stochastic process

\[
\log(p_t) = z_t + \eta_t + \Delta z_t
\]

\[p_t = \text{price of the traded asset at time } t\]

Objective: to increase the total wealth at the expense of the rest of the market.
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- **Comparison with the minority model**
  \[ z_t \leftrightarrow x_m \]
  \[ \Delta_t = \sum_i \left( m^{(i)}_t + p_t \times s^{(i)}_t \right) - \sum_i \left( m^{(i)}_0 + p_0 \times s^{(i)}_0 \right) \leftrightarrow P \]
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  Loglinear law
  \[ z_{t+1} - z_t = \frac{\omega_t}{\lambda} + \eta_t \]
  \( \lambda \) is the liquidity
  Not valid for large orders
  \[ z_{t+1} - z_t = \frac{\omega_t}{\lambda_0 + \lambda_1 |\omega_t|^{1/2}} + \eta_t \]
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- **Agent strategies**
  
  Two types of informations are taken into account:
  
  The *misprice*
  
  \[ z v_t - z_t = \log(v_t) - \log(p_t) \]
A market-like game

and the *price trend*

\[ z_t - z_{t-1} = \log(p_t) - \log(p_{t-1}) \]
A market-like game

- and the *price trend*

\[ z_t - z_{t-1} = \log(p_t) - \log(p_{t-1}) \]

- # Non-decreasing function \( f(x) \) such that \( f(-\infty) = 0 \) and \( f(\infty) = 1 \)

Two examples

\[
\begin{align*}
  f_1(x) &= \theta(x) \\
  f_2(x) &= \frac{1}{1 + \exp(-\beta x)}
\end{align*}
\]

Four-component vector \( \gamma \)

\[
\gamma_t = \begin{pmatrix}
  f(zv_t - z_t) f(z_t - z_{t-1}) \\
  f(zv_t - z_t) (1 - f(z_t - z_{t-1})) \\
  (1 - f(zv_t - z_t)) f(z_t - z_{t-1}) \\
  (1 - f(zv_t - z_t)) (1 - f(z_t - z_{t-1}))
\end{pmatrix}
\]
A market-like game

and the *price trend*

\[ z_t - z_{t-1} = \log(p_t) - \log(p_{t-1}) \]

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\end{pmatrix}
\]

Strategy of each investor:

four-component vector \( \alpha^{(i)} \) with entries \(-1, 0, \) or \(1\)

Investment of agent \( i \) : \( \alpha^{(i)} \cdot \gamma \)
A market-like game

- **Examples:**
  - Fundamental (value-investing strategy)
    \[ \alpha^{(i)} = (1, 1, -1, -1) \]
  - Pure trend-following (technical trading)
    \[ \alpha^{(i)} = (1, -1, 1, -1) \]
A market-like game

- Examples:
  - Fundamental (value-investing strategy)
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- Total number of strategies is \( 3^4 = 81 \). Strategies labelled by a number
  \[
  n^{(i)} = \sum_{k=0}^{3} 3^k \left( \alpha_k^{(i)} + 1 \right)
  \]
  (Fundamental strategy = no.72 and pure trend-following = no.60)
A market-like game

- **Examples:**
  - Fundamental (value-investing strategy)
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  (Fundamental strategy = no.72 and pure trend-following = no.60)
- **Evolution dynamics:**
  - After \( r \) time steps \( s \) agents copy the strategy of the \( s \) best performers plus
  - Mutation probability
    (In the figures: \( r = 50, s = 10, \lambda_0 = 10000 \), \( 45 = (0, 1, -1, -1) \),
    \( 18 = (-1, 1, -1, -1) \), \( 73 = (1, 1, -1, 0) \), \( 75 = (1, 1, 0, -1) \))
A market-like game

Market game simulation with evolution. Initial condition: all traders in the fundamental strategy
A market-like game

Market game simulation with evolution. Initial condition: 50% fundamental and 50% trend-following
A market-like game

Market game simulation with evolution. Initial condition: random strategies
A market-like game

Market game simulation without evolution. 50% of fundamental strategies and 50% trend-following
A market-like game

Lyapunov exponents for the log-price \((z_t)\) dynamics. The Jacobian of

\[
\begin{pmatrix}
  z_t \\
  z_{t-1}
\end{pmatrix}
\rightarrow
\begin{pmatrix}
  z_{t+1} \\
  z_t
\end{pmatrix}
\]

is

\[
M_t = \begin{pmatrix}
  1 + \frac{\partial}{\partial z_t} \frac{\sum_i \omega^{(i)}}{\lambda + \lambda_1 |\sum_i \omega^{(i)}|} & \frac{\partial}{\partial z_{t-1}} \frac{\sum_i \omega^{(i)}}{\lambda_0 + \lambda_1 |\sum_i \omega^{(i)}|} \\
  1 & 0
\end{pmatrix}
\]
A market-like game

- Lyapunov exponents for the log-price ($z_t$) dynamics. The Jacobian of

$$
\begin{pmatrix}
  z_t \\
  z_{t-1}
\end{pmatrix}
\rightarrow
\begin{pmatrix}
  z_{t+1} \\
  z_t
\end{pmatrix}
$$

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$$
M_t = 
\begin{pmatrix}
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\end{pmatrix}
$$

- Lyapunov spectrum obtained from

$$
\lim_{N \to \infty} \left| M_t^T M_{t+N-1} \cdots M_t^T M_t \cdots M_{t+N-1} \right|^{1/2N}
$$
A market-like game

- Lyapunov exponents for the log-price \((z_t)\) dynamics. The Jacobian of

\[
\begin{pmatrix}
Z_t \\
Z_{t-1}
\end{pmatrix}
\rightarrow
\begin{pmatrix}
Z_{t+1} \\
Z_t
\end{pmatrix}
\]

is

\[
M_t = \begin{pmatrix}
1 + \frac{\partial}{\partial z_t} \frac{\sum_i \omega^{(i)}}{\lambda_0 + \lambda_1} \left| \sum_i \omega^{(i)} \right| & \frac{\partial}{\partial z_{t-1}} \frac{\sum_i \omega^{(i)}}{\lambda_0 + \lambda_1} \left| \sum_i \omega^{(i)} \right| \\
1 & 0
\end{pmatrix}
\]

- Lyapunov spectrum obtained from

\[
\lim_{N \to \infty} \left| M_{t+N-1}^T \cdots M_t^T M_t \cdots M_t + N-1 \right|^{1/2N}
\]

- Lyapunov exponents computed for \(f = f_2\) for several values of \(\beta\) and a 50 – 50 admixture of fundamental and trend-following strategies. Typically one Lyapunov number equal to zero and the other smaller than but very close to one.