# Random sampling and superoscillations: "Beating" Nyquist's rate

Rui Vilela Mendes
CMAF and IPFN, IST
http://label2.ist.utl.pt/vilela/

# Contents

# The Shannon sampling theorem

## Theorem

*If a function x(t) contains no frequencies higher than B hertz, it is completely determined by giving its values at a series of points spaced 1/(2B) seconds apart. (! sampling for infinite time)*

**Proof:**

$$f(t) = \frac{1}{2\pi} \int_{-2\pi B}^{2\pi B} F(\omega) e^{i\omega t} d\omega$$

$F(\omega)$ may be writen as a Fourier series

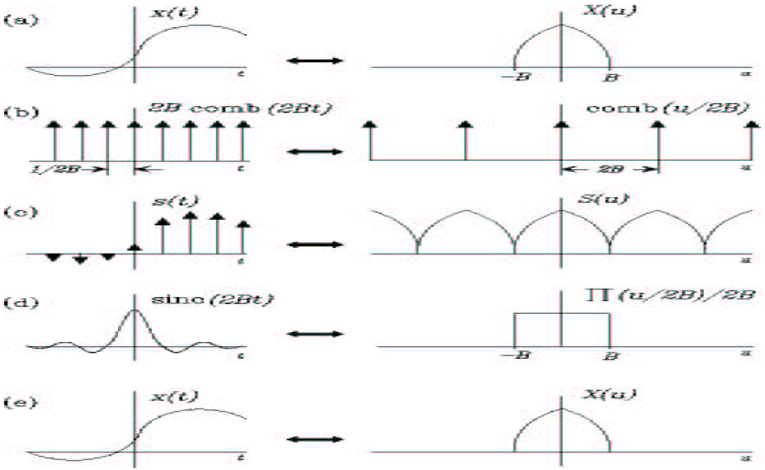
$$F(\omega) = \sum_{n=-\infty}^{n=\infty} \frac{x_n}{2B} e^{i\frac{n}{2B}\omega}$$

By substitution and integration in $\omega$

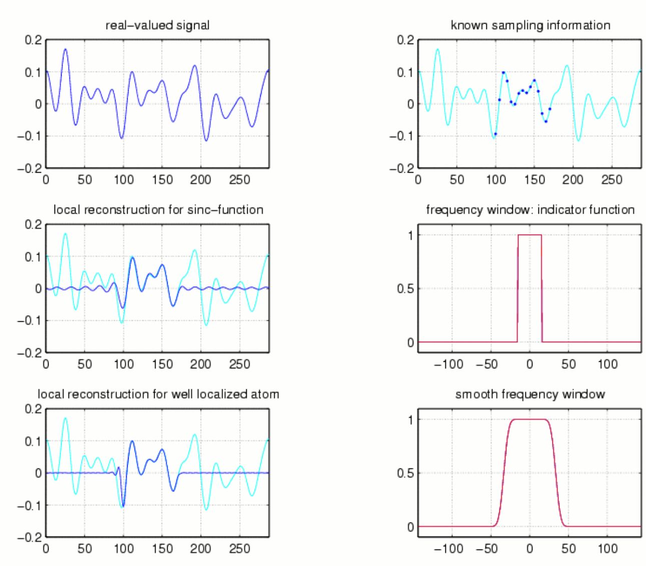
$$f(t) = \frac{1}{2\pi} \int_{-2\pi B}^{2\pi B} \sum_{n=-\infty}^{n=\infty} \frac{x_n}{2B} e^{i\frac{n}{2W}\omega} e^{i\omega t} d\omega = \sum_{n=-\infty}^{n=\infty} x_n \frac{\sin \pi (2Bt - n)}{\pi (2Bt - n)}$$

$x_n$ being $x_n = f\left(\frac{n}{2B}\right)$

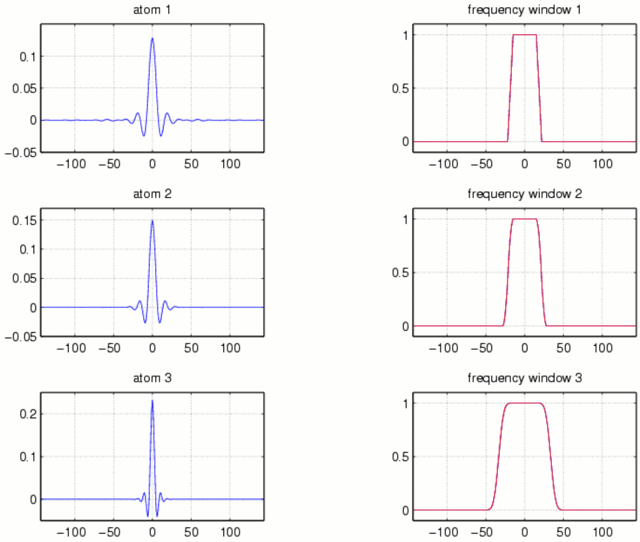
# The Shannon sampling theorem: graphical illustration



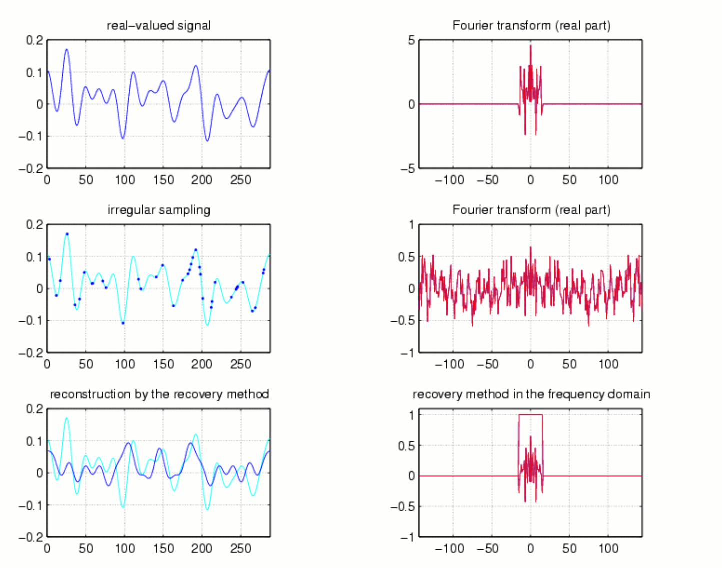
# Shannon sampling: local reconstruction



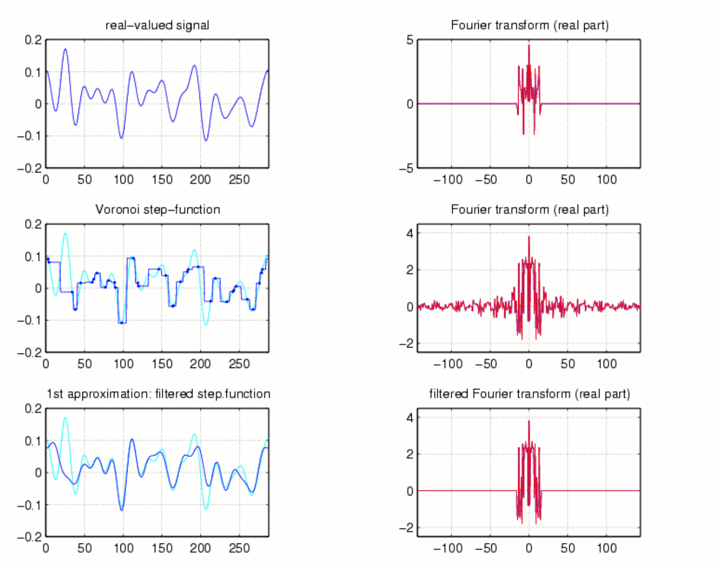
# Shannon sampling: alternatives to the sinc



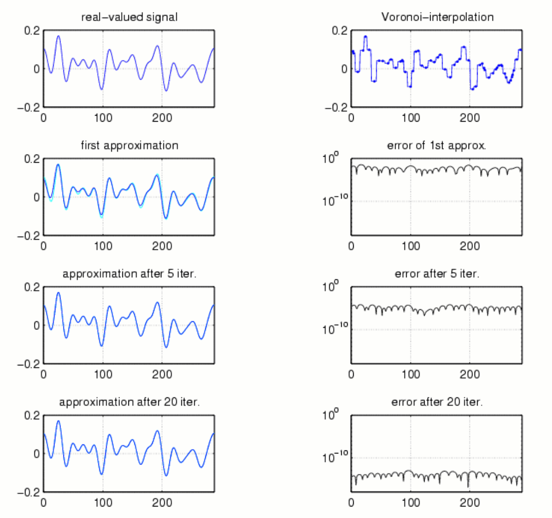
# Shannon sampling: irregular sampling



# Irregular sampling: The Voronoi-Allebach algorithm

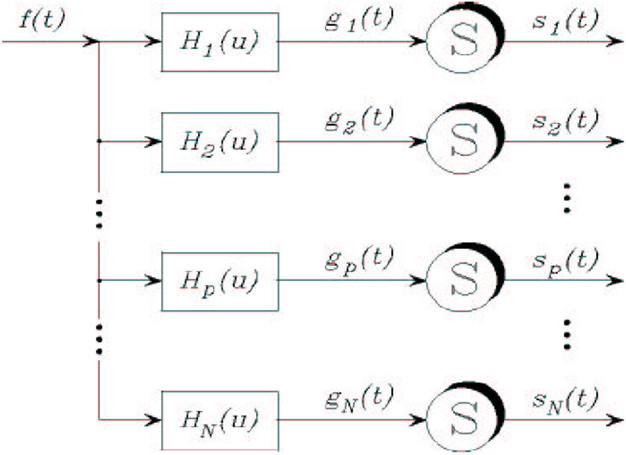


# Irregular sampling: The Voronoi-Allebach algorithm




Convergence, but band-limited by the largest gap

# The Papoulis generalization




Sampling of $N$ filtered copies of the signal at $1/N$ rate.

# The Papoulis generalization

Let $\{H_p(\omega); p = 1, \cdots, N\}$ be the set of filters. The outputs are

$$g_p(t) = f(t) * h_p(t) \qquad p = 1, \cdots, n$$

When the signals are sampled at the $1/N$ the Nyquist rate: $T_N = N/2B$

$$s_p(t) = \sum_{n=-\infty}^{\infty} g_p(nT_N)\,\delta(t - nT_N)$$

Then

$$f(t) = \sum_{p=1}^{N} \sum_{n=-\infty}^{\infty} g_p(nT_N)\,k_p(t - nT_N)$$

$$k_p(t) = \int_{-B}^{B} K_p(\omega, t)\, e^{i2\pi\omega t} d\omega$$

with the $K_p(\omega, t)$ being the solutions of set of equations $(0 \leq m \leq N)$

$$\frac{2B}{N} \sum_{p=1}^{N} K_p(\omega, t)\, H_p\left(\omega - 2m\frac{B}{N}\right) = e^{-i2\pi mtN/T}$$

# The Papoulis generalization

Because: The original signal has band $B$ and when filtered each $g_p(t)$ should still have band larger than $\frac{B}{N}$. When sampled at $T_N = NT$ intervals its Fourier transform $S_p(\omega)$ is periodic of period $2B/N$. Therefore it is an aliased replication of $G_p(\omega)$. In the interval $\left[B - \frac{2B}{N}, B\right]$ there are $N$ portions of replicated spectrum

$$S_p(\omega) = \frac{2B}{N} \sum_{n=0}^{N-1} H_p\left(\omega - \frac{2n}{N}B\right) F\left(\omega - \frac{2n}{N}B\right)$$

in matrix form

$$\left|\begin{array}{c} S_1(\omega) \\ S_2(\omega) \\ \vdots \\ S_N(\omega) \end{array}\right| = \left|\begin{array}{ccc} H_1(\omega) & \cdots & H_1\left(\omega - \frac{2(N-1)B}{N}\right) \\ H_2(\omega) & \cdots & H_2\left(\omega - \frac{2(N-1)B}{N}\right) \\ \vdots & \vdots & \vdots \\ H_N(\omega) & \cdots & H_N\left(\omega - \frac{2(N-1)B}{N}\right) \end{array}\right| \left|\begin{array}{c} F(\omega) \\ F\left(\omega - \frac{2B}{N}\right) \\ \vdots \\ F\left(\omega - \frac{2(N-1)B}{N}\right) \end{array}\right|$$

Solution for $F$ if the matrix $H$ is not singular (independence of the filters)

# Problems with Shannon sampling: Irregular sampling and subNyquist rates

- **Regular versus irregular sampling**
  Sampling at irregular intervals for band-limited functions poses special problems. Controlled by the largest gap.

# Problems with Shannon sampling: Irregular sampling and subNyquist rates

- **Regular versus irregular sampling**
  Sampling at irregular intervals for band-limited functions poses special problems. Controlled by the largest gap.
- However irregular sampling of an appropriate type, instead of being a nuisance, may be of help for the asymptotically exact reconstruction of signals.

# Problems with Shannon sampling: Irregular sampling and subNyquist rates

- **Regular versus irregular sampling**
  Sampling at irregular intervals for band-limited functions poses special problems. Controlled by the largest gap.
- However irregular sampling of an appropriate type, instead of being a nuisance, may be of help for the asymptotically exact reconstruction of signals.
- **Question:** Can we reconstruct signal at an average sampling rate slower than Nyquist's rate? And at irregular intervals?

# Problems with Shannon sampling: Irregular sampling and subNyquist rates

- **Regular versus irregular sampling**
  Sampling at irregular intervals for band-limited functions poses special problems. Controlled by the largest gap.
- However irregular sampling of an appropriate type, instead of being a nuisance, may be of help for the asymptotically exact reconstruction of signals.
- **Question:** Can we reconstruct signal at an average sampling rate slower than Nyquist's rate? And at irregular intervals?
- **Answer:** Yes, but in a different functional space.

# Exact asymptotic reconstruction of almost-periodic functions

- **Almost-periodic functions**
  *Functions that can be approximated by trigonometric polynomials:*
  For any $\varepsilon > 0$ , exists a finite number set $(\omega_1, B_1, \alpha_1) \cdots (\omega_n, B_n, \alpha_n)$ such that

$$g(x) = \sum_{j=1}^{n} B_j e^{i(\omega_j x + \alpha_j)}; \qquad \sup_{x\in\mathbb{R}} |f(x) - g(x)| \leq \varepsilon$$

# Exact asymptotic reconstruction of almost-periodic functions

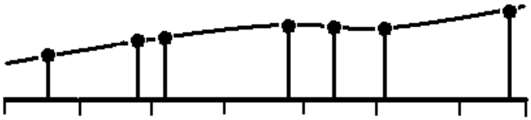
- **Almost-periodic functions**
  *Functions that can be approximated by trigonometric polynomials:*
  For any $\varepsilon > 0$ , exists a finite number set $(\omega_1, B_1, \alpha_1) \cdots (\omega_n, B_n, \alpha_n)$ such that

$$g(x) = \sum_{j=1}^{n} B_j e^{i(\omega_j x + \alpha_j)}; \qquad \sup_{x \in \mathbb{R}} |f(x) - g(x)| \leq \varepsilon$$

**Theorem**

*(Collet) Let $x_n = n\lambda + X_n$ with $X_n$ being a sequence of i.i.d. random variables uniformly distributed in $[0, \lambda]$. Then, almost every configuration $\{x_n\}$ of the point process has the property that if $F$ is any complex almost periodic function satisfying $F(x_n) = 0 \quad \forall n \in \mathbb{Z}$, then $F \equiv 0$.*

# Exact asymptotic reconstruction of almost-periodic functions



- Consider $F(x) = f(x) - g(x)$ where $g(x)$ is an almost-periodic function coinciding with the unknown function at the sampled points. If $f(x)$ is also almost-periodic it equals $g(x)$. Approximation by trigonometric polynomials and Collet's theorem provides a basis for asymptotically exact reconstruction algorithms at any (random) rate.

# Fourier and fractional Fourier transform

- Approximation by trigonometric polynomials is adequate for signals with stable frequency components, not so for signals with complex time-frequency structure.

# Fourier and fractional Fourier transform

- Approximation by trigonometric polynomials is adequate for signals with stable frequency components, not so for signals with complex time-frequency structure.
- Obtain inspiration from the fractional Fourier transform

# Fourier and fractional Fourier transform

- Approximation by trigonometric polynomials is adequate for signals with stable frequency components, not so for signals with complex time-frequency structure.
- Obtain inspiration from the fractional Fourier transform
- The Fourier operator $\mathcal{F}_1$ and its inverse

$$(\mathcal{F}_1 f)(\omega) = F(\omega) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\, dt$$

$$f(t) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{i\omega t} F(\omega)\, d\omega$$

In the $(t, \omega)$ plane, $\mathcal{F}_1$ is a rotation of the signal by $\alpha_1 = \frac{\pi}{2}$.

# Fourier and fractional Fourier transform

- Approximation by trigonometric polynomials is adequate for signals with stable frequency components, not so for signals with complex time-frequency structure.
- Obtain inspiration from the fractional Fourier transform
- The Fourier operator $\mathcal{F}_1$ and its inverse

$$\left(\mathcal{F}_1 f\right)(\omega) = F(\omega) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\, dt$$

$$f(t) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{i\omega t} F(\omega)\, d\omega$$

  In the $(t, \omega)$ plane, $\mathcal{F}_1$ is a rotation of the signal by $\alpha_1 = \frac{\pi}{2}$.
- The fractional Fourier transform is a rotation in the $(t, \omega)$ plane, by a non-integer multiple of $\frac{\pi}{2}$, $\alpha_b = b\frac{\pi}{2}$,

$$\left(\mathcal{F}_b f\right)(\zeta) = \frac{e^{-\frac{i}{2}\left(\operatorname{sgn}(\sin\alpha_b)\frac{\pi}{2} - \alpha_b\right)}}{\left(2\pi\left|\sin\alpha_b\right|\right)^{1/2}} \int_{-\infty}^{\infty} e^{\left(-i\frac{t\zeta}{\sin\alpha_b} + \frac{i}{2}\cot\alpha_b\left(t^2 + \zeta^2\right)\right)} f(t)\, dt$$

# Chirps

- The inverse of $\mathcal{F}_b$ is $\mathcal{F}_{-b}$.

# Chirps

- The inverse of $\mathcal{F}_b$ is $\mathcal{F}_{-b}$.
- The Fourier transform decomposes $f$ into harmonics $(\{e^{-i\omega t}\})$.

# Chirps

- The inverse of $\mathcal{F}_b$ is $\mathcal{F}_{-b}$.
- The Fourier transform decomposes $f$ into harmonics $(\{e^{-i\omega t}\})$.
- The kernel of the fractional Fourier transform is a linear chirp $e^{-i(\omega - ct)t}$ with $\omega = \frac{\zeta}{\sin\alpha_b}$ and $c = \frac{1}{2}\cot\alpha_b$.

# Chirps

- The inverse of $\mathcal{F}_b$ is $\mathcal{F}_{-b}$.
- The Fourier transform decomposes $f$ into harmonics $(\{e^{-i\omega t}\})$.
- The kernel of the fractional Fourier transform is a linear chirp $e^{-i(\omega - ct)t}$ with $\omega = \frac{\zeta}{\sin \alpha_b}$ and $c = \frac{1}{2}\cot \alpha_b$.
- This suggests that a more general basis to expand a signal, with arbitrary features in the $\omega - t$ plane, is a basis of linear chirps. Choice of a basis is always a critical issue in the reconstruction of sampled signals.

# Asymptotic reconstruction in chirp space

- **A uniform approximation result for random sampling in chirp space**
  Define the space $\mathcal{LC}$ of *linear chirp functions* as the space of functions $f$ such that $\forall \varepsilon > 0$ $\exists$ a finite number of real number sets $(\omega_1, c_1, \alpha_1, B_1), ..., (\omega_k, c_k, \alpha_k, B_k)$ such that

$$g(x) = \sum_{j=1}^{k} B_j e^{i\{\omega_j + c_j(x-\alpha_j)\}x}$$

and

$$\sup_{x\in\mathbb{R}} |f(x) - g(x)| \leq \varepsilon$$

(the space of almost periodic functions corresponds to the $c_j = 0$ case).

- **A uniform approximation result for random sampling in chirp space**
  Define the space $\mathcal{LC}$ of *linear chirp functions* as the space of functions $f$ such that $\forall \varepsilon > 0$ $\exists$ a finite number of real number sets $(\omega_1, c_1, \alpha_1, B_1), \ldots, (\omega_k, c_k, \alpha_k, B_k)$ such that

$$g(x) = \sum_{j=1}^{k} B_j e^{i\{\omega_j + c_j(x-\alpha_j)\}x}$$

  and

$$\sup_{x\in\mathbb{R}} |f(x) - g(x)| \leq \varepsilon$$

  (the space of almost periodic functions corresponds to the $c_j = 0$ case).
- *The space $\mathcal{LC}$ of linear chirp functions is strictly larger than the space of almost periodic functions. (It suffices to consider $e^{ix^2}$)*

# Asymptotic reconstruction in chirp space

**Theorem**

*(E. Carlen, RVM) Let $x_n = n\lambda + X_n$ with $X_n$ being a sequence of i.i.d. random variables uniformly distributed in $[0,\lambda]$. Then, almost every configuration $\{x_n\}$ of the point process has the property that if $f$ is a function in the linear chirp space $\mathcal{LC}$ satisfying*
*$f\left(x_n\right) = 0 \quad \forall n \in \mathbb{Z}, \quad$ then $f \equiv 0$.*

For the proof one needs the following :

**Lemma**

*For almost every configuration $\left\{x_n\right\}$ of the random process, one has*

$$\lim_{L\to\infty} \frac{1}{2L+1} \sum_{-L\leq n\leq L} e^{i\left(\omega x_n + c x_n^2\right)} = 0$$

*for real $\omega$ and $c$ with $\omega \neq 0$.*

- **Proof of the theorem:**
Let $f(x_n) = 0,\ \forall n \in \mathbb{Z}$ and $g(x)$ be its $\varepsilon-$approximation by linear chirp polynomials. Then

$$\left| \lim_{L\to\infty} \frac{1}{2L+1} \sum_{-L\leq n\leq L} e^{-i\left\{\omega x_n + c x_n^2\right\}} g(x_n) \right|$$
$$= \left| \lim_{L\to\infty} \frac{1}{2L+1} \sum_{-L\leq n\leq L} e^{-i\left\{\omega x_n + c x_n^2\right\}} \left(g(x_n) - f(x_n)\right) \right| \leq \varepsilon$$

for all $\omega$ and $c$. Inserting now $g(x) = \sum_{j=1}^{k} B_j e^{i\{\omega_j + c_j(x-\alpha_j)\}x}$ in the left-hand side of the above equation one obtains

$$\left| \lim_{L\to\infty} \frac{1}{2L+1} \sum_{-L\leq n\leq L} \sum_{j=1}^{k} B_j e^{i\left\{(\omega_j - c_j\alpha_j - \omega)x_n + (c_j - c)x_n^2\right\}} \right| \leq \varepsilon$$

Choosing $\omega = \omega_j - c_j\alpha_j$ , $c = c_j$ and using the lemma, one concludes that for almost every configuration $\{x_n\}$,

# Asymptotic reconstruction in chirp space

- $$|B_j| \leq \varepsilon$$

  for all $j$ in the linear chirp approximation.

# Asymptotic reconstruction in chirp space

- $$|B_j| \leq \varepsilon$$
  for all $j$ in the linear chirp approximation.
- Because this result holds for all $\varepsilon$ and the linear chirp basis functions are kernels to the fractional Fourier transform, one concludes that the function $f$ has zero fractional Fourier spectrum. Therefore it is the zero function.

# Asymptotic reconstruction in chirp space

- $$|B_j| \leq \varepsilon$$
  for all $j$ in the linear chirp approximation.
- Because this result holds for all $\varepsilon$ and the linear chirp basis functions are kernels to the fractional Fourier transform, one concludes that the function $f$ has zero fractional Fourier spectrum. Therefore it is the zero function.
- As in the case of functions in the almost periodic space, the above result may be used to estimate functions in the linear chirp space by random sampling. If from a time series $h(x_n)$, one obtains, by the appropriate algorithm, a linear chirp approximation $g(x)$ coinciding with the sampled function on a typical sequence $\{x_n\}$, that is
  $$f(x_n) = g(x_n) - h(x_n) = 0$$
  then, in the above defined space, one knows that $g(x) = h(x)$ for all $x$.

# Asymptotic reconstruction in chirp space

- Reconstruction by random sampling may be carried out by the following algorithm:

# Asymptotic reconstruction in chirp space

- Reconstruction by random sampling may be carried out by the following algorithm:
- 1) Compute

$$F(f,c) = \frac{1}{N}\sum_{n=1}^{N} s(t_n)\exp\left\{-i\left(2\pi f t_n + c t_n^2\right)\right\}$$

for the random sampled signal $s(t)$.
Find the dominant maximum of $|F(f,c)|$ in the $(f,c)$ plane. Let $(f_1, c_1)$ be the location of this maximum and

$$A_1 = F(f_1, c_1)$$

# Asymptotic reconstruction in chirp space

- Reconstruction by random sampling may be carried out by the following algorithm:
- 1) Compute

$$F(f,c) = \frac{1}{N}\sum_{n=1}^{N} s(t_n)\exp\left\{-i\left(2\pi f t_n + c t_n^2\right)\right\}$$

  for the random sampled signal $s(t)$.
  Find the dominant maximum of $|F(f,c)|$ in the $(f,c)$ plane. Let $(f_1, c_1)$ be the location of this maximum and

$$A_1 = F(f_1, c_1)$$

- 2) Subtract $A_1 \exp\left\{i\left(2\pi f_1 t + c t^2\right)\right\}$ from the signal

$$s_1(t_n) = s(t_n) - A_1 \exp\left\{i\left(2\pi f_1 t + c_1 t^2\right)\right\}$$

# Asymptotic reconstruction in chirp space

- Reconstruction by random sampling may be carried out by the following algorithm:
- 1) Compute

$$F\left(f,c\right)=\frac{1}{N}\sum_{n=1}^{N}s\left(t_n\right)\exp\left\{-i\left(2\pi f t_n+ct_n^2\right)\right\}$$

  for the random sampled signal $s\left(t\right)$.
  Find the dominant maximum of $\left|F\left(f,c\right)\right|$ in the $\left(f,c\right)$ plane. Let $\left(f_1,c_1\right)$ be the location of this maximum and

$$A_1=F\left(f_1,c_1\right)$$

- 2) Subtract $A_1\exp\left\{i\left(2\pi f_1 t+ct^2\right)\right\}$ from the signal

$$s_1\left(t_n\right)=s\left(t_n\right)-A_1\exp\left\{i\left(2\pi f_1 t+c_1 t^2\right)\right\}$$

- 3) Repeat step 1) for $s_1\left(t\right)$ looking for another dominant maximum away from $\left(f_1,c_1\right)$. Let $\left(f_2,c_2\right)$ be the location of this maximum and $A_2=F\left(f_2,c_2\right)$.

# Asymptotic reconstruction in chirp space

- 4) Compute

$$s_2\left(t_n\right) = s_1\left(t_n\right) - A_2 \exp\left\{i\left(2\pi f_2 t + c_2 t^2\right)\right\}$$

# Asymptotic reconstruction in chirp space

- 4) Compute

$$s_2(t_n) = s_1(t_n) - A_2 \exp\left\{i\left(2\pi f_2 t + c_2 t^2\right)\right\}$$

- Repeat the process until no new maxima appear. Then repeat the whole process starting from 1) until a stable estimate of $s(t)$ is obtained.

# Asymptotic reconstruction in chirp space

- 4) Compute

$$s_2(t_n) = s_1(t_n) - A_2 \exp\left\{ i\left(2\pi f_2 t + c_2 t^2\right)\right\}$$

- Repeat the process until no new maxima appear. Then repeat the whole process starting from 1) until a stable estimate of $s(t)$ is obtained.
- This, of course, is the kind of procedure that one naively expects to lead to an estimate of the signal in chirp space. What our result improves upon is not on this or similar algorithms but on the guarantee of the asymptotic convergence of the random sampling approximation.

# Asymptotic reconstruction in chirp space

- 4) Compute

$$s_2(t_n) = s_1(t_n) - A_2 \exp\left\{i\left(2\pi f_2 t + c_2 t^2\right)\right\}$$

- Repeat the process until no new maxima appear. Then repeat the whole process starting from 1) until a stable estimate of $s(t)$ is obtained.
- This, of course, is the kind of procedure that one naively expects to lead to an estimate of the signal in chirp space. What our result improves upon is not on this or similar algorithms but on the guarantee of the asymptotic convergence of the random sampling approximation.
- The power of random sampling may be illustrated by a simple example. Let

$$s(t) = \sum_{i=1}^{3} A_i \exp\left\{i\left(2\pi f_i t + c_i t^2\right)\right\}$$

be a 3-chirp signal.

# Asymptotic reconstruction in chirp space

- When the function $|F(f, c)|$ is computed either by random or regular sampling above Nyquist's rate, the identification of the maxima in the $(f, c)$ −plane is quite similar.

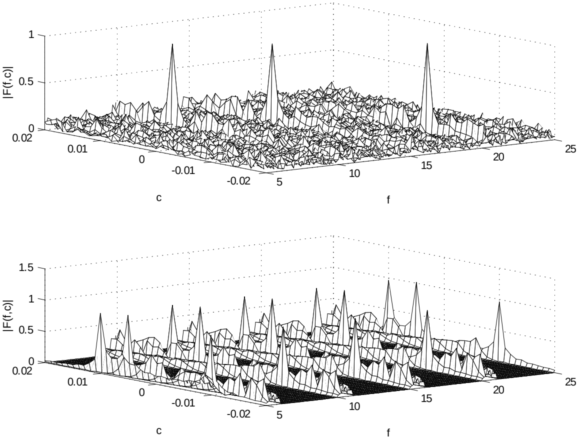
# Asymptotic reconstruction in chirp space

- When the function $|F(f, c)|$ is computed either by random or regular sampling above Nyquist's rate, the identification of the maxima in the $(f, c)$ −plane is quite similar.
- However, for (average) sampling rates below Nyquist's the difference is quite remarkable.

# Asymptotic reconstruction in chirp space

- When the function $|F(f, c)|$ is computed either by random or regular sampling above Nyquist's rate, the identification of the maxima in the $(f, c)$ −plane is quite similar.
- However, for (average) sampling rates below Nyquist's the difference is quite remarkable.
- The figure compares the behavior of $|F(f, c)|$ for random and regular sampling at the same average rate, equal to $\frac{1}{4}$ the Nyquist rate. The accurate identification of the chirp parameters by random sampling is quite impressive, whereas for regular sampling the result is pure nonsense. What one sees in the regular sampling case are the beatings between the signal frequencies and the sampling frequency. Notice that to make the regular sampling as unbiased as possible the initial time of the sequence has been randomized.

# Asymptotic reconstruction in chirp space



# Questions

- Quasi-periodic, linear chirp space, nonlinear chirps
  Collet's counter-example
  Where is the boundary in function space for reconstruction by random sampling?

# Questions

- Quasi-periodic, linear chirp space, nonlinear chirps
  Collet's counter-example
  Where is the boundary in function space for reconstruction by random sampling?
- Reconstruction with nonuniform probability distributions

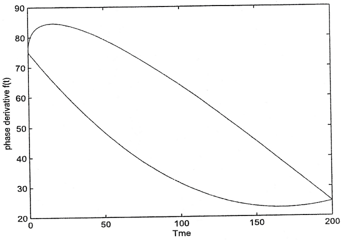
# Tomograms and random sampling

Tomographic reconstruction of the superposition of two nonlinear chirps in the interval $[0, T]$

$$Y(t) = y_1(t) + y_2(t) = A_1 e^{i\Phi_1(t)} + A_2 e^{i\Phi_2(t)}$$

$$\begin{aligned} \Phi_1(t) &= a_1 t^2 + c_1 t^{\frac{3}{2}} + b_1 t \\ \Phi_2(t) &= a_2 t^2 + c_2 t^3 + b_2 t \end{aligned}$$

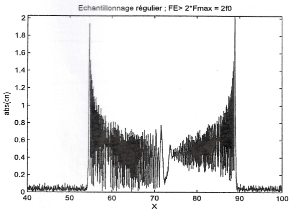
**Phase derivative of the chirps**

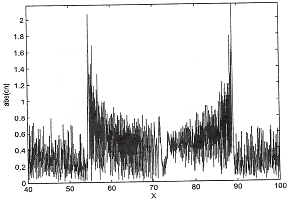


# Tomograms and random sampling

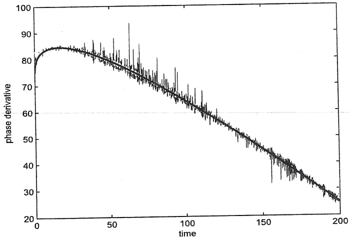
- Two reconstructions are analysed:
  1) Regular sampling
  T=200s, 20000 points $f = 314rd/s$ $f_{Sh} = 150$
  T=200s, 2000 points $f = 31.4rd/s$
  2) Random sampling
  T=200s, 20000 points $f = 314rd/s$ $f_{Sh} = 150$
  T=200s, 2000 points $f = 31.4rd/s$

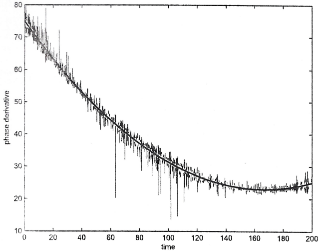
- Two reconstructions are analysed:
  1) Regular sampling
  T=200s, 20000 points $f = 314 rd/s$ $f_{Sh} = 150$
  T=200s, 2000 points $f = 31.4 rd/s$
  2) Random sampling
  T=200s, 20000 points $f = 314 rd/s$ $f_{Sh} = 150$
  T=200s, 2000 points $f = 31.4 rd/s$
- **Tomograms cut at $\theta = 0.442\pi$ for sampling with 20000 points (regular or random) and random sampling at 200 points**



**Phase derivative reconstruction after separation with random sampling**



# References


- R. J. Marks; *Introduction to Shannon Sampling and Interpolation Theory*, Springer 1991.
- M. Unser; *Sampling-50 years after Shannon*, Proc. IEEE 88 (2000) 569-587.
- *A first guided tour on the irregular sampling problem*, http://www.math.ucdavis.edu/~strohmer/research/sampling/irsampl.htr
- P. Collet; *Sampling almost periodic functions with random probes of finite density*; Proc. Roy. Soc. London A452 (1996) 2263-2277.
- E. Carlen, R. Vilela Mendes; *Signal reconstruction by random sampling in chirp space*, Nonlinear Dynamics 56 (2009) 223-229.

# Superoscillations

- Question: *Can we send Beethoven fifth symphony using a 5KHz band and receiving in real time a 25KHz signal?*

# Superoscillations

- Question: *Can we send Beethoven fifth symphony using a 5KHz band and receiving in real time a 25KHz signal?*
- Let $B = 25KHz$ and $T \simeq$ the duration of Beethoven's fifth. Using Nyquist-Shannon's theorem, the symphony is converted into a sequence of $2BT$ real numbers.

# Superoscillations

- Question: *Can we send Beethoven fifth symphony using a 5KHz band and receiving in real time a 25KHz signal?*
- Let $B = 25KHz$ and $T \simeq$ the duration of Beethoven's fifth. Using Nyquist-Shannon's theorem, the symphony is converted into a sequence of $2BT$ real numbers.
- **Theorem** (Kempf): *Each Hilbert space of bandlimited signals contains signals such that the Fourier transform of $F(\omega)$, i.e. the signal $f(t)$, passes through any finite number of arbitrarily prespecified values.*

# Superoscillations

- Question: *Can we send Beethoven fifth symphony using a 5KHz band and receiving in real time a 25KHz signal?*
- Let $B = 25KHz$ and $T \simeq$ the duration of Beethoven's fifth. Using Nyquist-Shannon's theorem, the symphony is converted into a sequence of $2BT$ real numbers.
- **Theorem** (Kempf): *Each Hilbert space of bandlimited signals contains signals such that the Fourier transform of $F(\omega)$, i.e. the signal $f(t)$, passes through any finite number of arbitrarily prespecified values.*
- Proof (sketch): In the Hilbert space of $B-$band limited functions, consider the operator $T$

$$T\phi(t) = t\phi(t)$$

and its self-adjoint extensions $T(\alpha)$. Each self-adjoint extension has linearly independent eigenvectors $\{\mathbf{t}_n(\alpha)\}$ such that

$$\phi(t_n) = (\mathbf{t}_n(\alpha), \phi)$$

# Superoscillations

- Given a function that at a set $\{t_i\}$ of points must have prespecified values $\phi_i$, that is
$$(\mathbf{t}_i, \phi) = \phi_i$$
its coefficients $(\mathbf{t}_n(\alpha), \phi)$ in the $\{\mathbf{t}_n(\alpha)\}$ basis are obtained by
$$\sum_{n=-\infty}^{n=\infty} (\mathbf{t}_i(\alpha), \mathbf{t}_n(\alpha)) (\mathbf{t}_n^*(\alpha), \phi) = \phi_i$$

# Superoscillations

- Given a function that at a set $\{t_i\}$ of points must have prespecified values $\phi_i$, that is
$$(\mathbf{t}_i, \phi) = \phi_i$$
its coefficients $(\mathbf{t}_n(\alpha), \phi)$ in the $\{\mathbf{t}_n(\alpha)\}$ basis are obtained by
$$\sum_{n=-\infty}^{n=\infty} (\mathbf{t}_i(\alpha), \mathbf{t}_n(\alpha)) (\mathbf{t}_n^*(\alpha), \phi) = \phi_i$$
- Linear independence of the $\{\mathbf{t}_n(\alpha)\}$ basis yields the existence of a solution.

# Superoscillations: An example

- Let the bandwidth be 1/2 Hz. Then any signal has the cardinal series form

$$f\left(t\right)=\sum_{n=1}^{N}a_{n}\frac{\sin\left(\left(t-n\right)\pi\right)}{\left(t-n\right)\pi}$$

# Superoscillations: An example

- Let the bandwidth be 1/2 Hz. Then any signal has the cardinal series form

$$f\left(t\right)=\sum_{n=1}^{N}a_{n}\frac{\sin\left(\left(t-n\right)\pi\right)}{\left(t-n\right)\pi}$$

- Now I want to encode in this signal, information about a signal with a higher band, that is, information concerning values at points less spaced in time. Let $\tau<1$

$$f\left(t\right)=\sum_{r=1}^{N}x_{r}\frac{\sin\left(\left(t-\tau r\right)\pi\right)}{\left(t-\tau r\right)\pi}$$

The signal should satisfy $f\left(n\tau\right)=a_{n}$, the prescribed values. Then

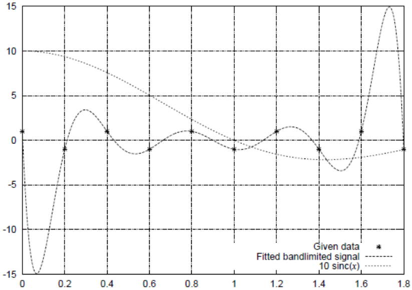
$$a_{n}=\sum_{r=1}^{N}x_{r}\frac{\sin\left(\left(n-r\right)\tau\pi\right)}{\left(n-r\right)\tau\pi}$$

All one has to do is to solve this system of equations to obtain the $x_r$'s

# Superoscillations: An example

$$x_r = \sum_{n=1}^{N} S_{rn}^{-1} a_n$$

$S_{rn}^{-1}$ being the inverse of the matrix $S_{nr} = \frac{\sin((n-r)\tau\pi)}{(n-r)\tau\pi}$



# Superoscillations

- Superoscillations are possible because they are a local phenomenon. The global behavior of a signal is not affected by the occurrence of superoscillations which occur over finite intervals. A bandlimited signal can oscillate at a rate higher than the Nyquist rate only on finite intervals, but not on infinite intervals.
- Instead of a reduced bandwidth, another possibility is to use superoscillating signals of the same bandwidth. The use of superoscillating signals will allow to compress messages into an arbitrarily short time interval.
- The price to be paid is that, for fixed message size, the energy expense grows polynomially with the compression and that, for fixed compression, the energy expense grows exponentially with the message size.

# Applications of superoscillations

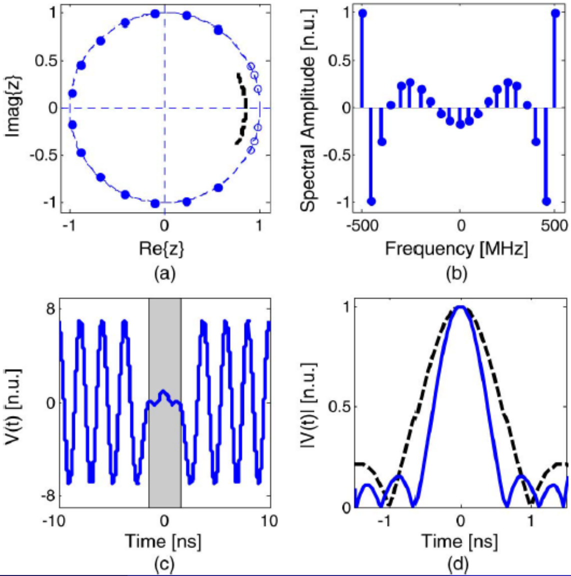
- **Superresolution imaging (ex. radar):** A superoscillatory waveform contains, across a finite time interval, faster variations than its highest constituent frequency component. Radar imaging using a superoscillatory pulse allows one to detect an object with a range resolution improved beyond the fundamental bandwidth limitation. In particular it reduces distance uncertainty.
- Construction of the superoscillation

$$\tilde{V}(\omega) = \sum_{n=0}^{N-1} a_n \delta(\omega - \omega_0 - n\Delta\omega)$$

$$V(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{V}(\omega) e^{i\omega t} d\omega = \frac{e^{i\omega_0 t}}{2\pi}\sum_{n=0}^{N-1} a_n z^n, \qquad z = e^{i\Delta\omega t}$$

$$V\left(z = e^{i\Delta\omega t}\right) = \frac{a_{N-1} e^{i\omega_0 t}}{2\pi}\prod_{n=1}^{N-1}(z - z_n)$$

# Applications of superoscillations



# Questions

- What happens when random sampling and superoscillations are used together?
- Will random sampling reconstruct an arbitrary superoscillation?

# References


- A. Kempf; *Black holes, bandwidths and Beethoven,* J. Math. Phys. 41, 2360 (2000)
- P. Ferreira and A. Kempf; *The energy expense of superoscillations,* Proceedings of EUSIPCO-2002, XI European Signal Processing Conference, Toulouse, France, Sep. 2002
- P. Ferreira; *Superoscillations: Faster Than the Nyquist Rate*, IEEE Transactions on Signal Processing 54 (2006) 3732.
- A. M. H. Wong and G. V. Eleftheriades; *Superoscillatory Radar Imaging: Improving Radar Range Resolution Beyond Fundamental Bandwidth Limitations*, IEEE Microwave and Wireless Components Letters 22 (2012) 147.