Networks and hypernetworks 1

Topology, dynamics and applications

Rui Vilela Mendes

http://label2.ist.utl.pt/vilela/

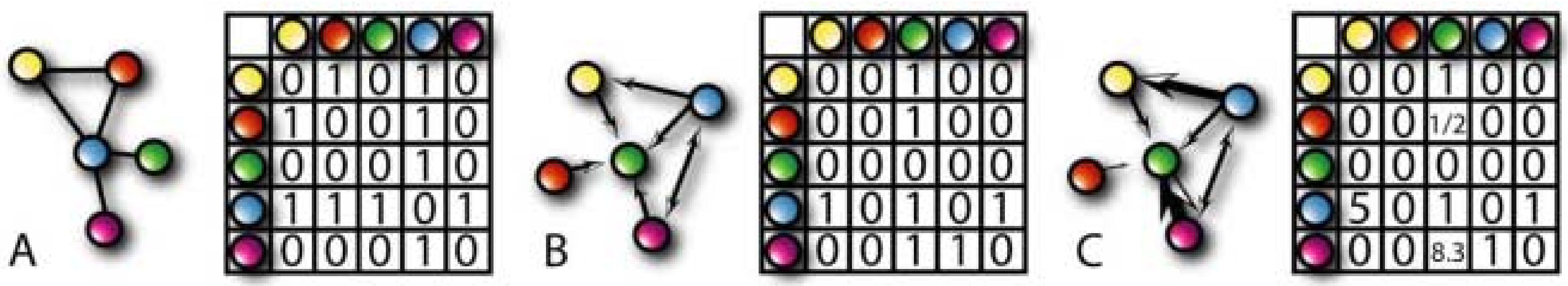
Network basic elements

- Nodes
- Links (directed, undirected, weighted, non-weighted)
- Simple or bipartite
- Adjacency matrix
- Networks are graphs

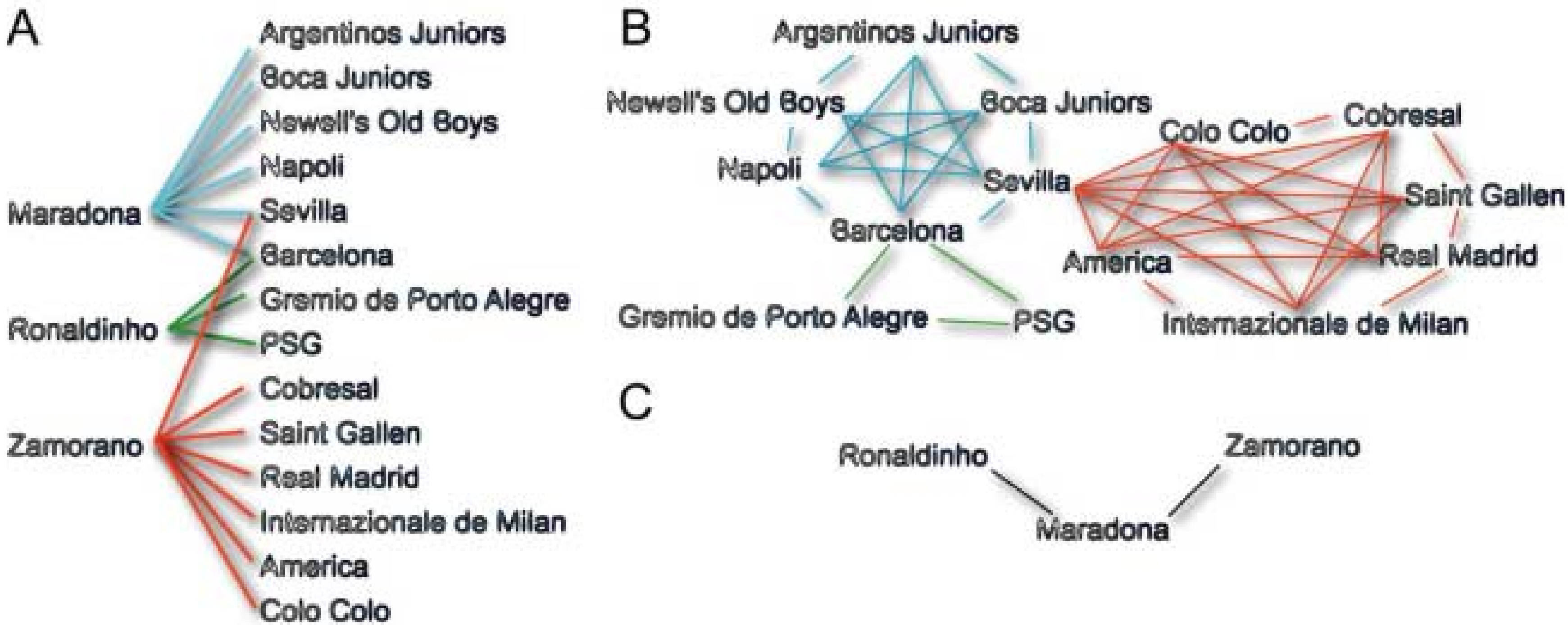
- Simple graph Symmetric

Directed

Directed Weighted



A bi-partite graph and their derived simple graphs

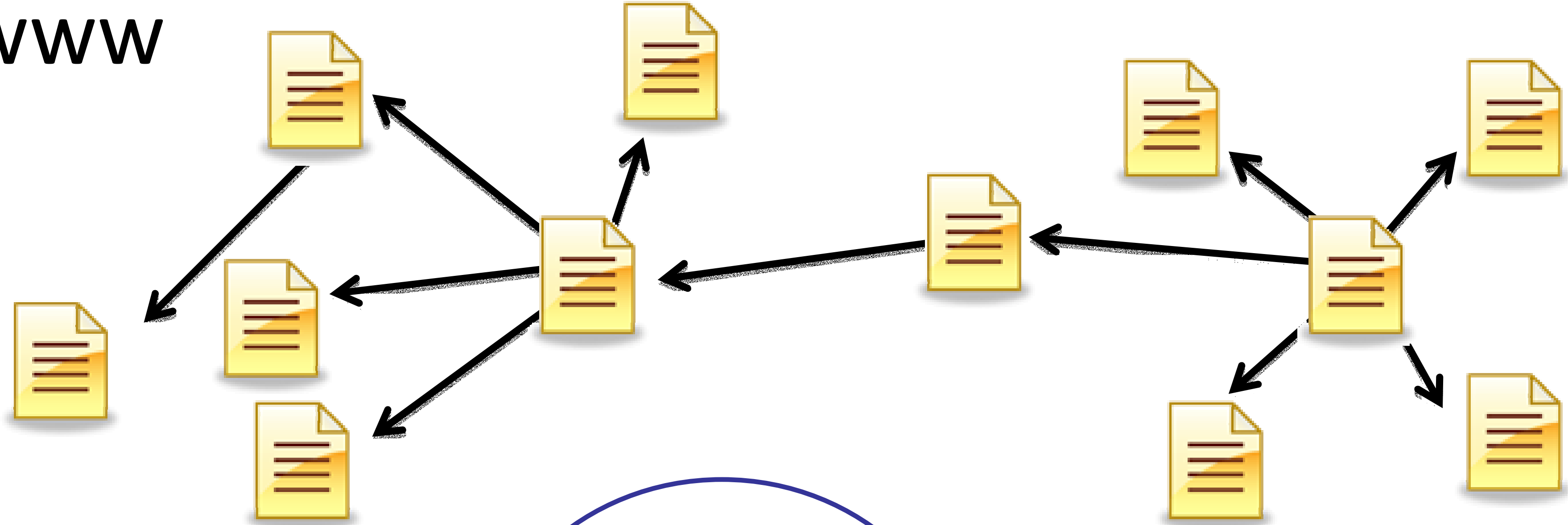


Where does one find networks?

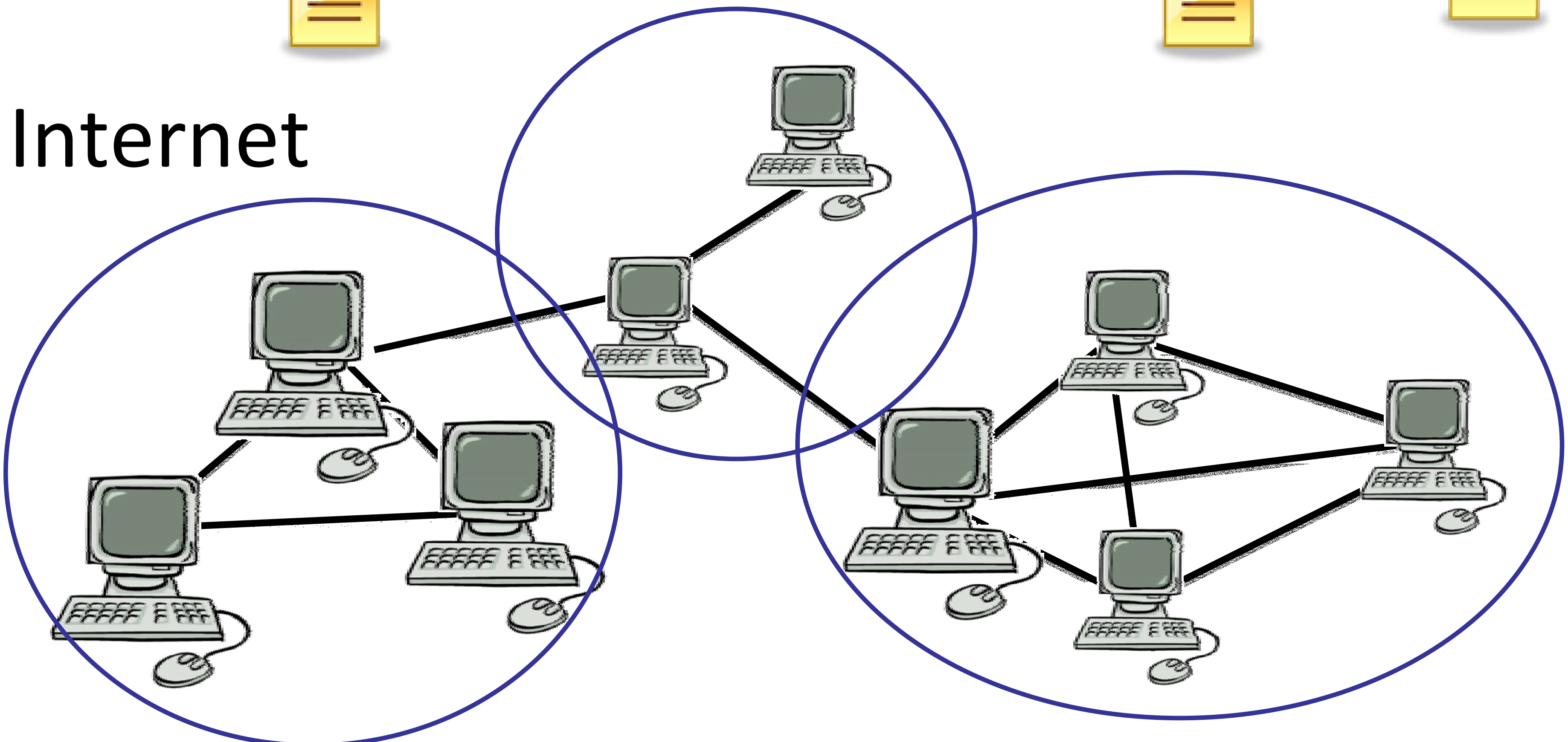
Where does one find networks?

- EVERYWHERE

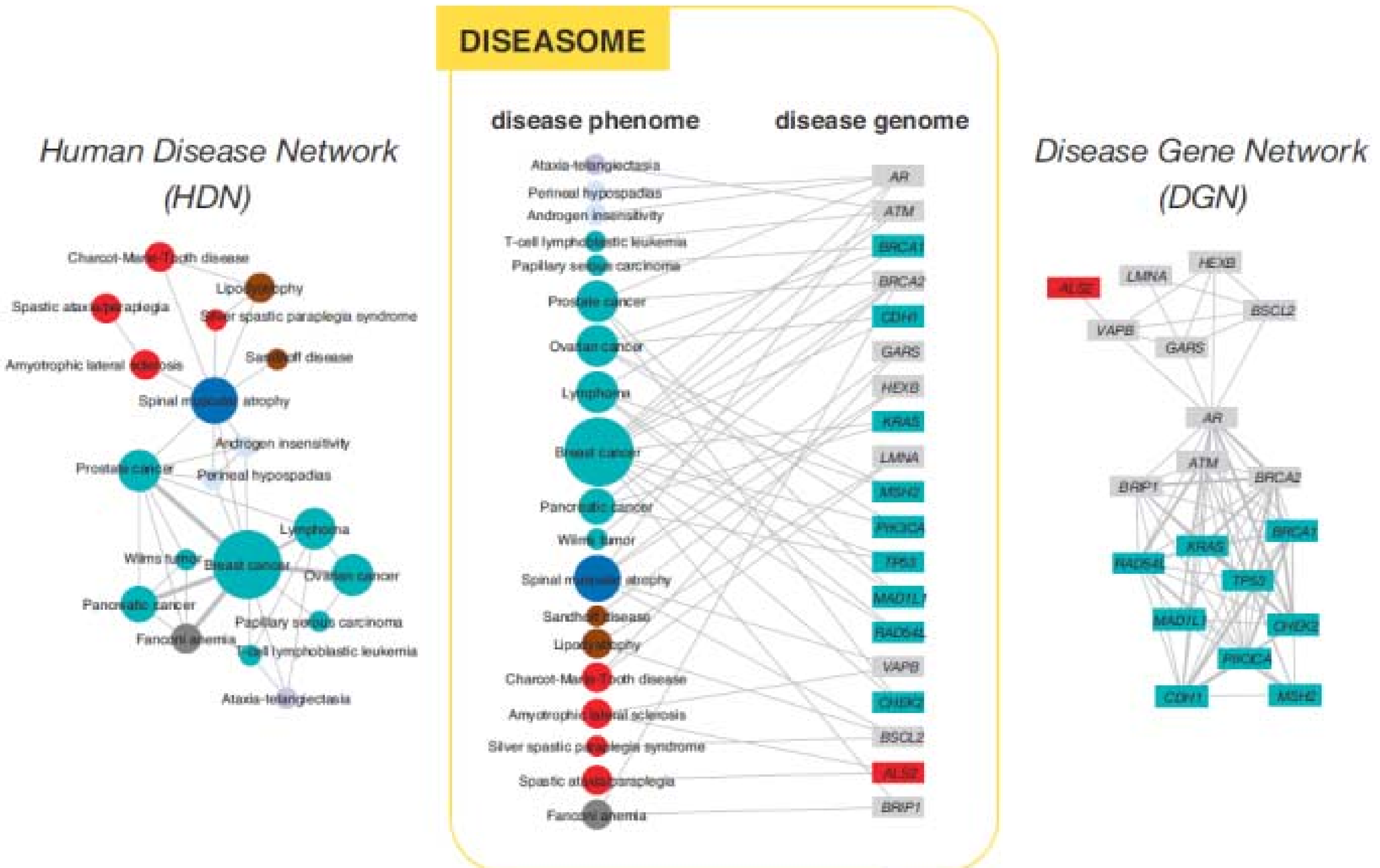
WWW

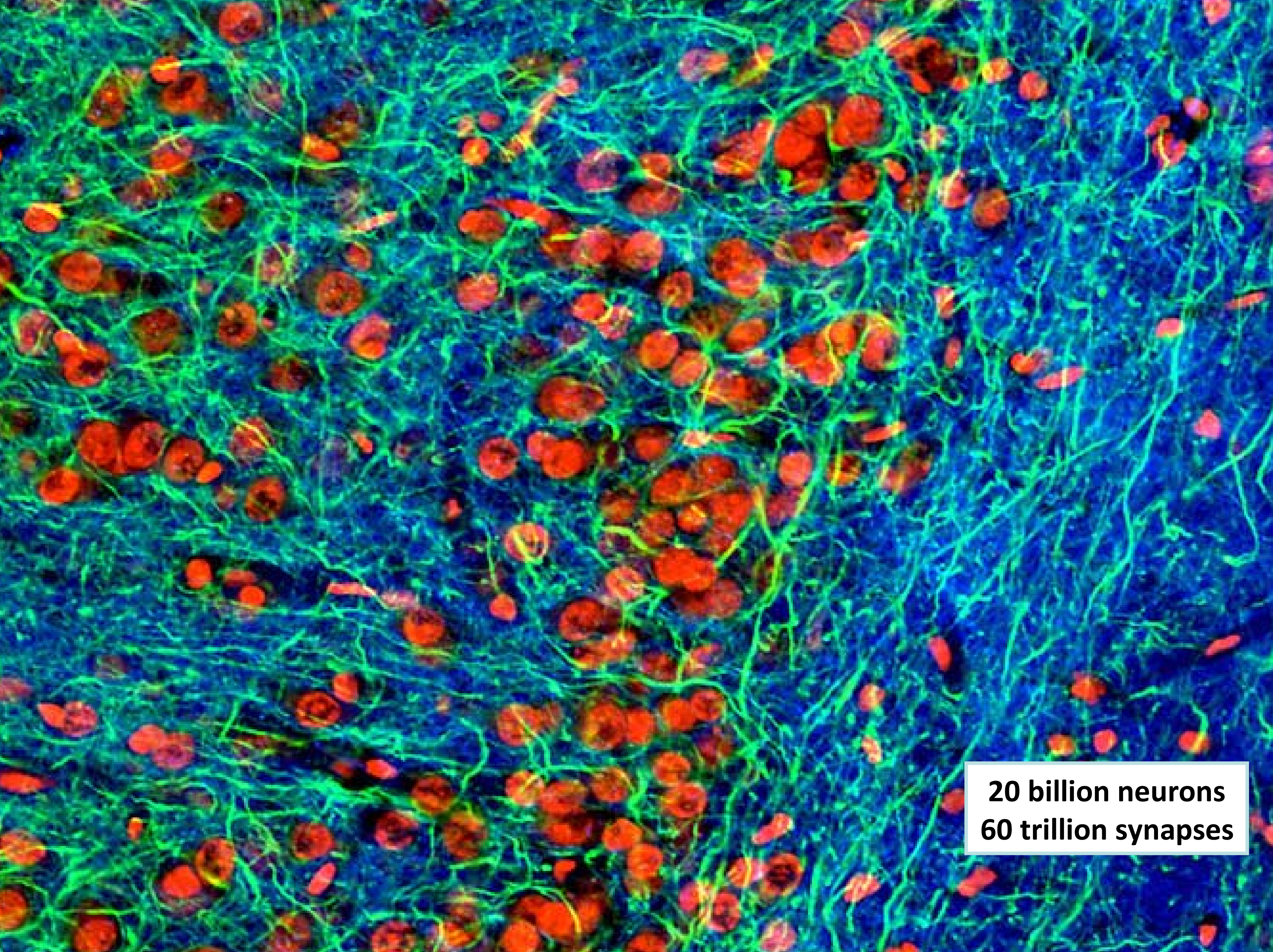


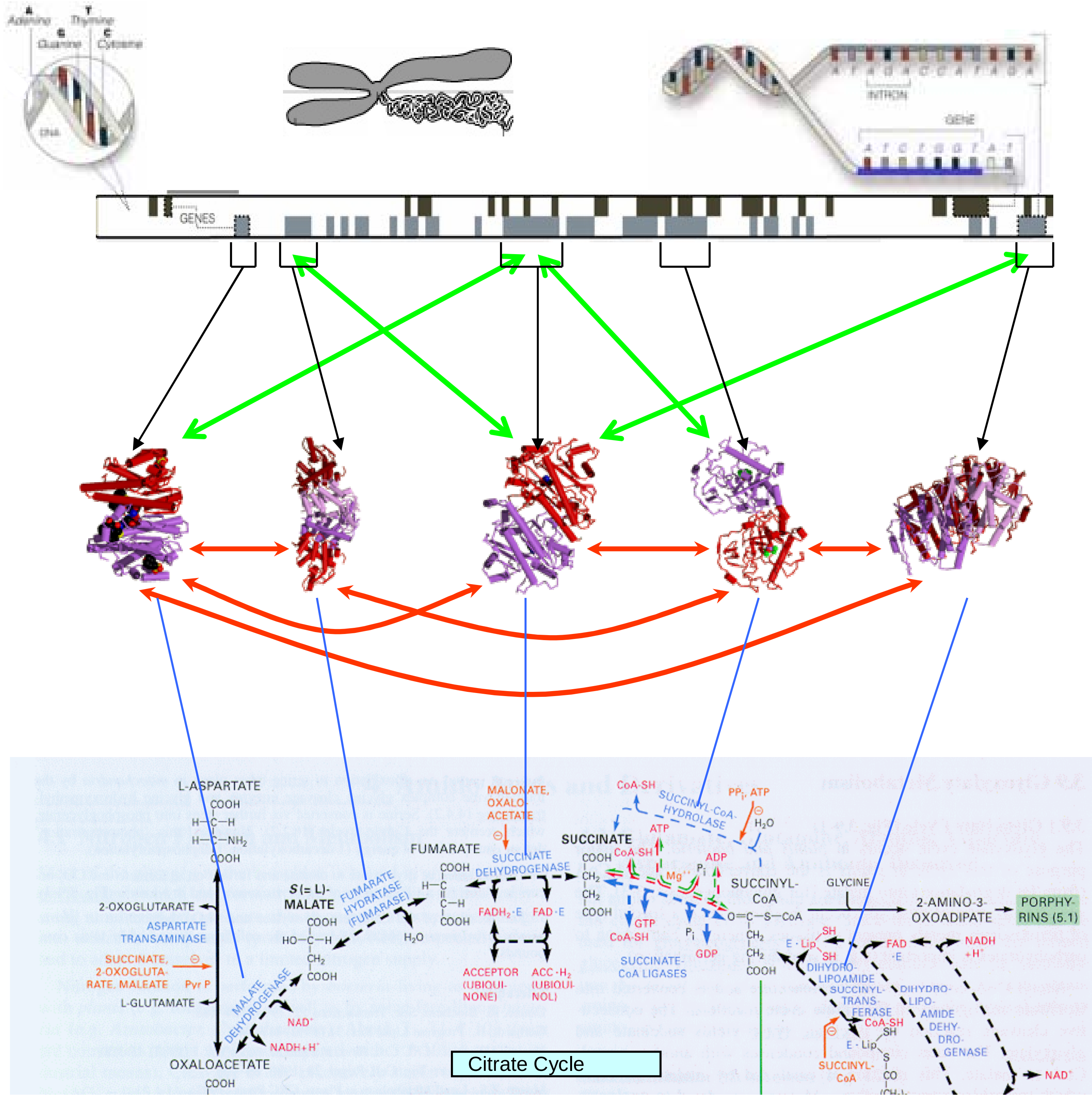
Internet



The disease network




20 billion neurons
60 trillion synapses



GENOME

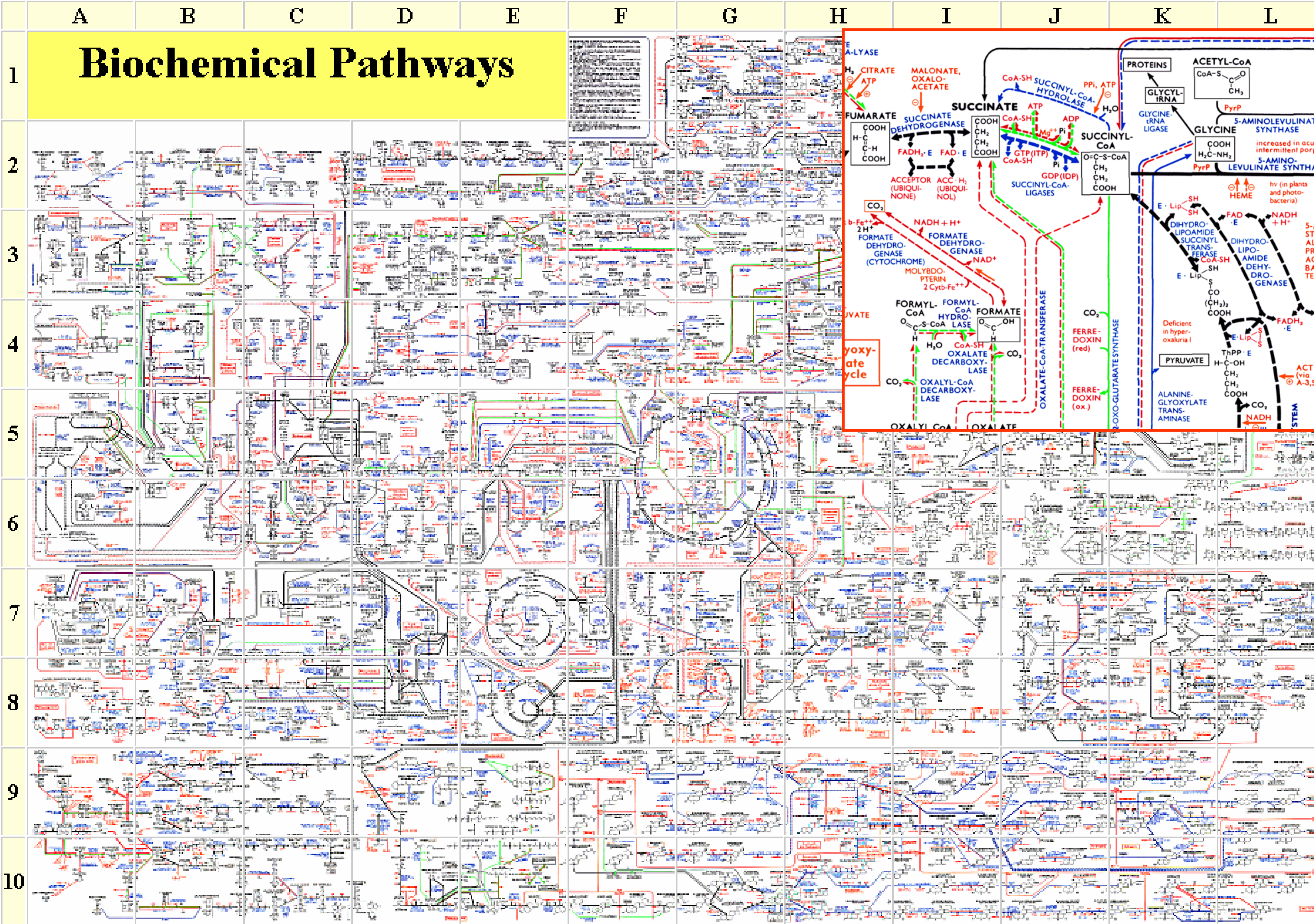
protein-gene interactions

PROTEOME

protein-protein interactions

METABOLISM

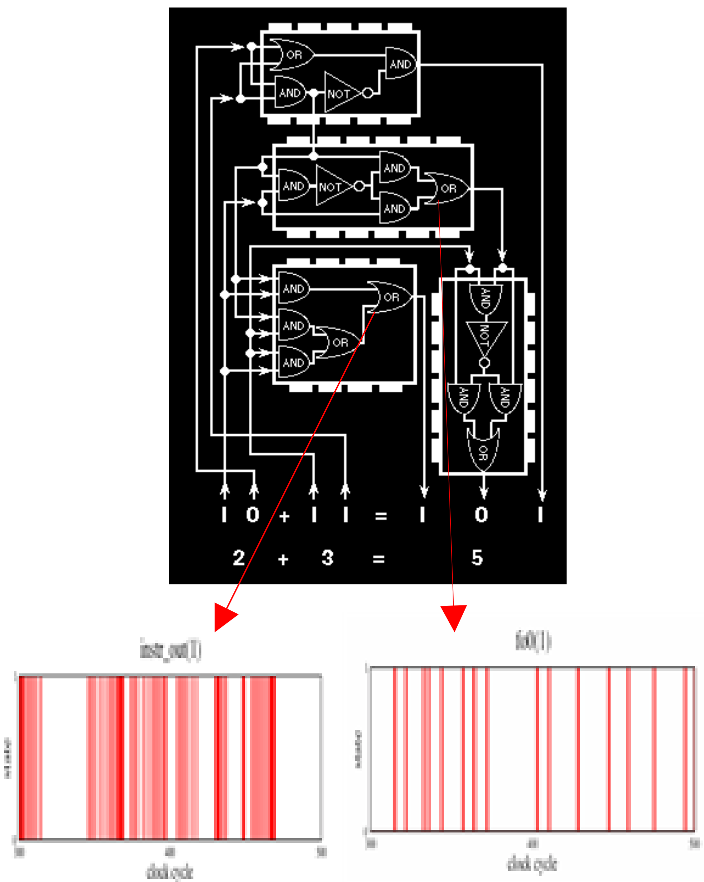
Bio-chemical reactions



Biochemical Pathways
A
B
C
D
E
F
G
H
I
J
K
L
1
2
3
4
5
6
7
8
9
10
SUCCINATE
FUMARATE
SUCCINATE DEHYDROGENASE
SUCCINYL-CoA
SUCCINYL-CoA HYDROLASE
SUCCINYL-CoA-LIGASES
PROTEINS
ACETYL-CoA
GLYCYL-tRNA
GLYCINE
5-AMINOLEVULINATE SYNTHASE
FORMATE
FORMATE DEHYDROGENASE
FORMYL-CoA
OXALATE DECARBOXYLASE
OXALYL-CoA DECARBOXYLASE
PYRUVATE
ALANINE-GLYOXYLATE TRANSAMINASE
HEME

Computer chip

•$f_i(t)$ =state of a given logic component *i* at clock cycle *t*.

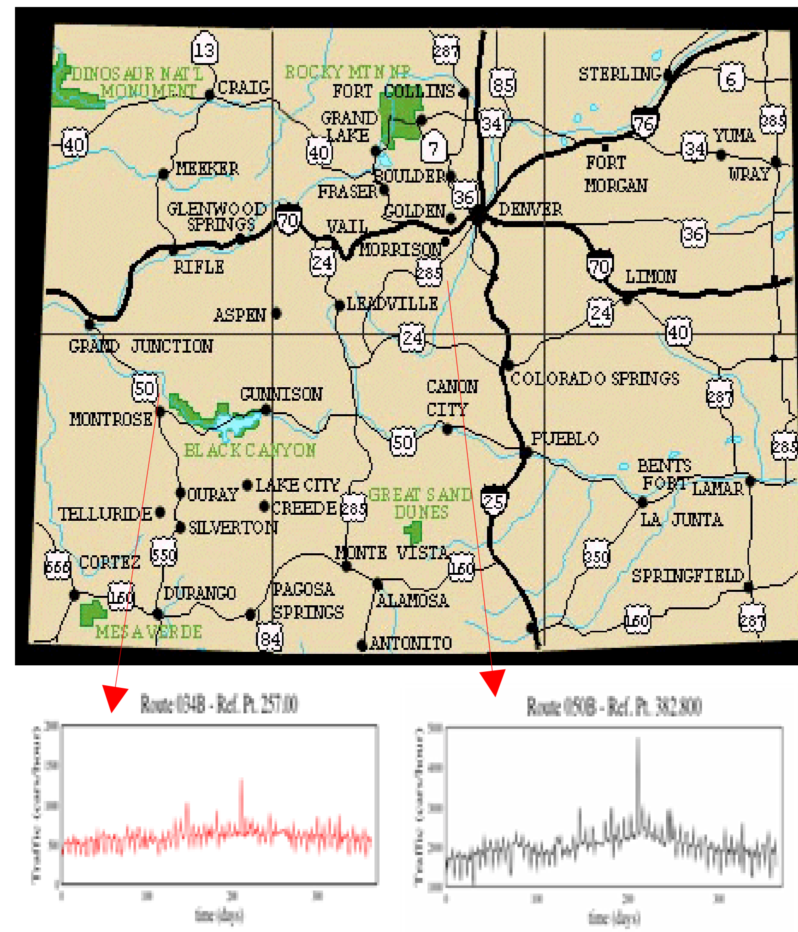


• 462 signal carriers

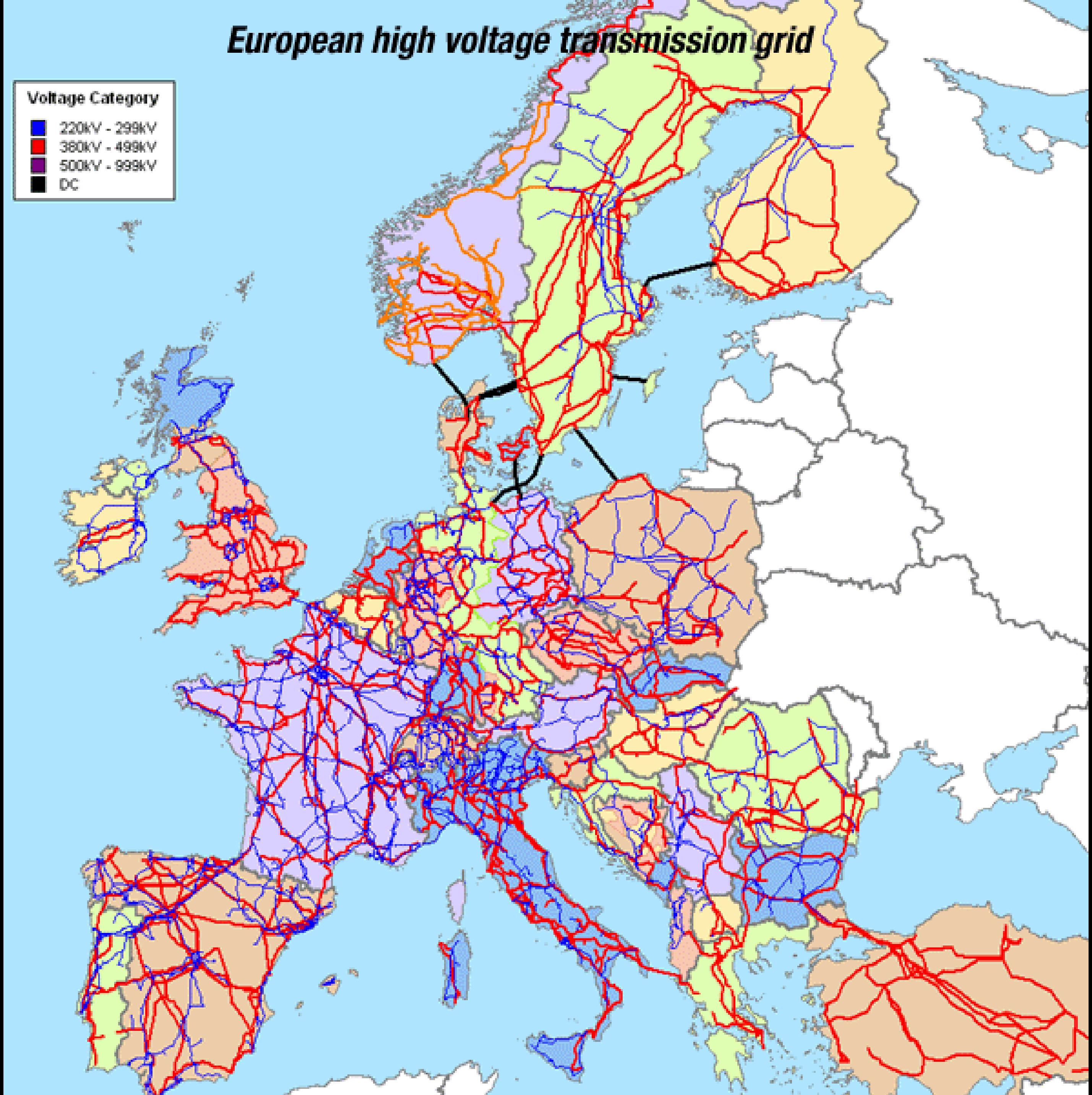
• 8,862 clock cycles.

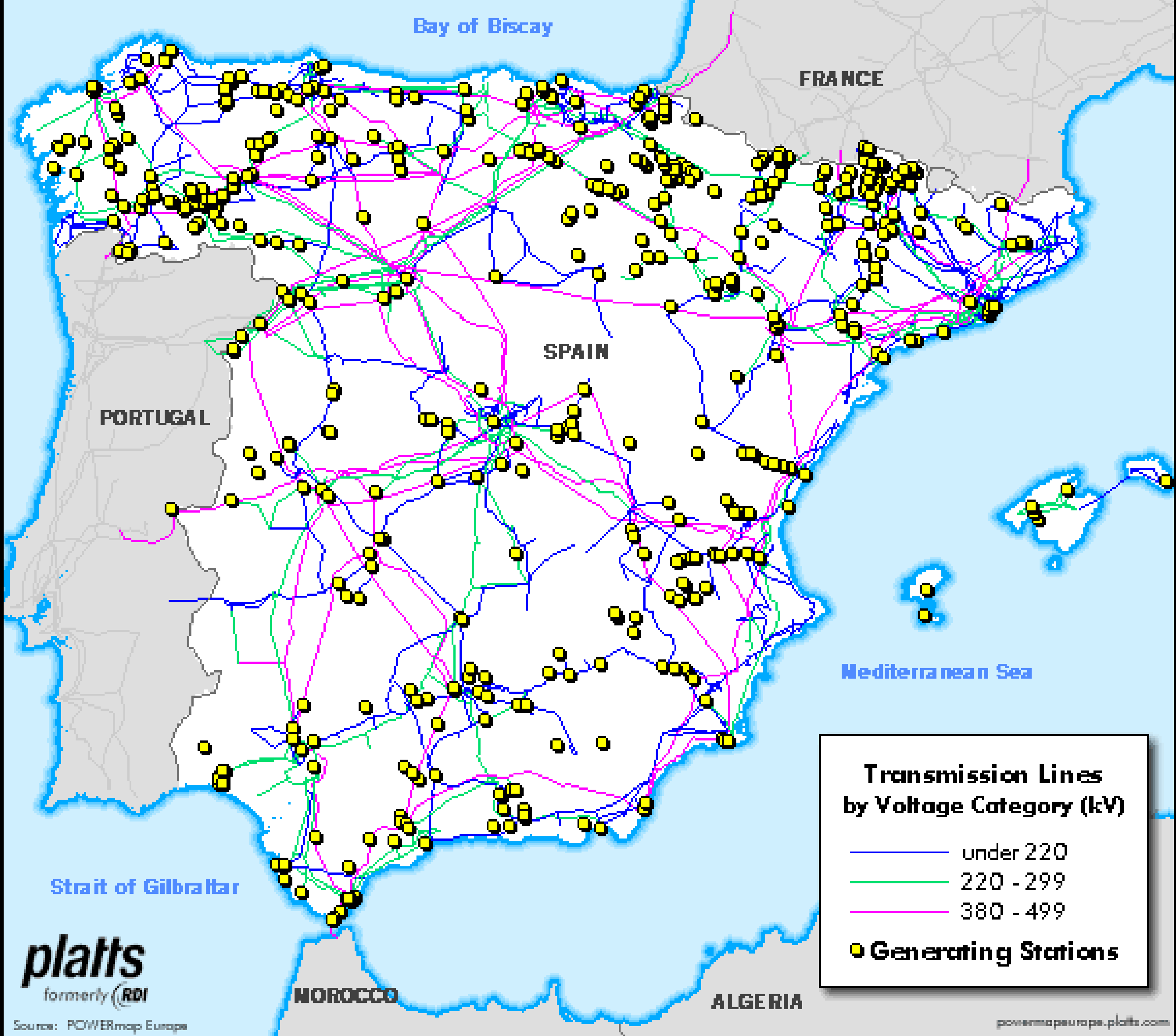
Highways

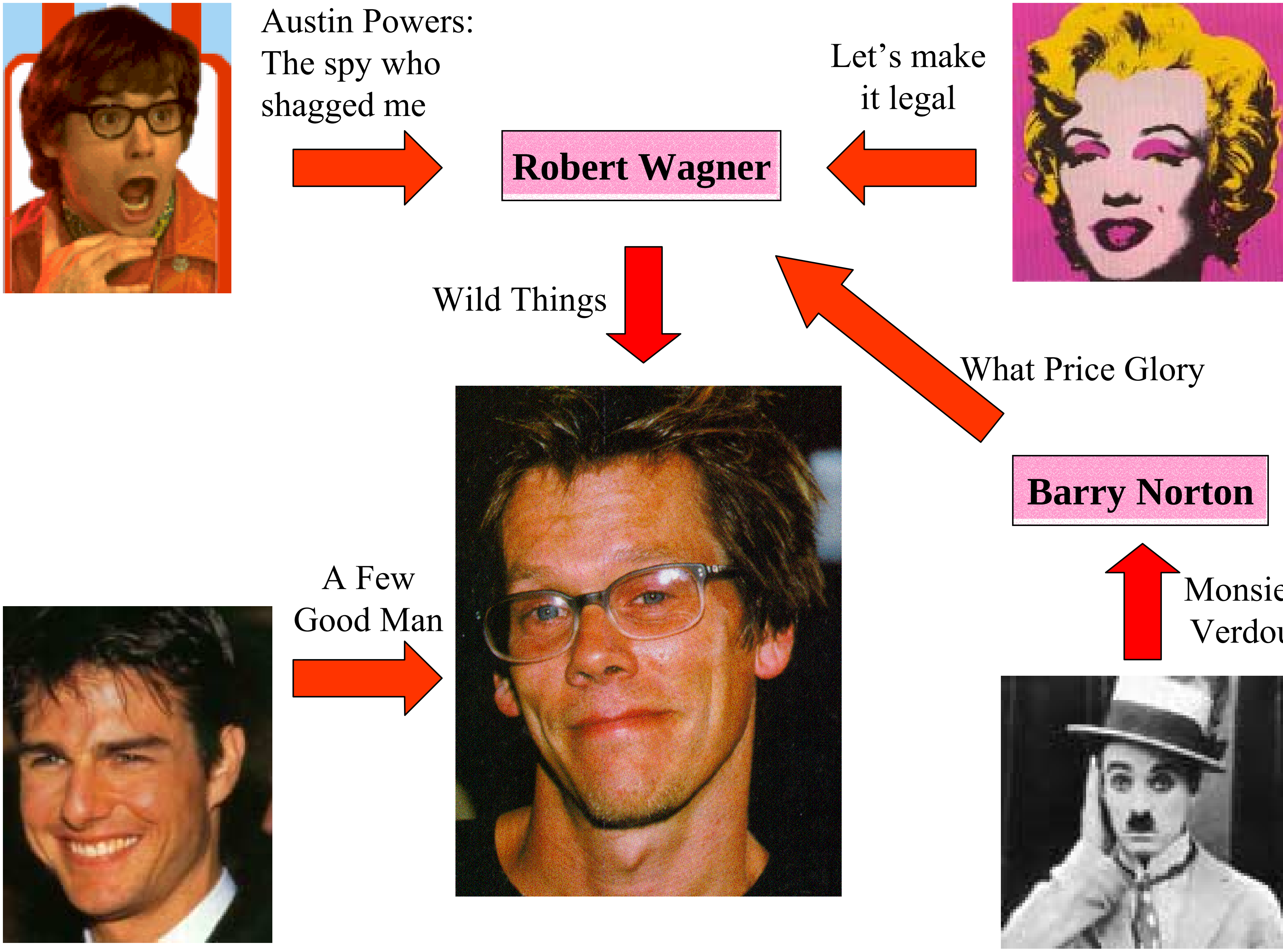
•$f_i(t)$ =traffic at a given point of a road *i* at day *t*.



•Daily traffic on 127 Colorado roads from 1998 to 2001.


European high voltage transmission grid
Voltage Category
220kV - 299kV
380kV - 499kV
500kV - 999kV
DC


Bay of Biscay
FRANCE
SPAIN
PORTUGAL
Mediterranean Sea
Strait of Gilbraltar
MOROCCO
ALGERIA
Transmission Lines
by Voltage Category (kV)
under 220
220 - 299
380 - 499
Generating Stations
platts
formerly RDI
Source: POWERmap Europe
powermapeurope.platts.com


Austin Powers: The spy who shagged me
Robert Wagner
Let's make it legal
Wild Things
What Price Glory
Barry Norton
A Few Good Man
Monsieur Verdoux

Characterizing networks: Topological parameters

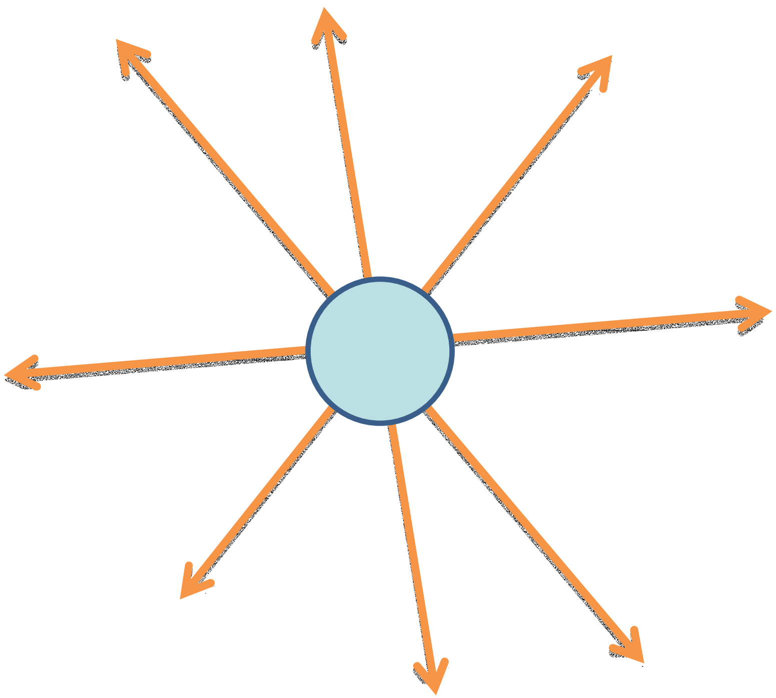
- Degree
- Path length
- Clustering
- Centrality
- Betweenness
- Eigenvalue centrality
- Motifs

Local and global measures

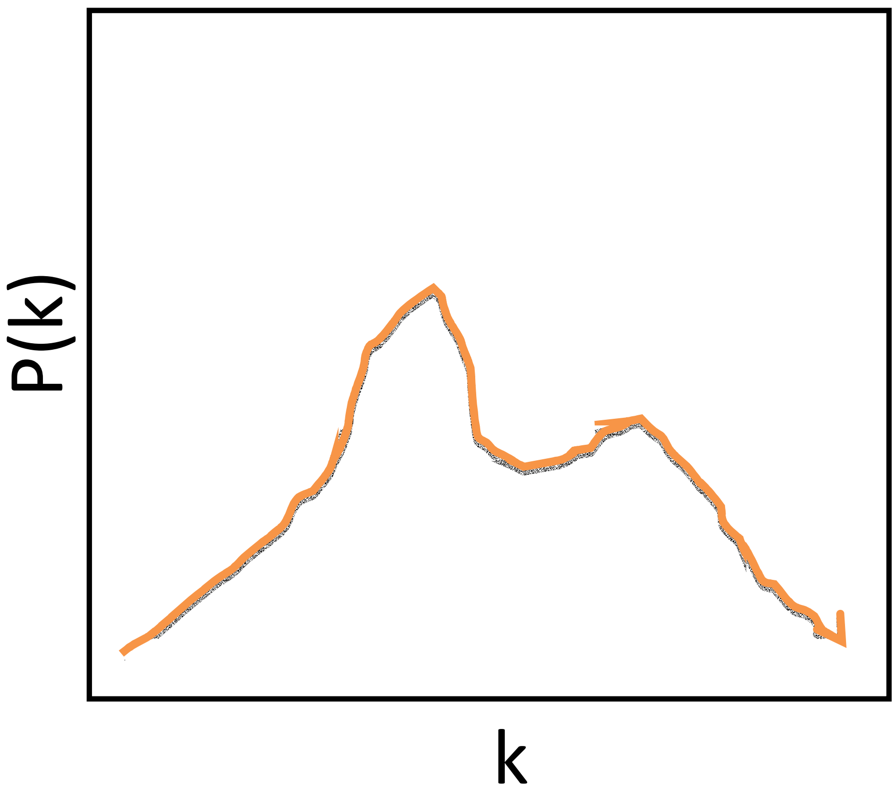
Local Measures
Centrality measures (degree, closeness, betweenness, eigenvector, page-rank)
Clustering measures (Clustering, Topological Overlap or Mutual Clustering)
Motifs

Global Measures
Degree Correlations, Correlation Profile.
Hierarchical Structure
Fractal Structure
Communities
Modularity

Degree (*k*)

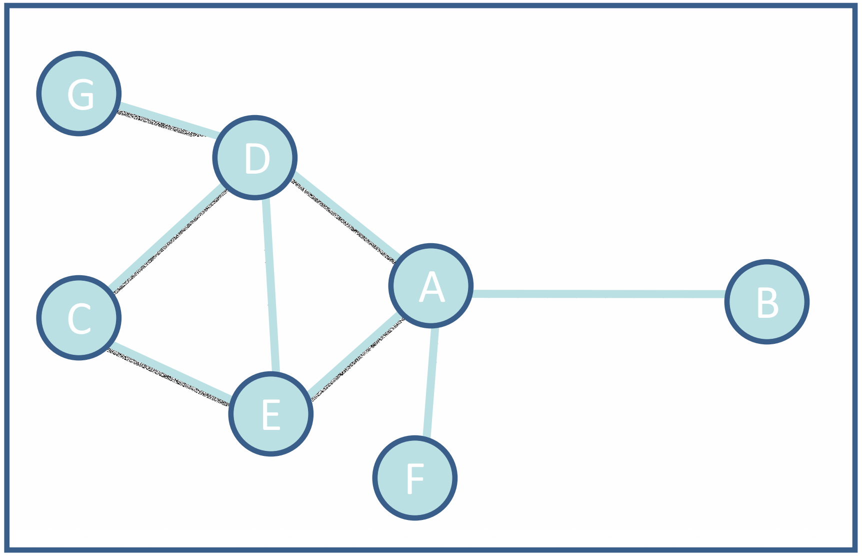


Degree Distribution



Average Path Length

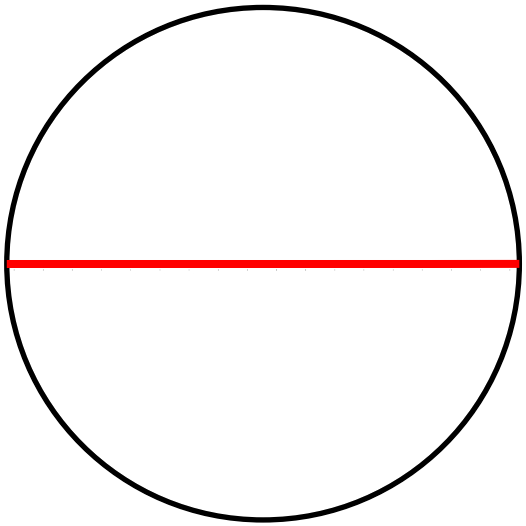
	A	B	C	D	E	F	G
A		1	2	1	1	1	2
B			3	2	2	2	3
C				1	1	3	2
D					1	2	1
E						2	2
F							3
G							

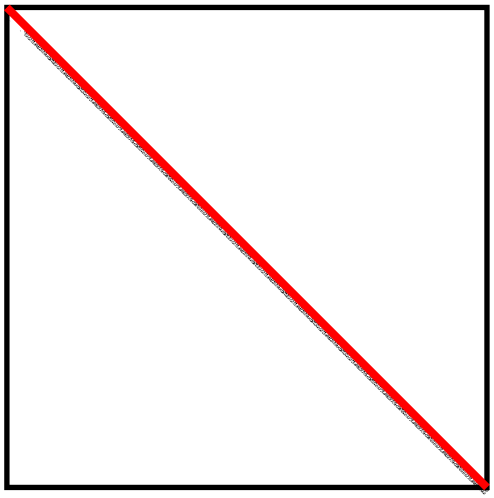


D(1)=8
D(2)=9
D(3)=4

L=(8+2x9+3x4)/(8+9+4) L=1.8

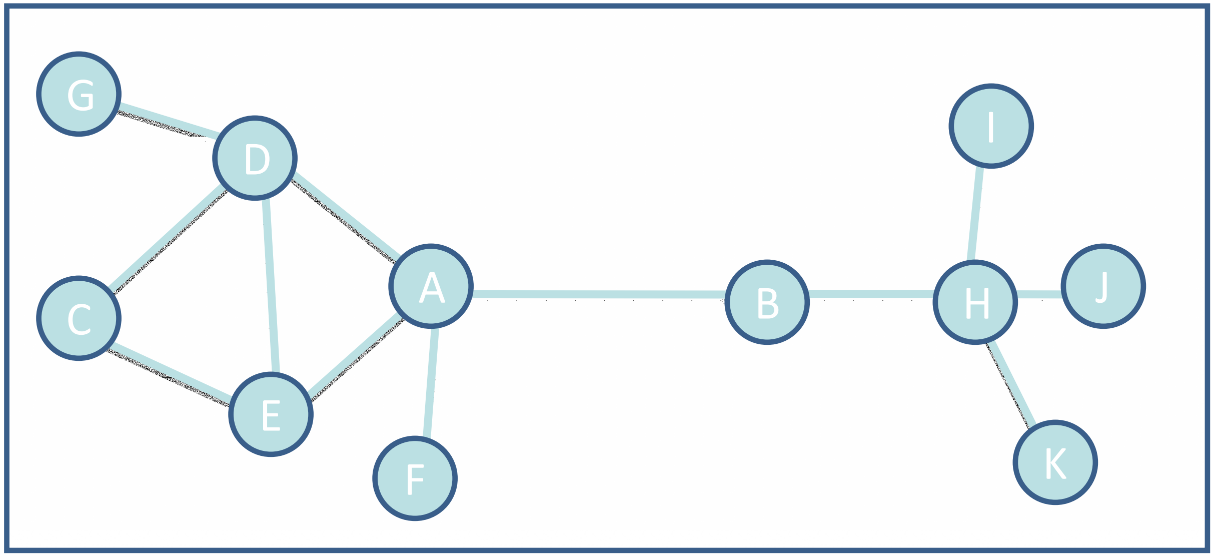
Diameter





Diameter=Maximum Distance Between Elements in a Set

Diameter=D(G,J)=D(C,J)=D(G,I)=…=5



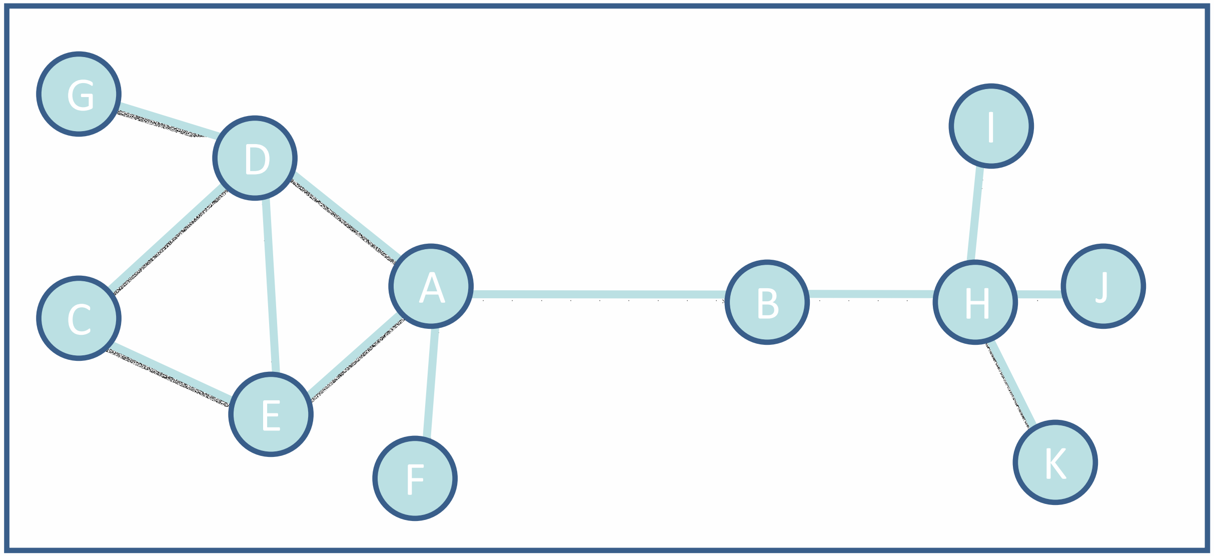
Clustering Coefficient

$$C_i=2\Delta/k(k-1)$$

$C_A=2/12=1/6$

$C_C=2/2=1$

$C_E=4/6=2/3$



Degree centrality C(v)

- N=number of nodes
- *For a node:*
  $C(v_i)=degree(v_i)/(N-1)$
- *For a graph:*
  $C(G)=sum_i\ (C(v^*)-C(v_i))/(N-2)$
- v* is the node with maximum C(v)
- N-2 is the maximum value of the sum (a star graph)

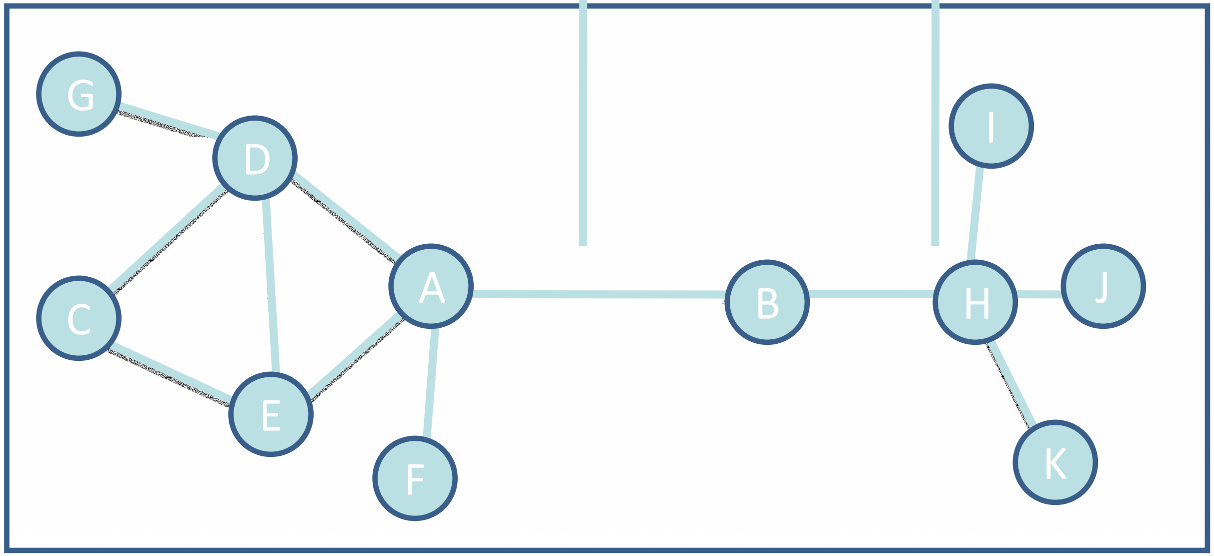
BETWEENNESS CENTRALITY

BC= number of shortest Paths that go through a node.

BC(G)=0

BC(A)=5*5+4=29

BC(B)=4*6=24



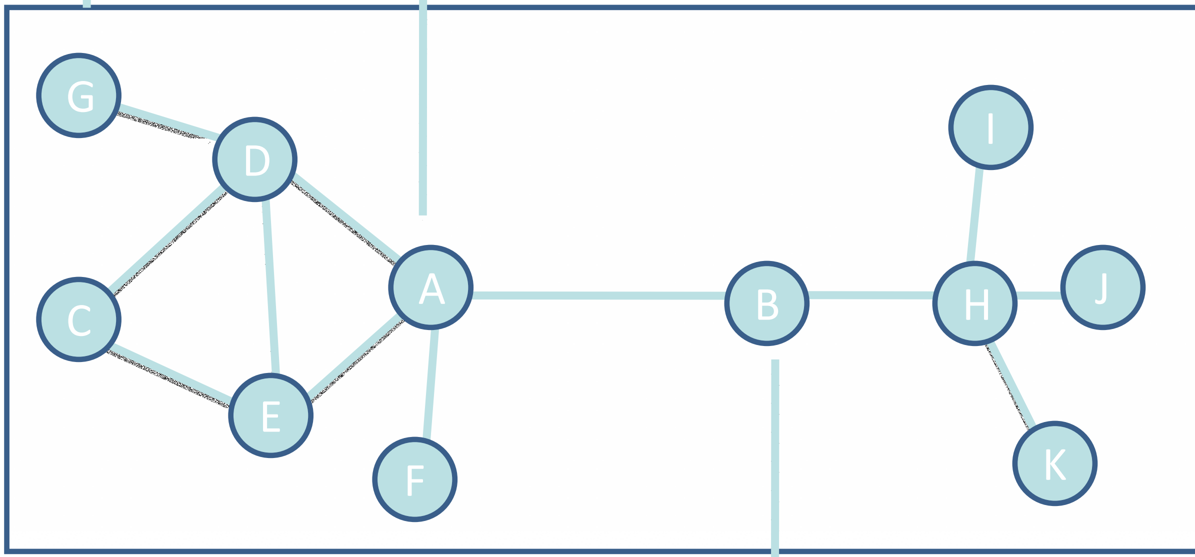
N=11

CLOSENESS CENTRALITY

C= Average Distance to neighbors

C(G)=1/10(1+2*3+2*3+4+3*5)
C(G)=3.2

C(A)=1/10(4+2*3+3*3)
C(A)=1.9



C(B)=1/10(2+2*6+2*3)
C(B)=2

N=11

EIGENVECTOR CENTRALITY

Consider the Adjacency Matrix A_{ij} = 1 if node i is connected to node j and 0 otherwise.

Consider the eigenvalue problem:

$$A\mathbf{x}=\lambda\mathbf{x}$$

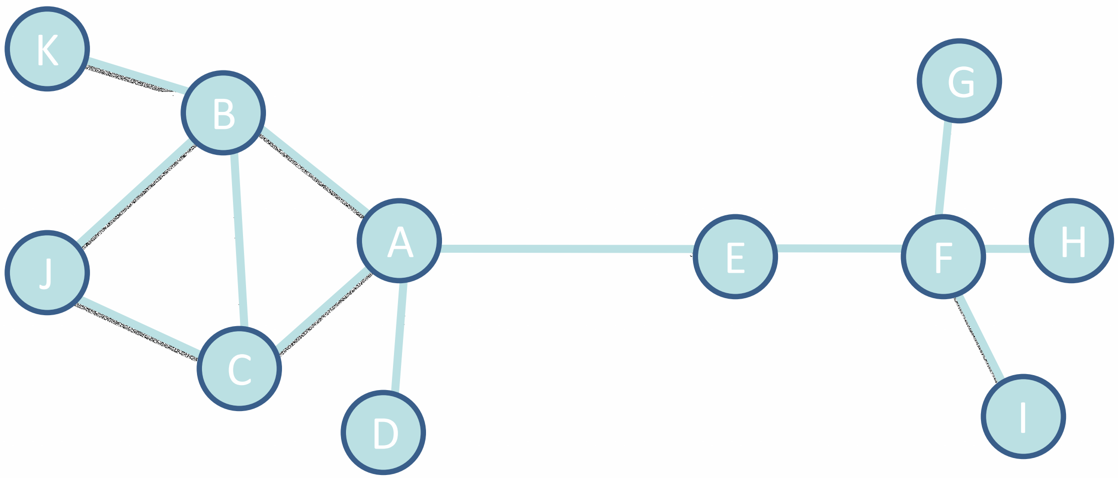
Then the eigenvector centrality of a node is defined as:

$$x_i = \frac{1}{\lambda} \sum_{j \in M(i)} x_j = \frac{1}{\lambda} \sum_{j=1}^{N} A_{i,j} x_j$$

where λ is the largest eigenvalue associated with A.

PAGE RANK

PR=Probability that a random walker with interspersed Jumps would visit that node.
PR=Each page votes for its neighbors.



PR(A)=PR(B)/4 + PR(C)/3 + PR(D)+PR(E)/2

A random surfer eventually stops clicking

$PR(X)=(1-d)/N + d(\sum PR(y)/k(y))$

PAGE RANK

PR=Probability that a random Walker would visit that node.
PR=Each page votes for its neighbors.

$$\mathbf{R} = \begin{bmatrix} PR(p_1) \\ PR(p_2) \\ \vdots \\ PR(p_N) \end{bmatrix}$$

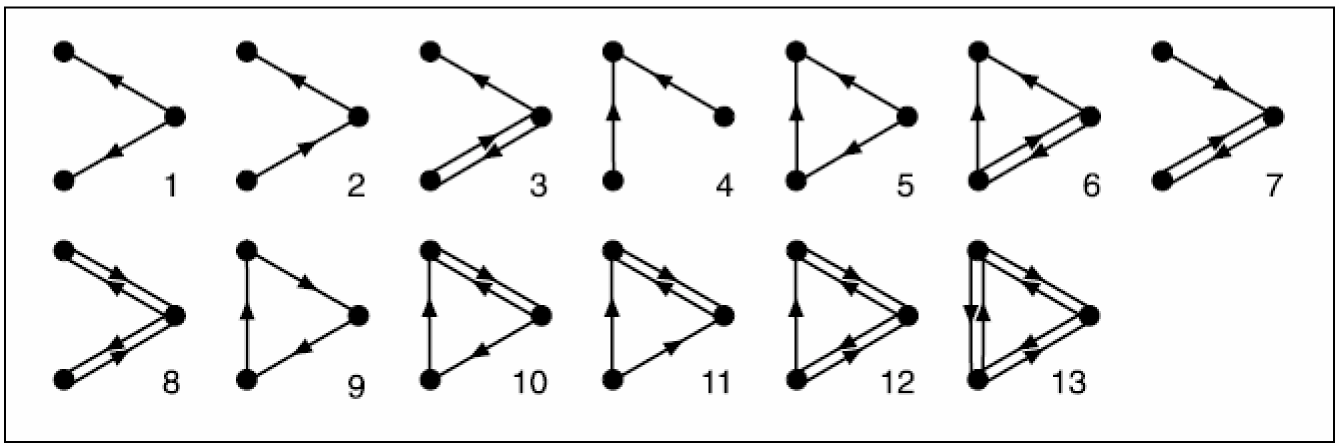
$$\mathbf{R} = \begin{bmatrix} (1-d)/N \\ (1-d)/N \\ \vdots \\ (1-d)/N \end{bmatrix} + d \begin{bmatrix} \ell(p_1,p_1) & \ell(p_1,p_2) & \cdots & \ell(p_1,p_N) \\ \ell(p_2,p_1) & \ddots & & \vdots \\ \vdots & & \ell(p_i,p_j) & \\ \ell(p_N,p_1) & \cdots & & \ell(p_N,p_N) \end{bmatrix} \mathbf{R}$$

$$\sum_{i=1}^{N} \ell(p_i,p_j) = 1,$$

Network Motifs: Simple Building Blocks of Complex Networks

R. Milo,[1] S. Shen-Orr,[1] S. Itzkovitz,[1] N. Kashtan,[1] D. Chklovskii,[2]
U. Alon[1]*

Motifs



Networks are graphs

- The first well-studied graphs:
- Random graphs

Erdos-Renyi Model (1959)

Random Graph Theory

Original Formulation:

N nodes, *n* links chosen randomly from the *N(N-1)/2* possible links.

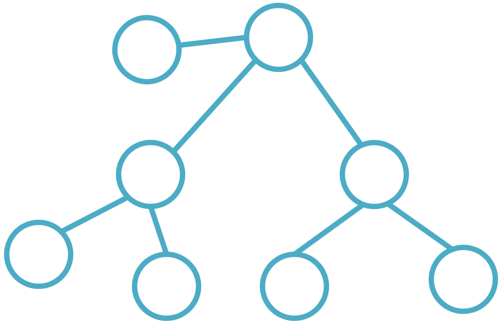
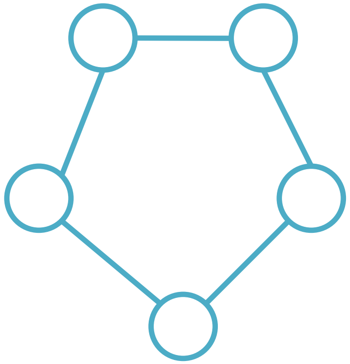
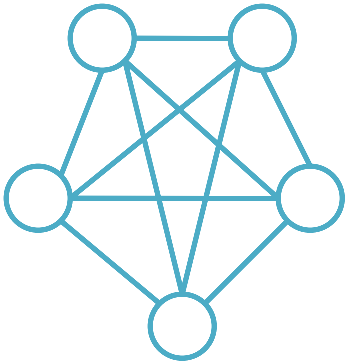
Alternative Formulation:

N nodes. Each pair is connected with probability *p*.
Average number of links *=p(N(N-1))/2;*

Random Graph Theory Works on the limit N->∞ and studies when properties on a graph emerge as a function of *p*.

Random Graph Theory: Erdos-Renyi (1959)

Subgraphs

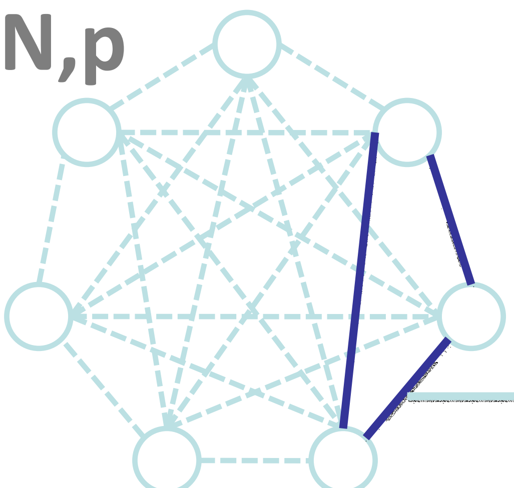
	Trees	**Cycles**	**Cliques**
			
			
Nodes:	k	k	k
Links:	k-1	k	k(k-1)/2

Random Graph Theory: Erdos-Renyi (1959), Bollobas (1985)

Among N nodes choose k

Which in the large N goes like

$G_{N,p}$



$F_{(k,l)}$

$$E = C^N_k \, p^l \, \frac{k!}{a} \cong N^k p^l / a$$

Each link occurs with Probability p

We can permute the nodes we choose in $k!$ ways, but have to remember not to double count isomorphisms (a)

Random Graph Theory: Erdos-Renyi (1959), Bollobas (1985)

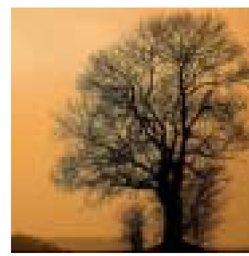
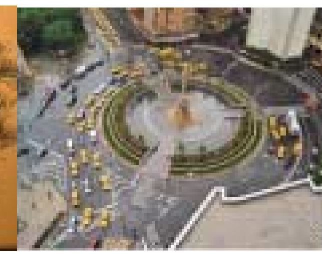
$$E \cong N^k \, p^l / a$$

One expects the number of subgraphs to be finite if:

$$p(N) \sim cN^{-k/l}$$

Which implies a number of subgraphs:

$$E = c^l / a = \lambda$$

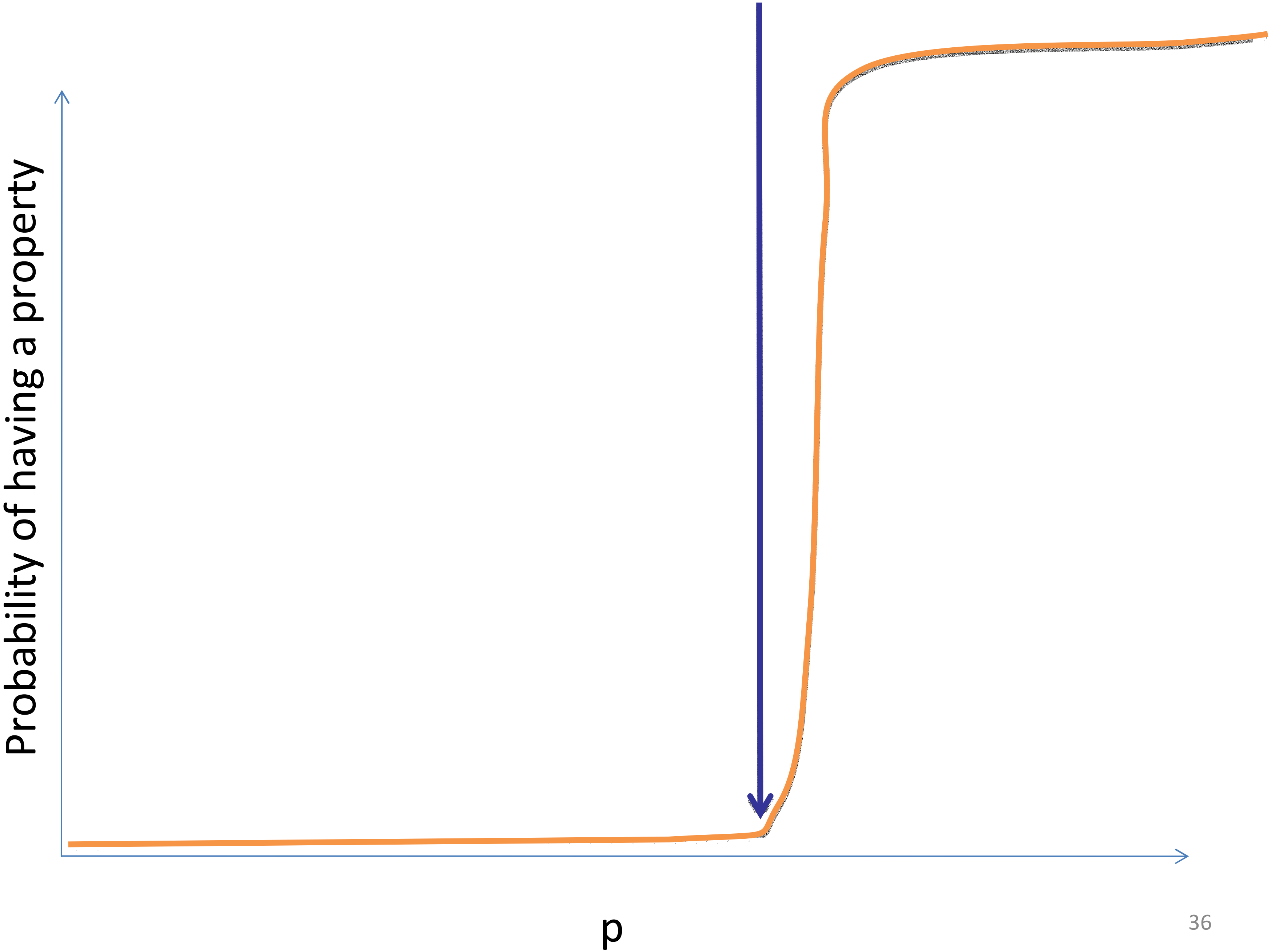




(a) The critical probability of having a tree of order k is $p_c(N) = cN^{-k/(k-1)}$;

(b) The critical probability of having a cycle of order k is $p_c(N) = cN^{-1}$;

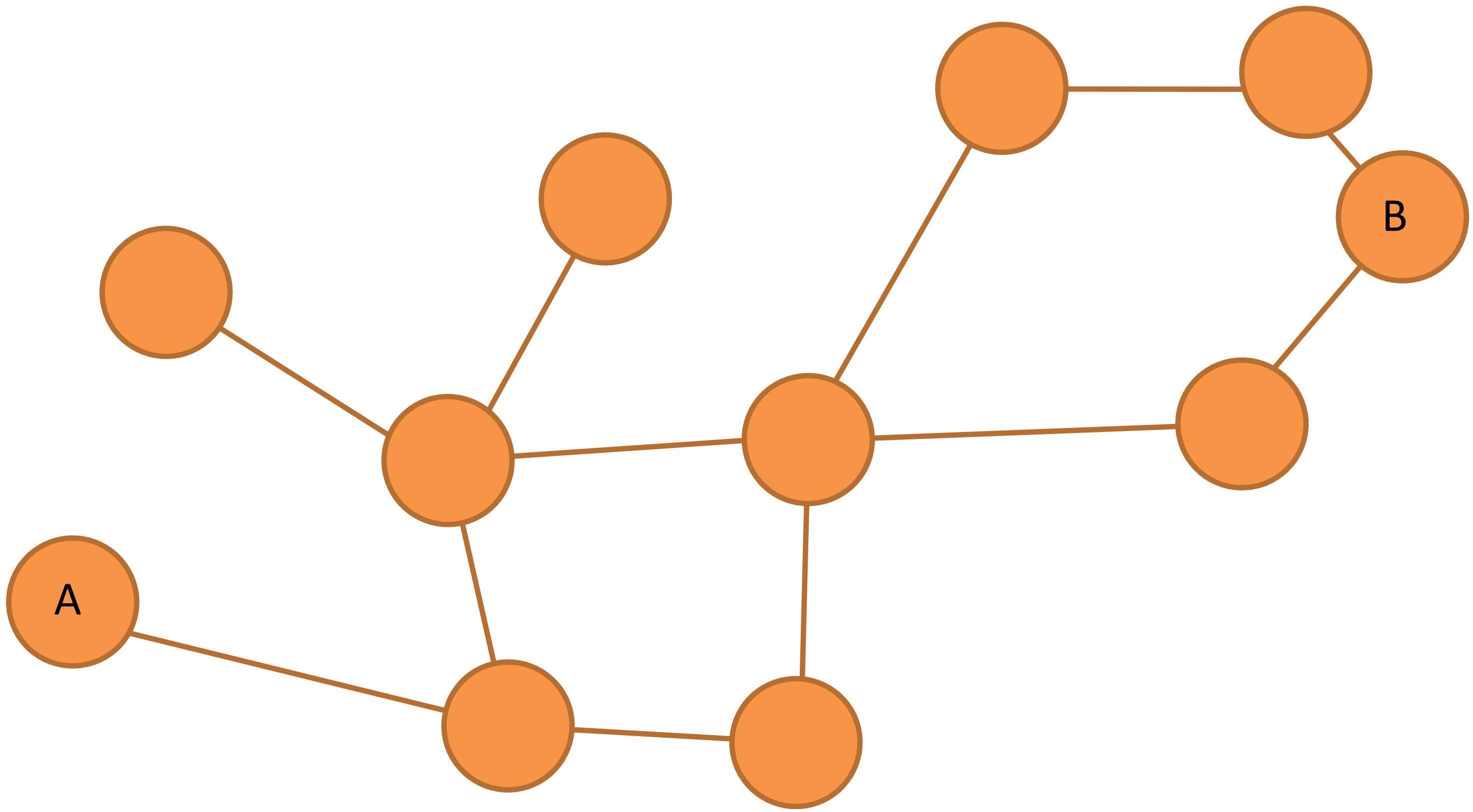
(c) The critical probability of having a complete subgraph of order k is $p_c(N) = cN^{-2/(k-1)}$.

Subgraphs appear suddenly (percolation threshold)

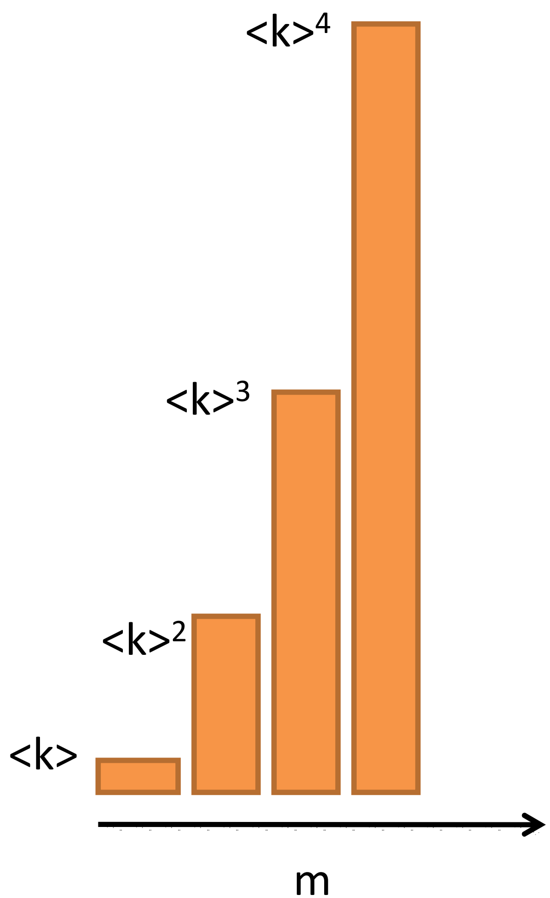


- Average degree <k>=p(N-1)~=pN
- Let $p(N) \sim N^z$
- For z<-3/2 almost all graphs contain only isolated nodes and edges
- When z passes -3/2, trees of order 3 appear
- When z reaches -4/3, trees of order 4 appear
- As z approaches -1 ,trees of larger and larger order
- But as long as z<-1, <k>~=pN~0 for large N, the graph is still an union of disjoint trees.
- At z=-1 a giant component appears
  A connected graph although <k>~=1

Distance Between A and B?



Number of nodes at distance **m** from a randomly chosen node



Hence the average path length is

$$N \sim <k>^{<l>}$$

$$\log(N)/\log(<k>) \sim <l>$$

- **Degree distribution**

$$P(k_i = k) = C_k^{N-1} p^k (1-p)^{N-1-k}$$

- X_k = Number of nodes with degree k

$$E(X_k) = NP(k_i = k) = NC_k^{N-1} p^k (1-p)^{N-1-k}$$

- Approaches a Poisson distribution

$$P(X_k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

- **Clustering coefficient**

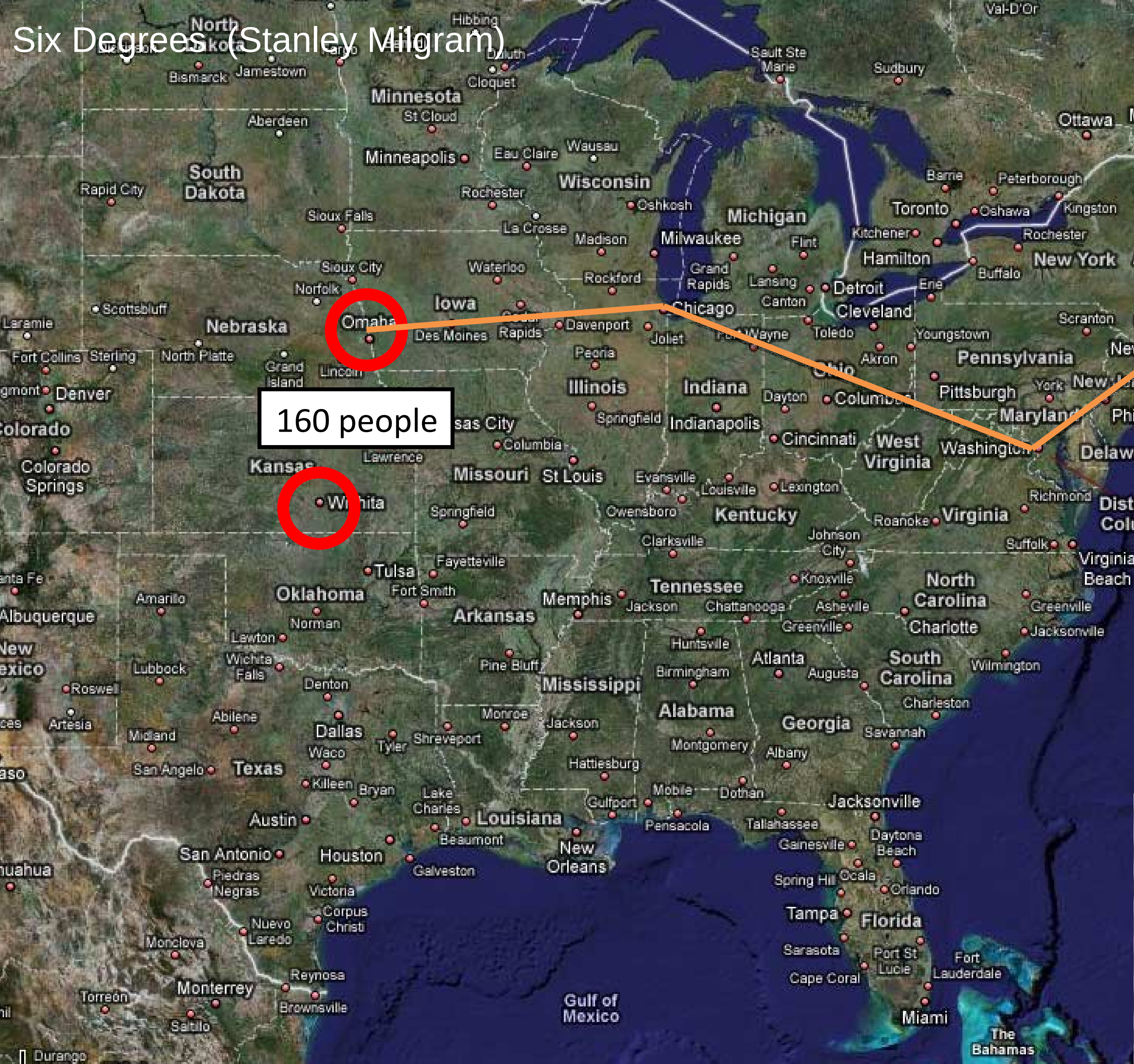
$$C_{rand} = p = \frac{\langle k \rangle}{N}$$

Are most naturally occuring networks random graphs?

- NO!

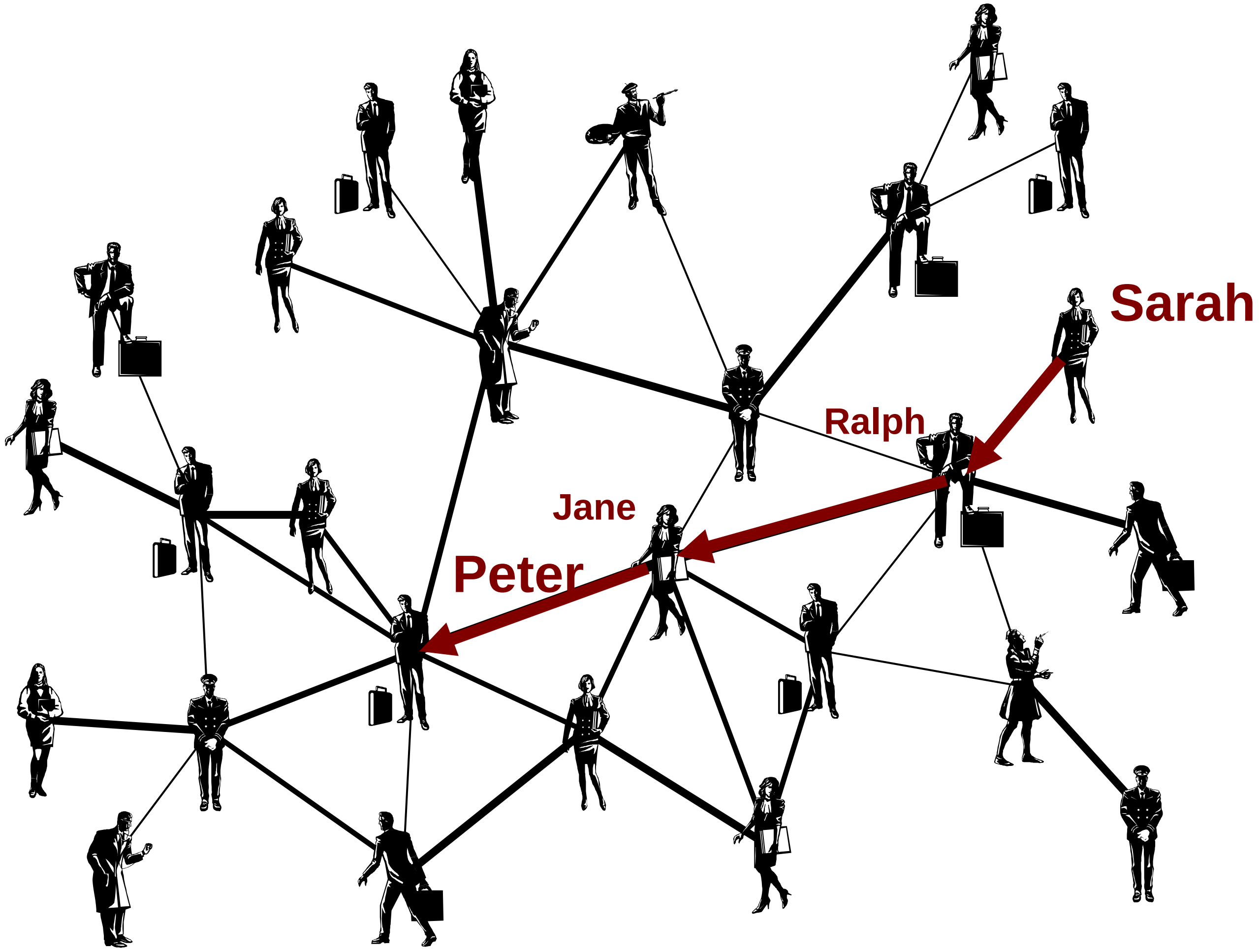
- Small worlds
- Clustered
- Scale free

Six Degrees (Stanley Milgram)



Stanley Milgram

Small worlds



Society:

Six degrees

S. Milgram 1967

F. Karinthy 1929

WWW:

19 degrees

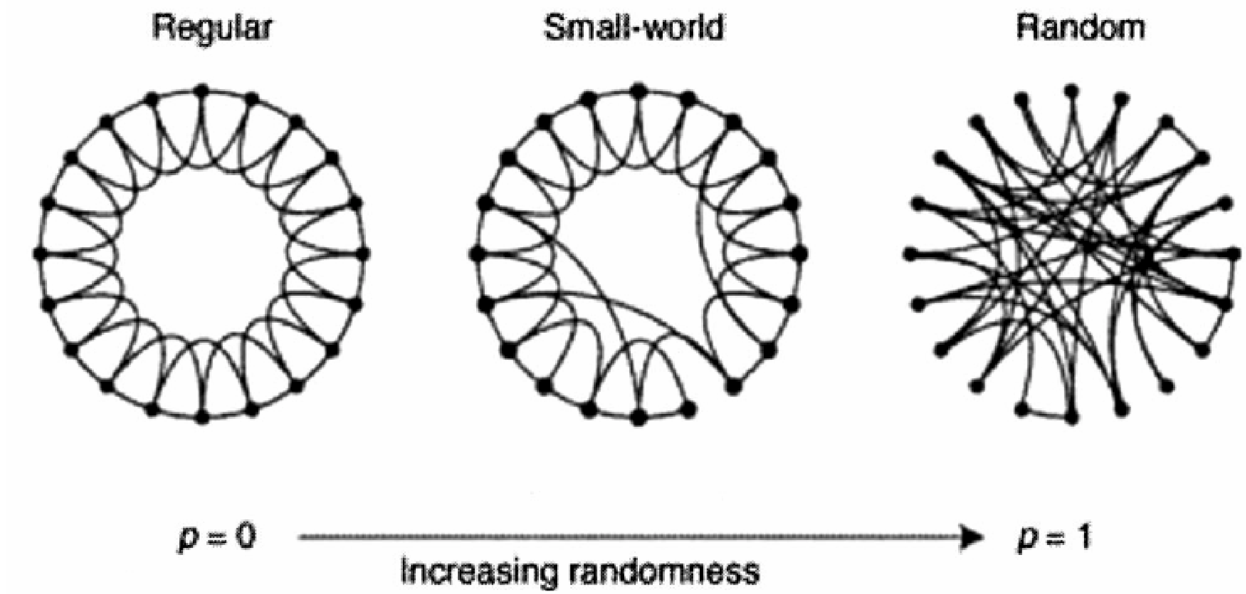
Albert *et al.* 1999

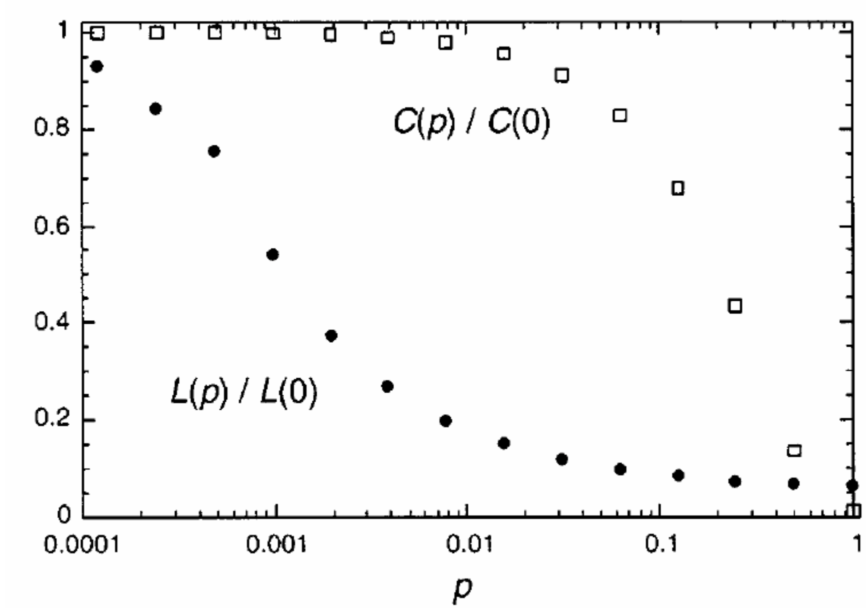
Small Worlds

- Stanley Milgram found that the average length of the chain connecting the sender and receiver was of length 5.5
  (The origin of the mythical “Six degres of separation)
- 19 degrees in the Web
- Random networks have short path length and small clustering
- Many natural networks have short path length, but high clustering
- The Watts-Strogatz model. Is it the model of natural networks?

Collective dynamics of 'small-world' networks

Duncan J. Watts* & Steven H. Strogatz





Duncan Watts

Collective dynamics of 'small-world' networks

letters to nature

Duncan J. Watts* & Steven H. Strogatz

Department of Theoretical and Applied Mechanics, Kimball Hall, Cornell University, Ithaca, New York 14853, USA

Table 1 Empirical examples of small-world networks

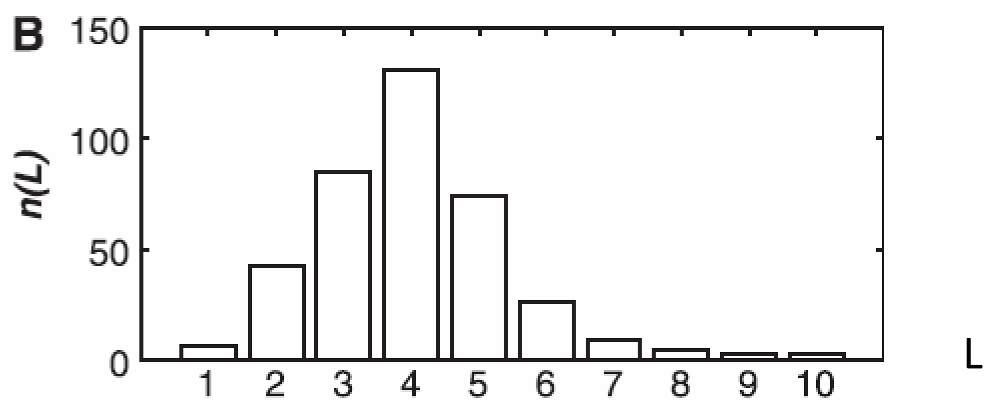
	L_{actual}	L_{random}	C_{actual}	C_{random}
Film actors	3.65	2.99	0.79	0.00027
Power grid	18.7	12.4	0.080	0.005
C. elegans	2.65	2.25	0.28	0.05

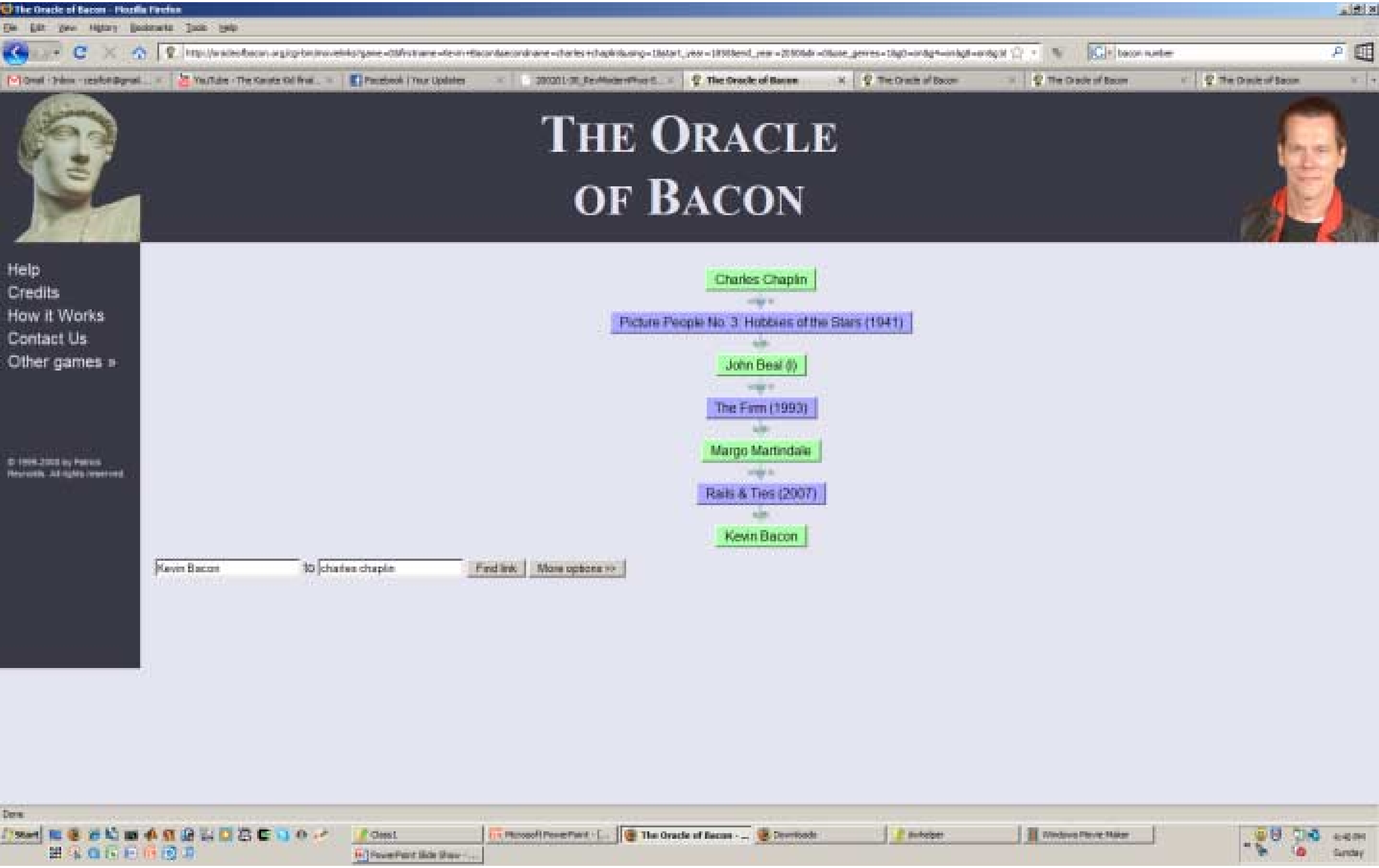
Other small worlds

An Experimental Study of Search in Global Social Networks

Peter Sheridan Dodds,[1] Roby Muhamad,[2] Duncan J. Watts[1,2]*

We report on a global social-search experiment in which more than 60,000 e-mail users attempted to reach one of 18 target persons in 13 countries by forwarding messages to acquaintances. We find that successful social search is




THE ORACLE OF BACON
Help
Credits
How it Works
Contact Us
Other games »
Charles Chaplin
Picture People No. 3: Hobbies of the Stars (1941)
John Beal (I)
The Firm (1993)
Margo Martindale
Rails & Ties (2007)
Kevin Bacon
Kevin Bacon
to
charles chaplin
Find link
More options >>

Total number of linkable actors: 953840
Average Kevin Bacon number: 2.946

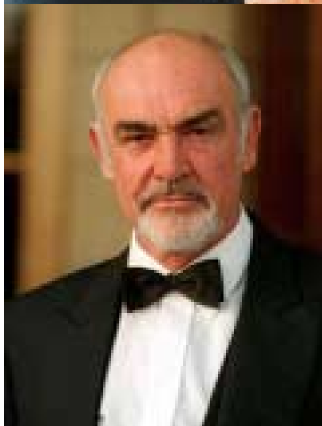
Kevin Bacon Number	# of People
0	1
1	2108
2	204188
3	601747
4	136178
5	8656
6	839
7	111
8	12

Kevin Bacon

Average Connery number: 2.731

Connery Number	# of people
0	1
1	2272
2	218560
3	380721
4	40263
5	3537
6	535
7	66
8	2

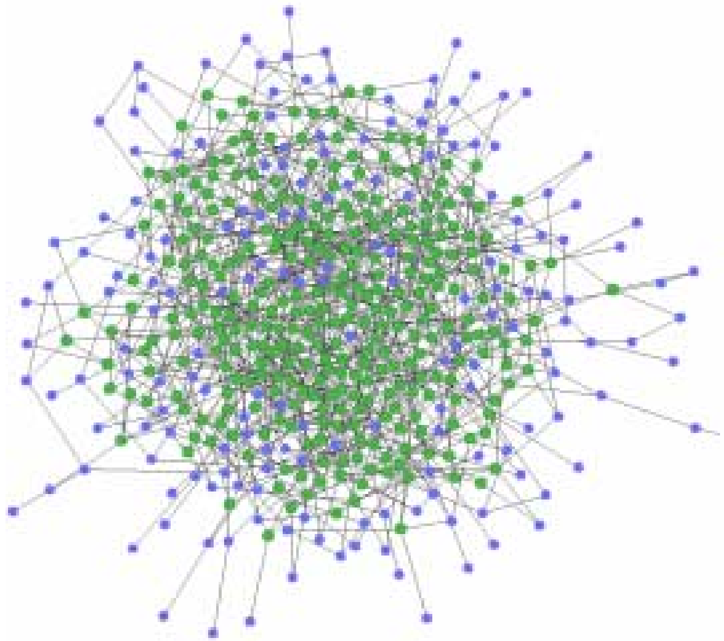
Sean Connery



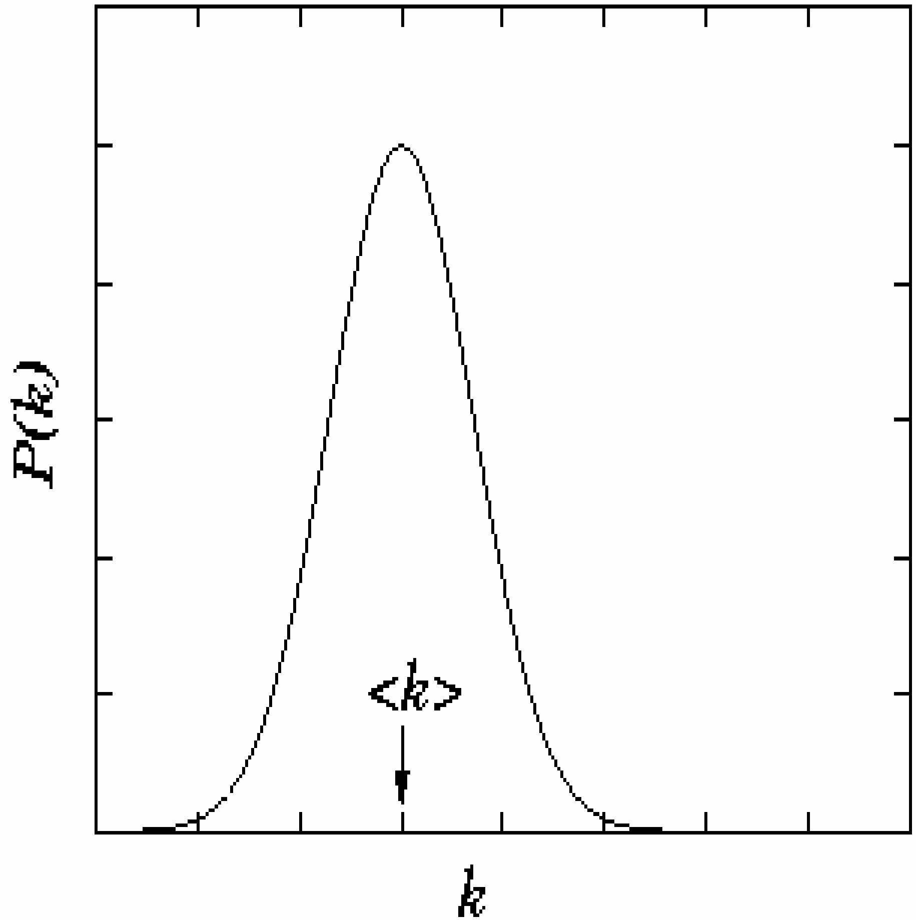
Is Watts-Strogatz the model of most natural occuring networks?

- No!
- Most naturally occuring networks have scale-free degree distribution
- They are small worlds but neither random, nor Watts and Strogatz
- Examples of scale-free networks

Erdös-Rényi model (1960)



Degree distribution (Poisson)



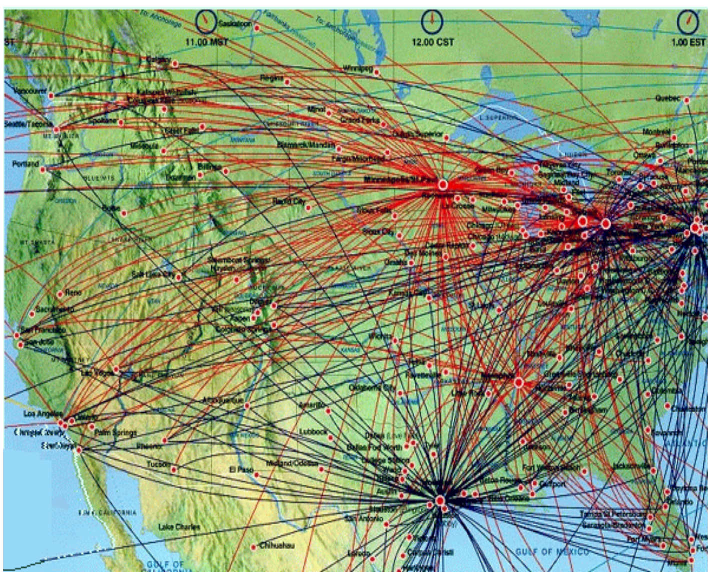
World Wide Web

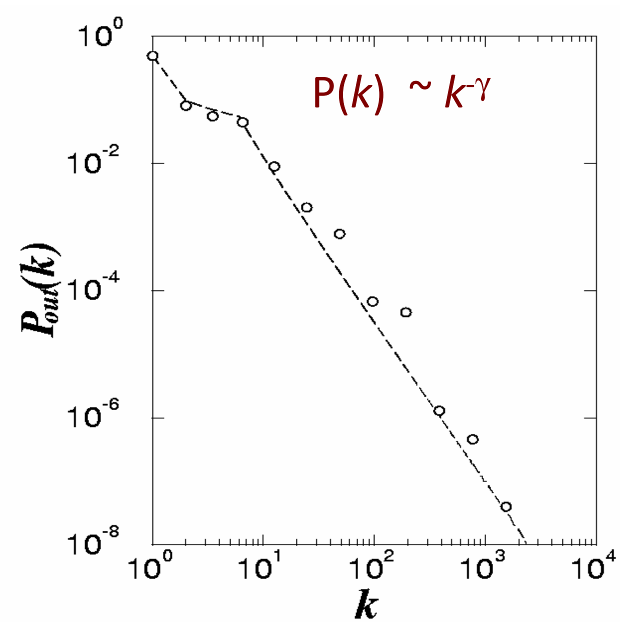
Nodes: WWW documents Links: URL links

Over 3 billion documents

ROBOT: `collects all URL's found in a document and follows them recursively`

Scale-free Network

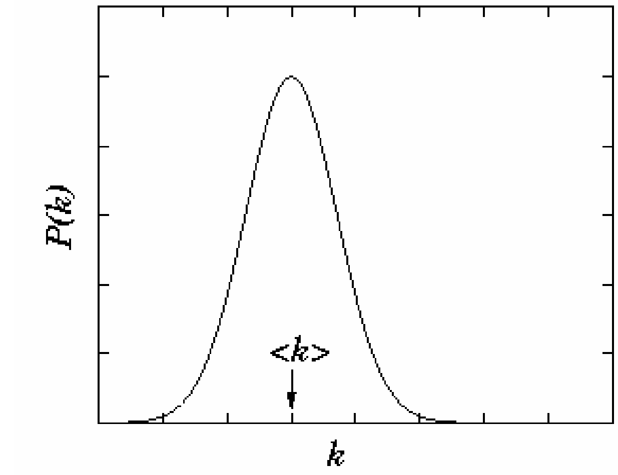




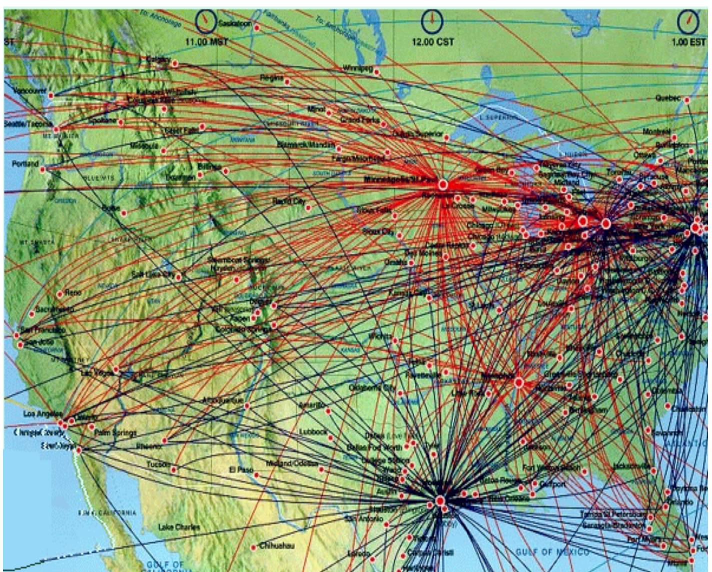
Found

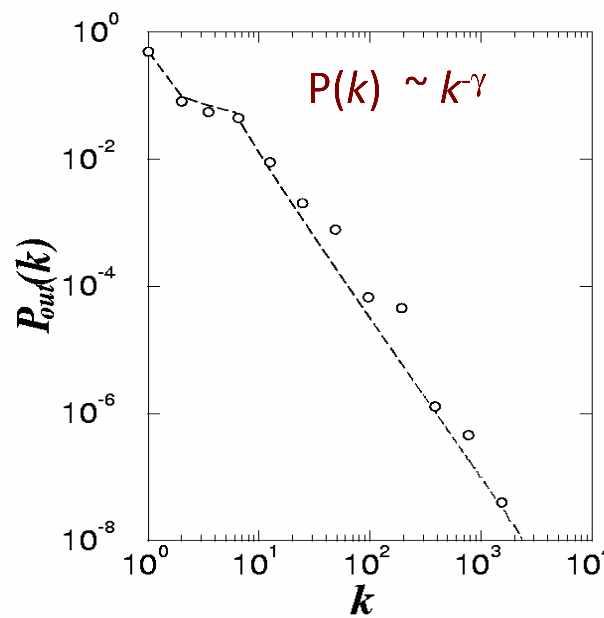
R. Albert, H. Jeong, A-L Barabasi, *Nature*, 401 130 (1999).

Exponential Network

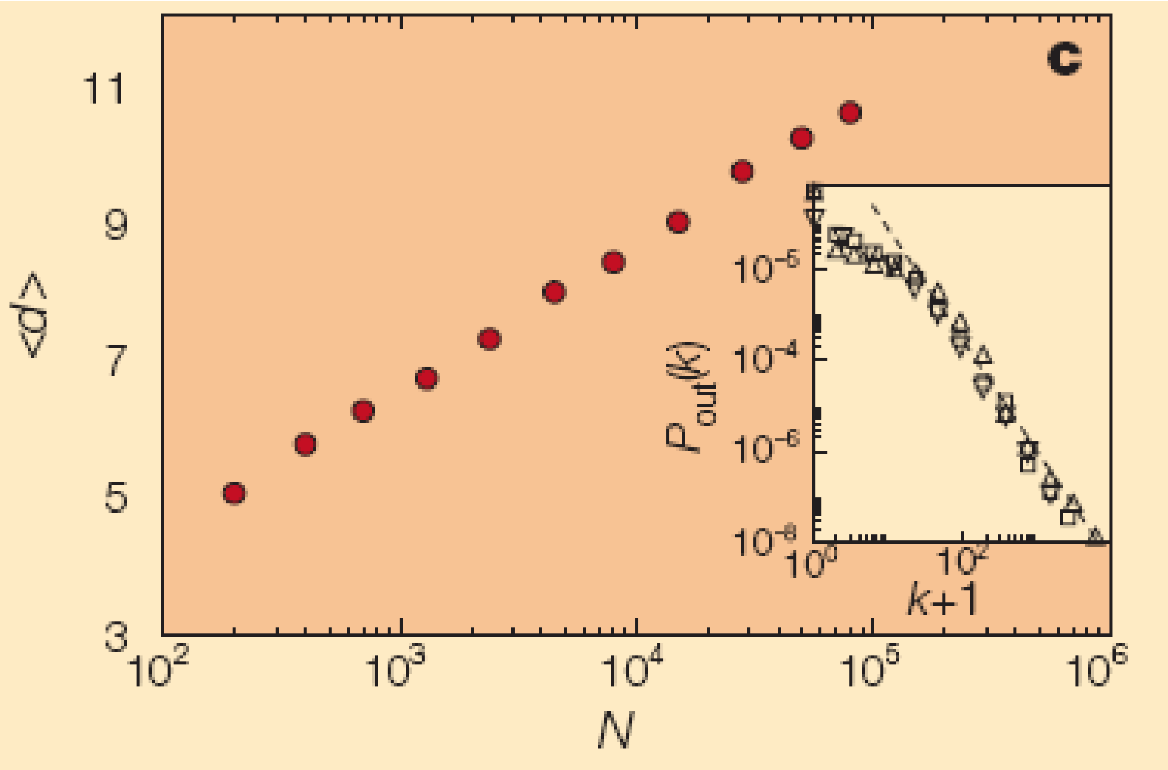


Scale-free Network





R. Albert, H. Jeong, A-L Barabasi, *Nature*, 401 130 (1999).



Réka Albert, Hawoong Jeong,
Albert-László Barabási

Diameter of the World-Wide Web

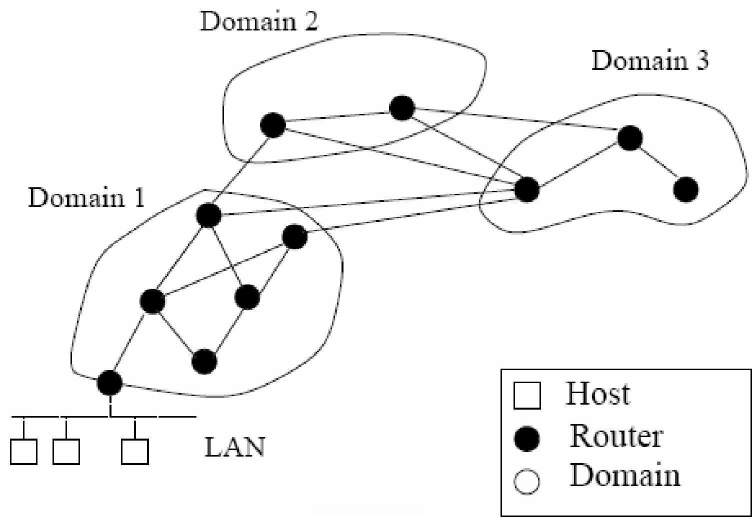
NATURE | VOL 401 | 9 SEPTEMBER 1999 | www.nature.com

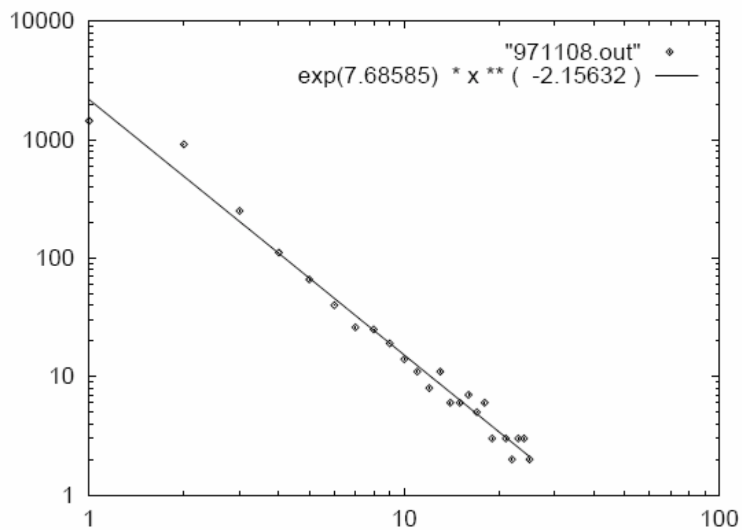
"On Power-Law Relationships of the Internet Topology",
Michalis Faloutsos, Petros Faloutsos, Christos Faloutsos, ACM SIGCOMM'99, Cambridge, Massachussets,pp 251-262, 1999

Abstract

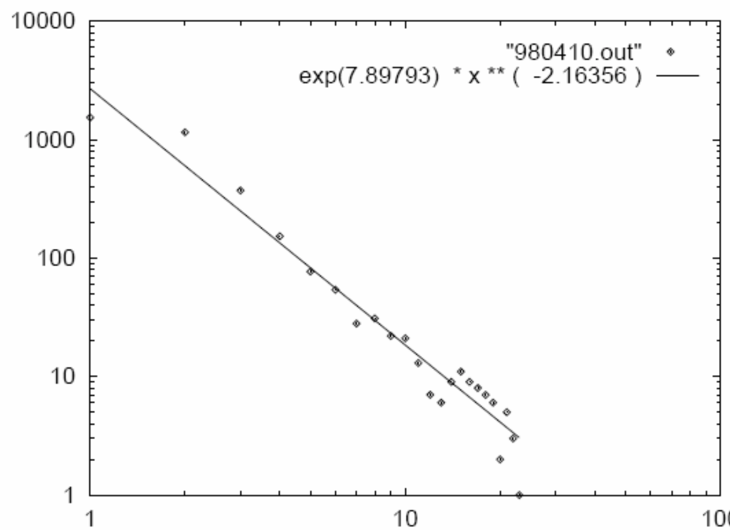
Despite the apparent randomness of the Internet, we discover some surprisingly simple power-laws of the Internet topology. These power-laws hold for three snapshots of the Internet, between November 1997 and December 1998, despite a 45% growth of its size during that period. We show that our power-laws fit the real data very well resulting in correlation coefficients of 96% or higher.

Our observations provide a novel perspective of the structure of the Internet. The power-laws describe concisely





(a) Int-11-97



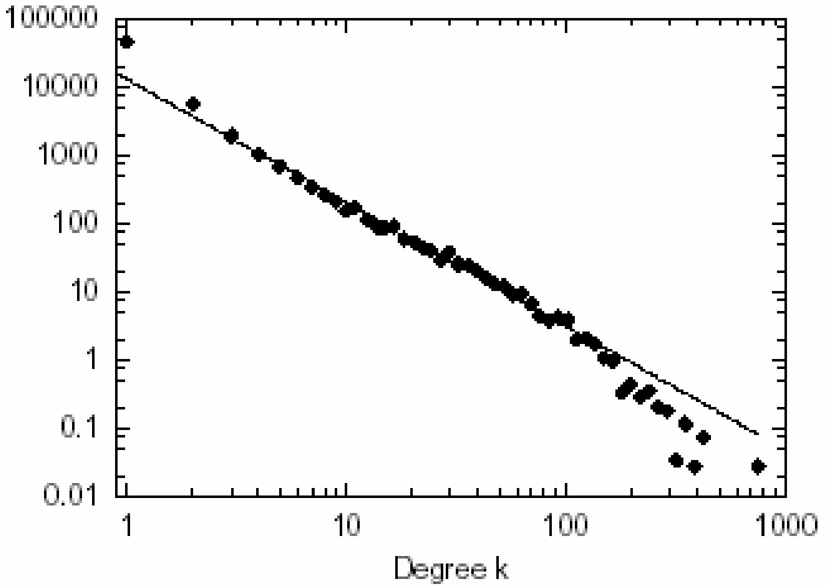
(b) Int-04-98

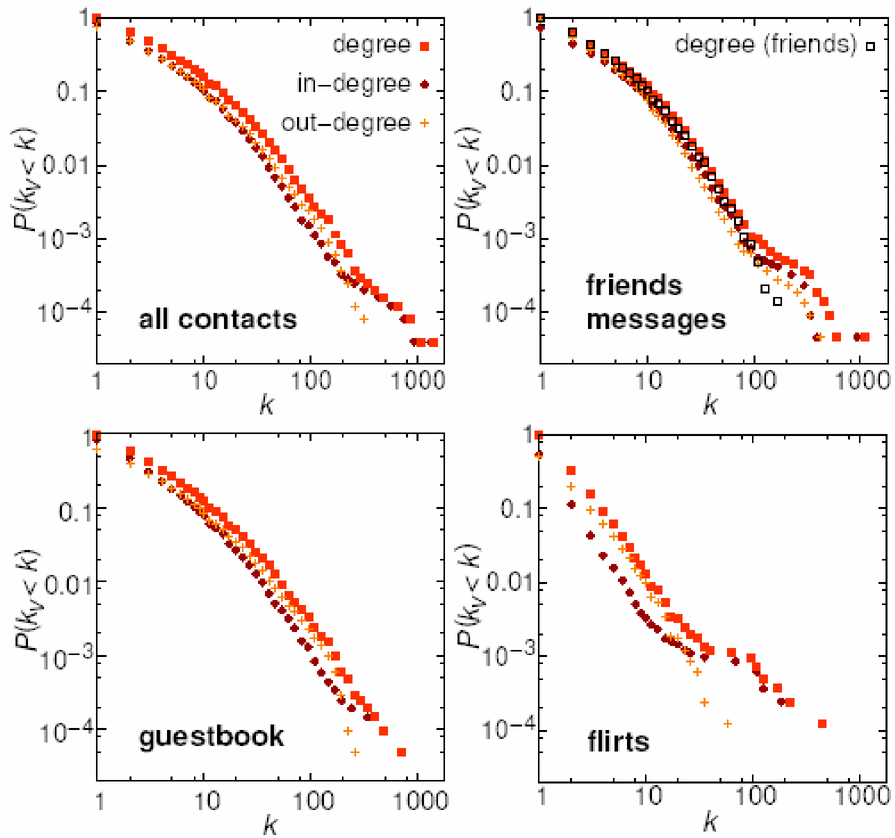
Online communities

Nodes: online user
Links: email contact

Pussokram.com online community; 512 days, 25,000 users.

Kiel University log files
112 days, N=59,912 nodes



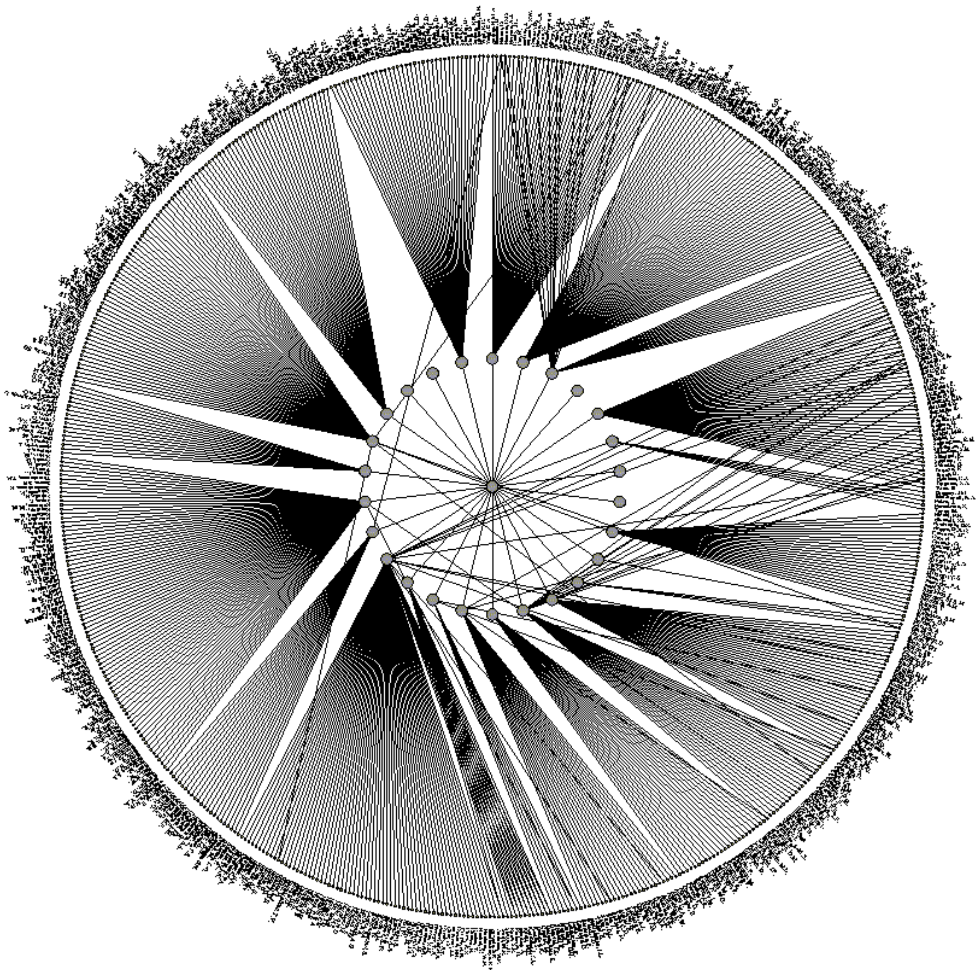


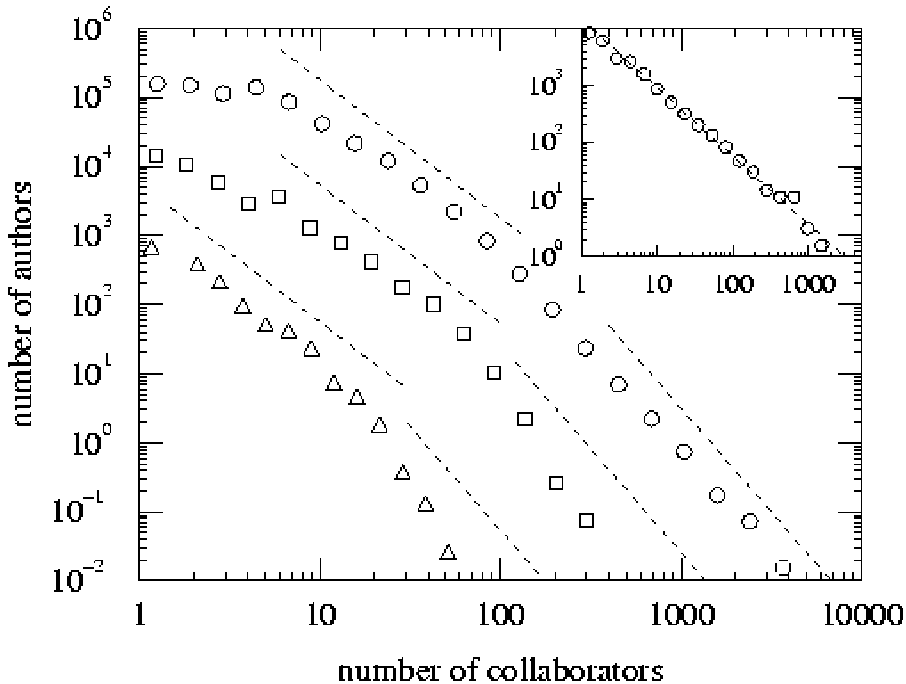
Ebel, Mielsch, Bornholdtz, PRE 2002.

Holme, Edling, Liljeros, 2002.

SCIENCE COAUTHORSHIP

Nodes: scientist (authors) Links: write paper together





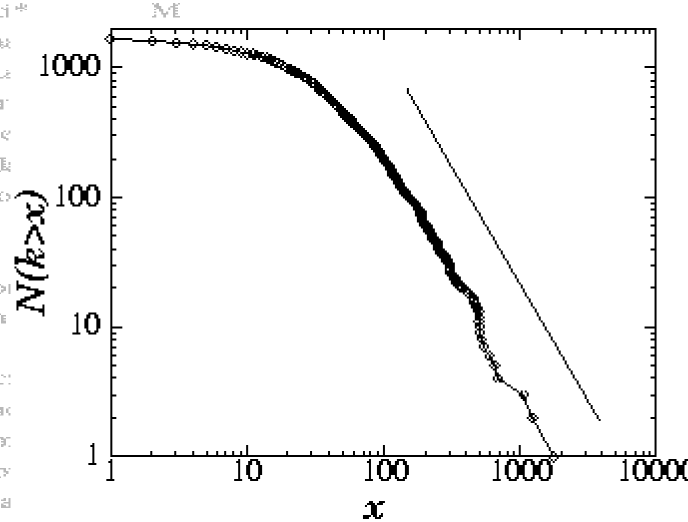
(Newman, 2000, A.-L. B. *et al* 2001)

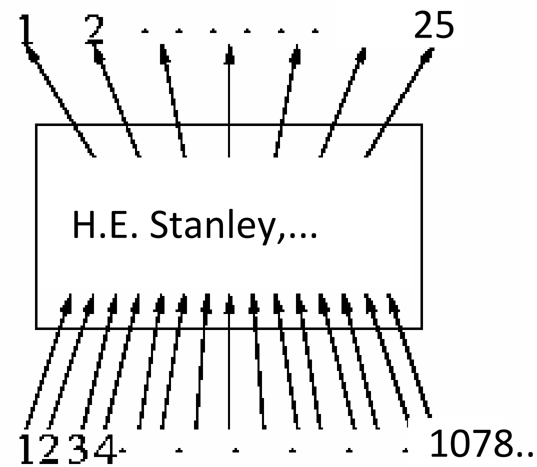
SCIENCE CITATION INDEX

1,000 Most Cited Physicists, 1981-June 1997
Out of over 500,000 Examined
(see http://www.sst.nrel.gov)

[Nodes]: papers
[Links]: citations

1736 PRL papers (1988)





$P(k) \sim k^{-\gamma}$

$(\gamma = 3)$

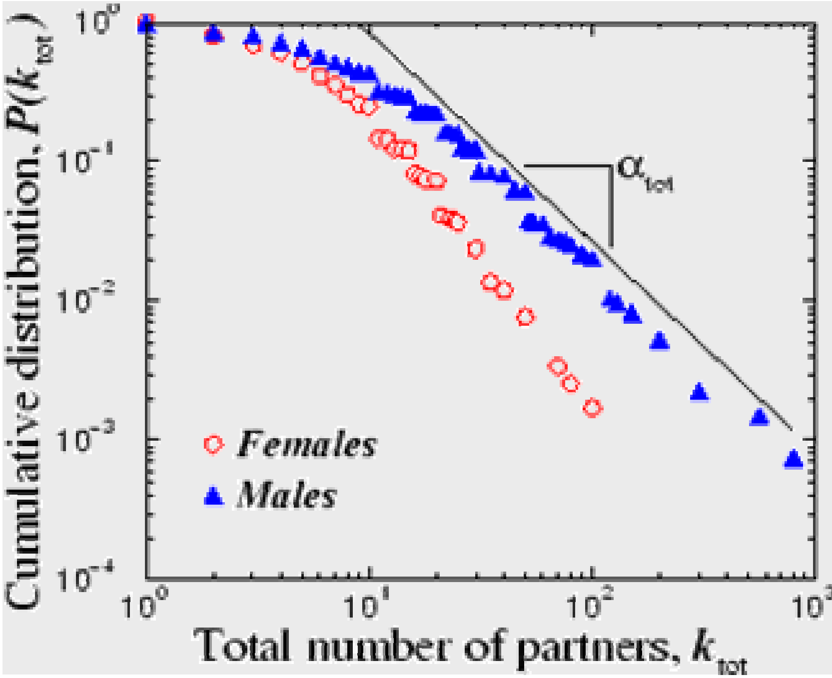
(S. Redner, 1998)

Author name		Institute	Country	Field	avg. cites	total art.	total cites	rank by total cit.
Witten	E	Princeton (U)	USA, NJ	High-energy (T)	168	138	23235	1
Gossard	AC	UCSB (U)	USA, CA	Semicc				2
Cava	RJ	Bell Labs (I)	USA, NJ	Superc				3
Batlogg			USA, NJ	Superc				4
Ploog			Germany	Semicc				5
Ellis		Euro Nuclear Cent.	Switzerland	Astropl				6
Fisk			USA, FL	Solid S				7
Cardona	M	Max Planck (NL)	Germany	Semicc				8
Nanopoulos	DV	Texas A&M (U)	USA, TX	High-e				9
Heeger				Polym				10
Lee*								11
Suzuki*								12
Anderson	PW	Princeton (U)	USA, NJ	Solid S				13
Suzuki*	M							14
Freema				Solid S				15
Tanaka								16
Muller				Superc				17
Schnee				Superc				18
Chemk				Optics (E)	60	162	9668	19
Morko				Semiconductors (E)	20	477	9668	19
Miller				Semiconductors (E)	67	144	9652	21
Chu				Superconductivity (E)			9453	22
Bednoi				Superconductivity (E)	110	85	9311	23
Cohen				Solid State (T)	33	284	9311	23
Meng				Superconductivity (E)	86	108	9300	25
Waszc				Superconductivity (E)	57		9170	26
Shiran				Superconductivity (E)			8841	27
Wiegn				Semiconductors (E)	85		8822	28
Vando				Magnetism (E)	67	129	8686	29
Uchida					28	301	8520	30
Hor	PH	Houston Univ. (U)	USA, TX	Superconductivity (E)	72	119	8512	31
Murphy	DW			Astronomy (E)	111	76	8439	32
Birgeneau	RJ	MIT (U)	USA, MA	Superconductivity (E)	41	286	8375	33
Jorgensen	JD	Argonne (NL)	USA, IL	Superconductivity (E)				34
Hinks	DG	Argonne (NL)	USA, IL	Superconductivity (E)	37	223	8263	35

* citation total may be skewed because of multiple authors with the same name

Swedish sex-web

Nodes: people (Females; Males)
Links: sexual relationships



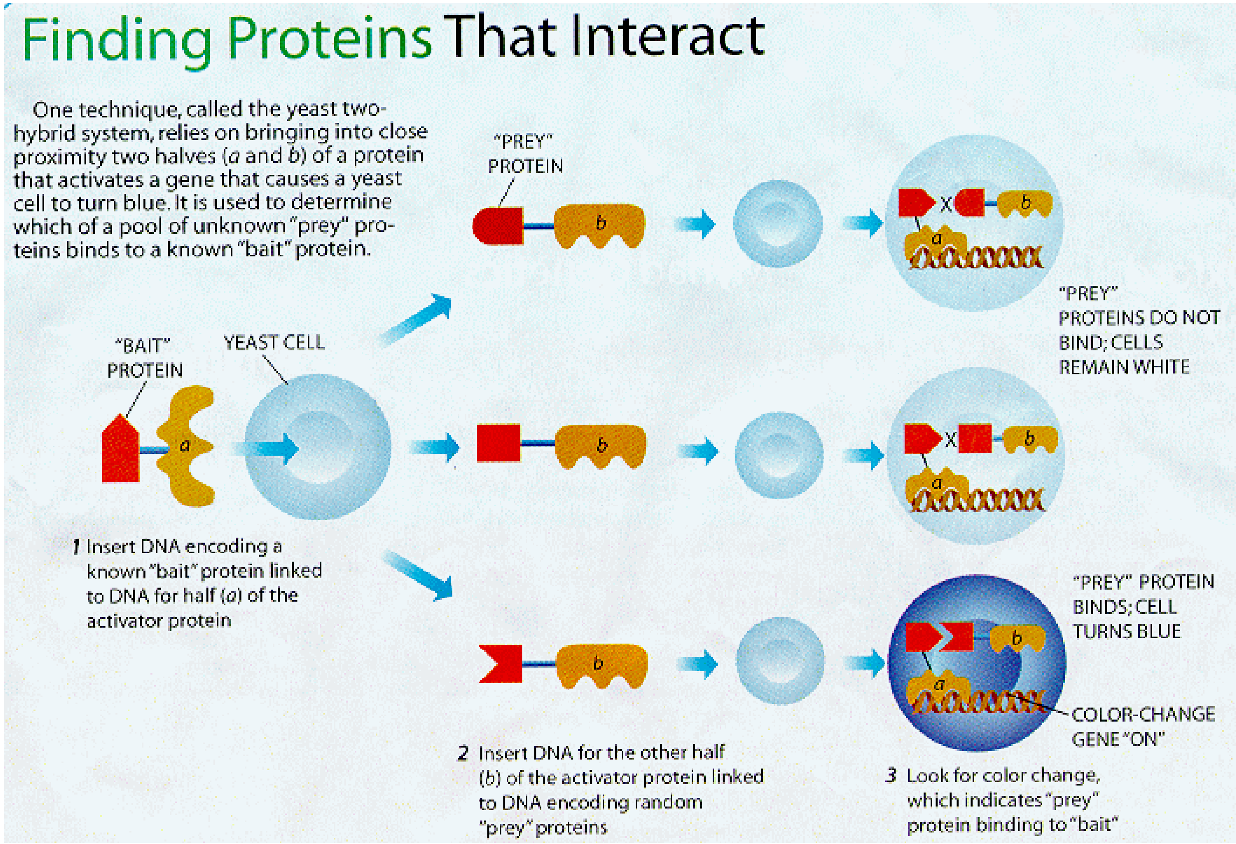
4781 Swedes; 18-74;
59% response rate.

Liljeros et al. *Nature* 2001

Yeast protein network

Nodes: proteins

Links: physical interactions (binding)

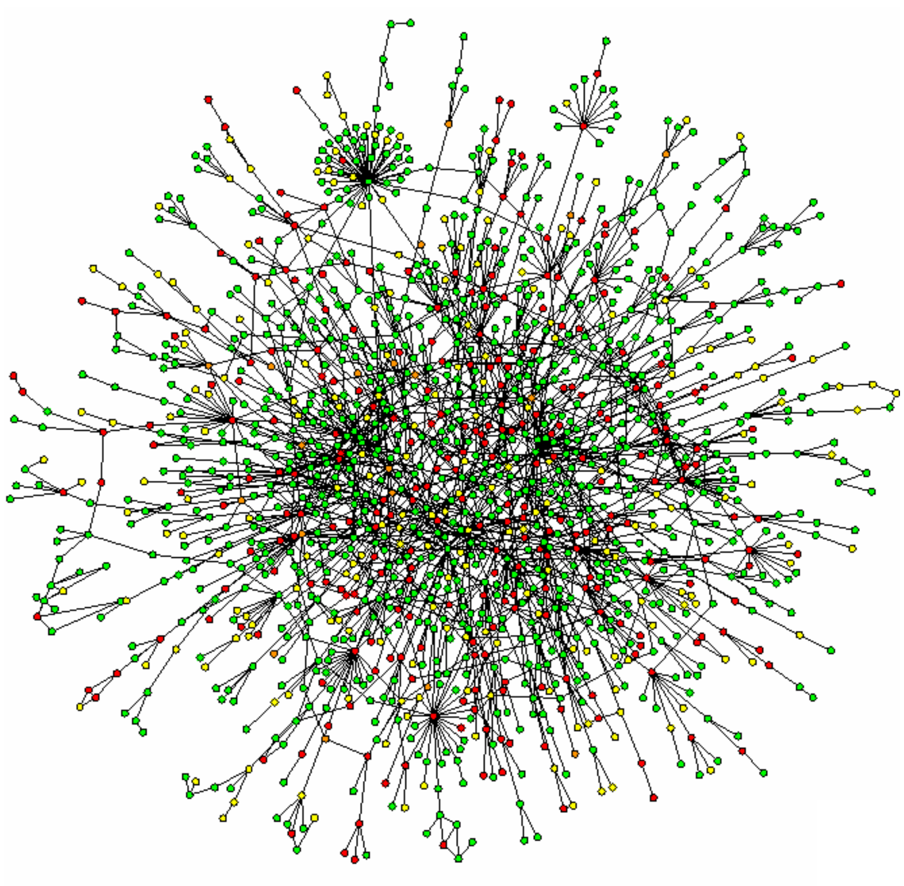


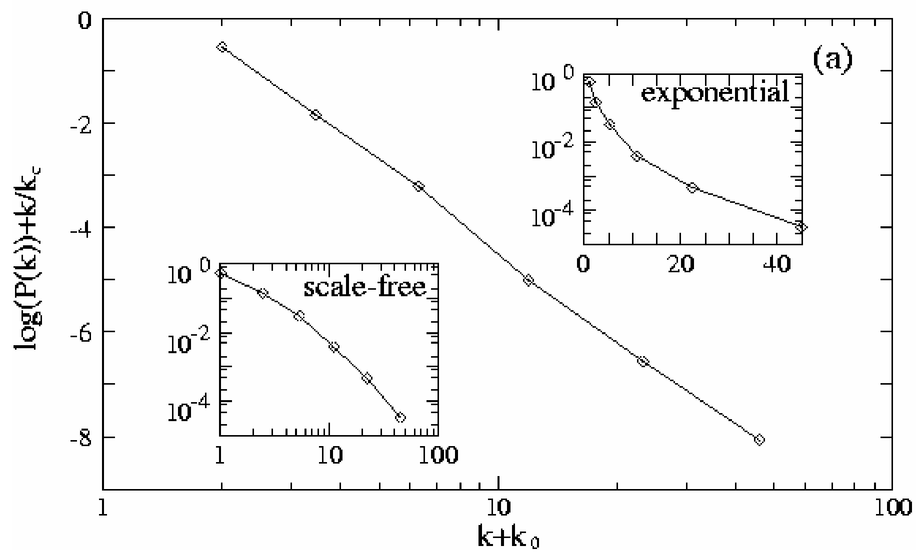
P. Uetz, *et al.* *Nature* **403**, 623-7 (2000).

Protein interaction network

Nodes: proteins

Links: physical interactions (binding)





$$P(k) \sim (k+k_0)^{-\gamma} \exp(-\frac{k+k_0}{k_\tau})$$

H. Jeong, S.P. Mason, A.-L. Barabasi, Z.N. Oltvai, *Nature* 411, 41-42 (2001)

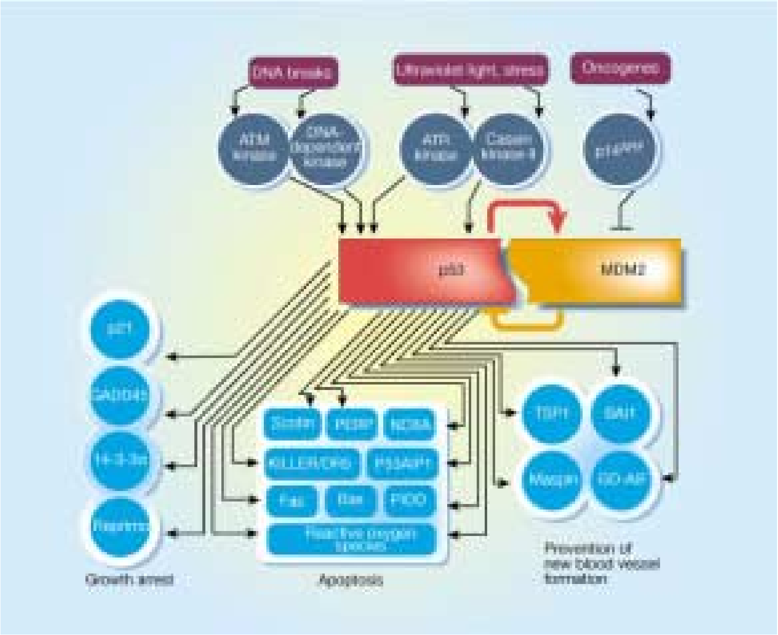
Nature **408** 307 (2000)

news and views feature

Surfing the p53 network

Bert Vogelstein, David Lane and Arnold J. Levine

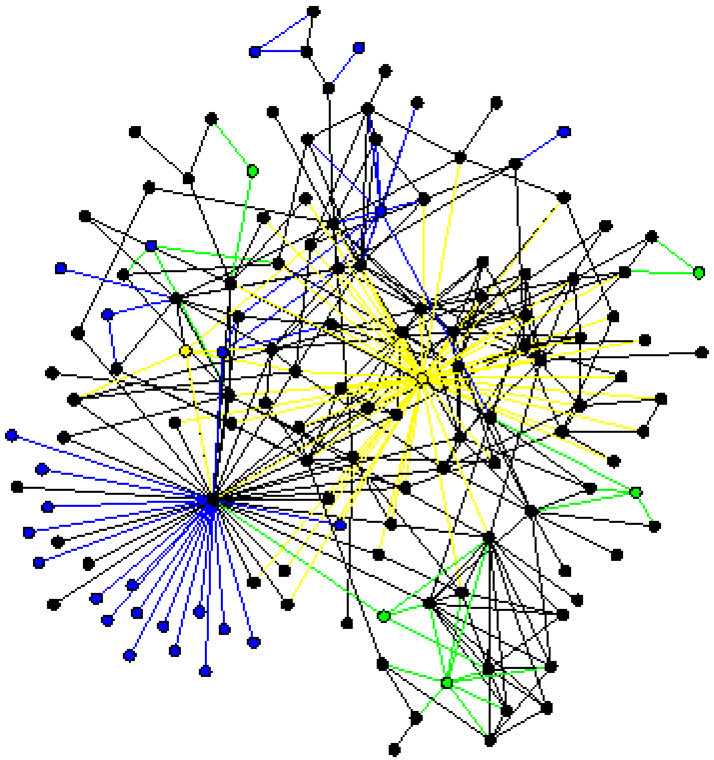
The p53 tumour-suppressor gene integrates numerous signals that control cell life and death. As when a highly connected node in the Internet breaks down, the disruption of p53 has severe consequences.

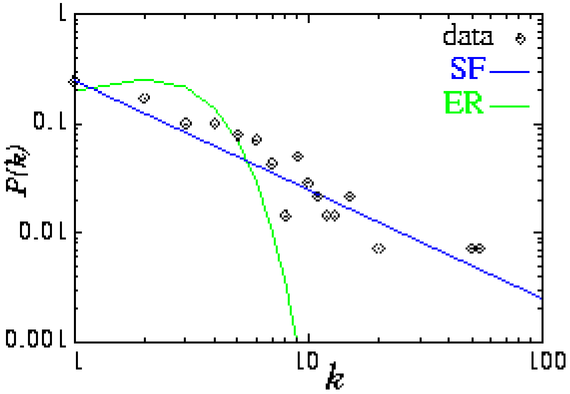


…

"One way to understand the p53 network is to compare it to the Internet.
The cell, like the Internet, appears to be a '**scale-free network**'."

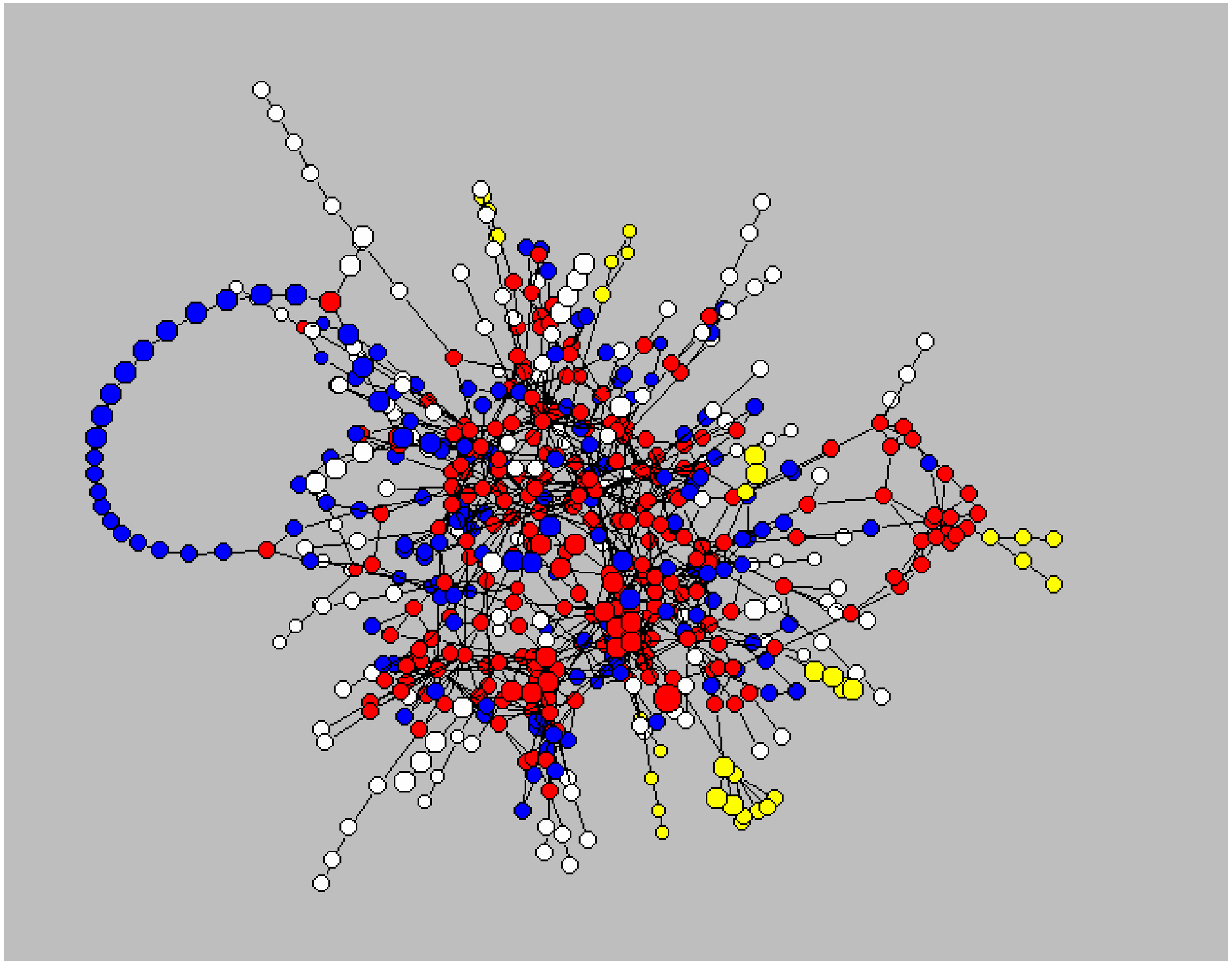
p53 network (mammals)



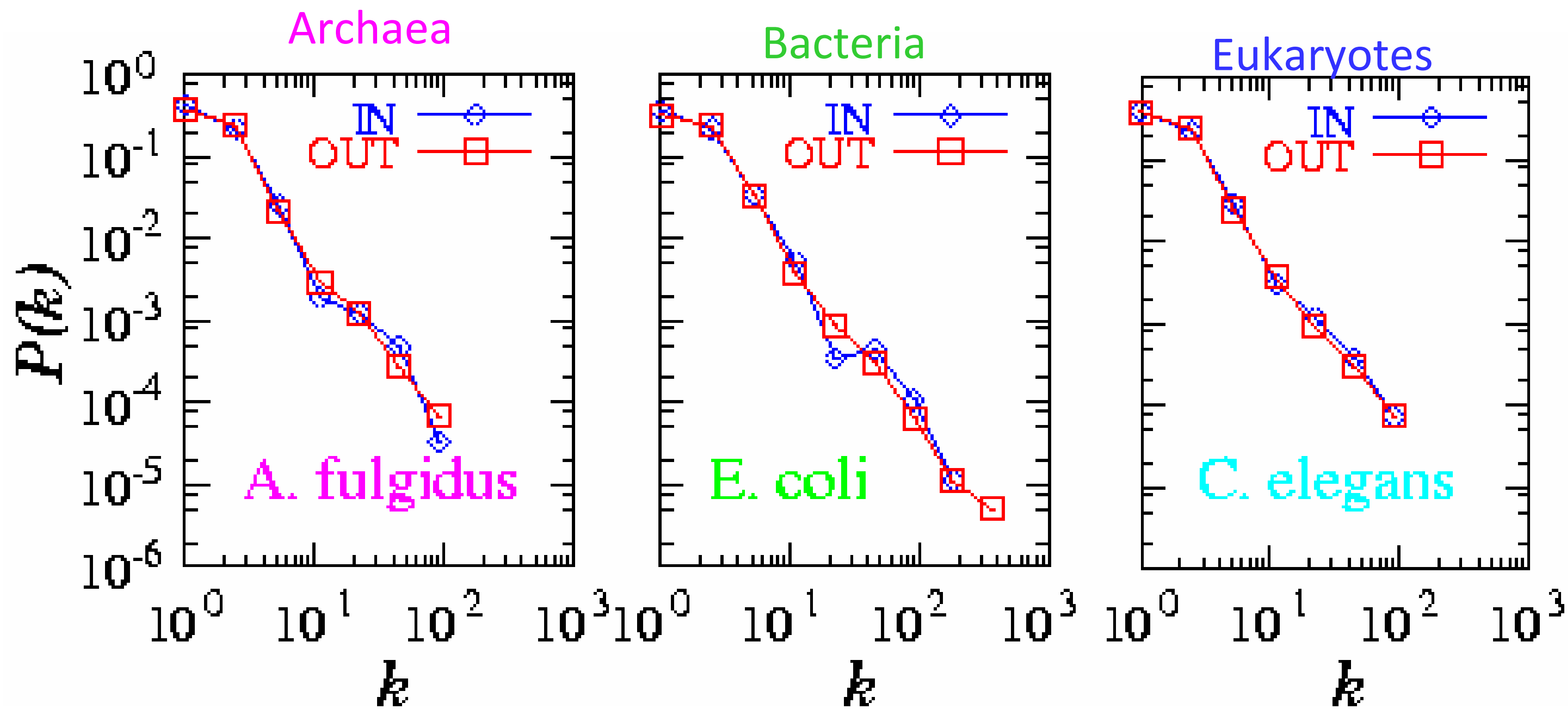


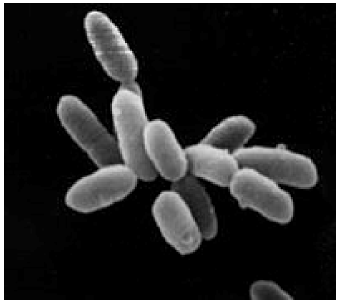
Metabolic Network

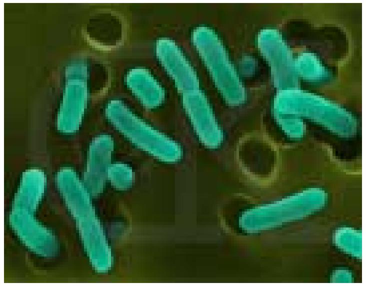
- Nodes: chemicals (substrates)
- Links: bio-chemical reactions

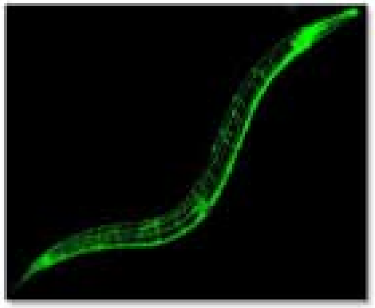


Metabolic network









Organisms from all three domains of life have scale-free metabolic networks!

H. Jeong, B. Tombor, R. Albert, Z.N. Oltvai, and A.L. Barabasi, *Nature*, 407 651 (2000)

Small World Features of Scale free Networks

Random Networks:

D~log(N)

(small world effect)

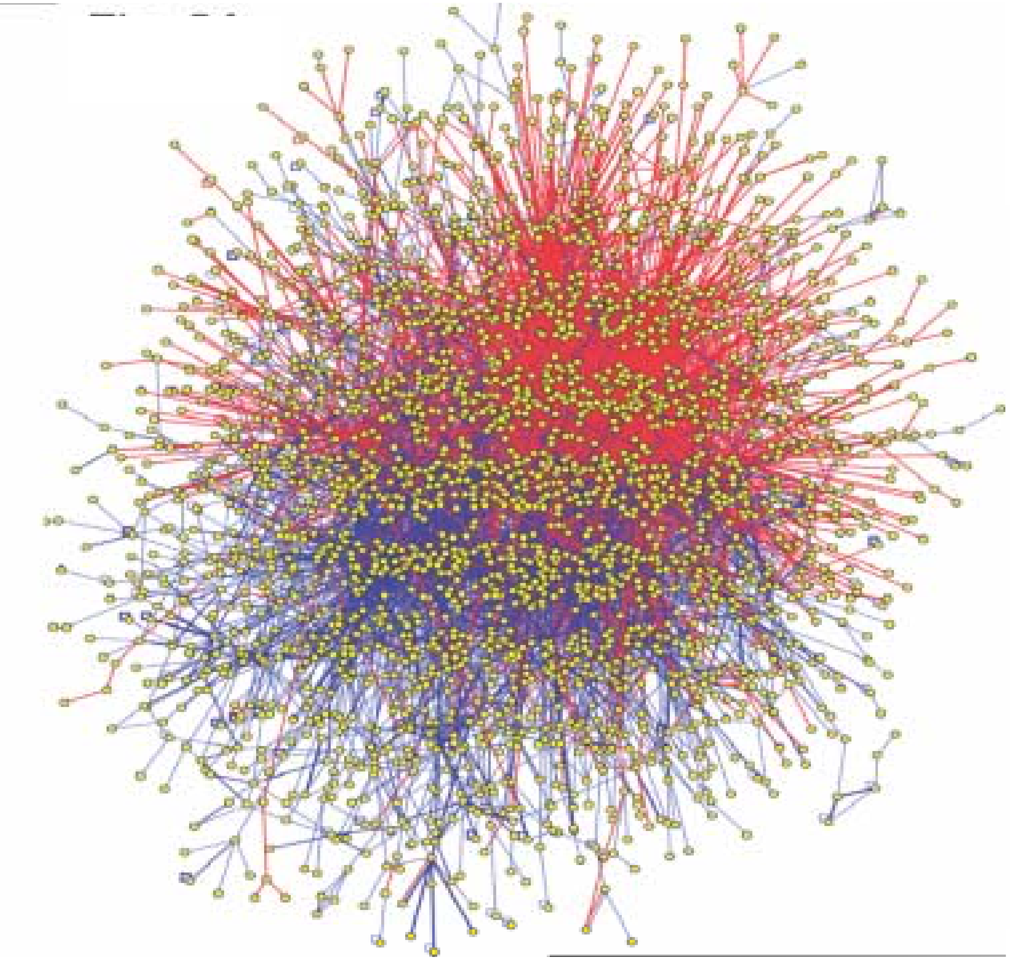
Scale-Free Networks: $P(k) \sim k^{-\gamma}$

$$D = \begin{cases} \log N & \gamma > 3 \\ \log \log N & 2 < \gamma < 3 \\ const & \gamma = 2 \end{cases}$$

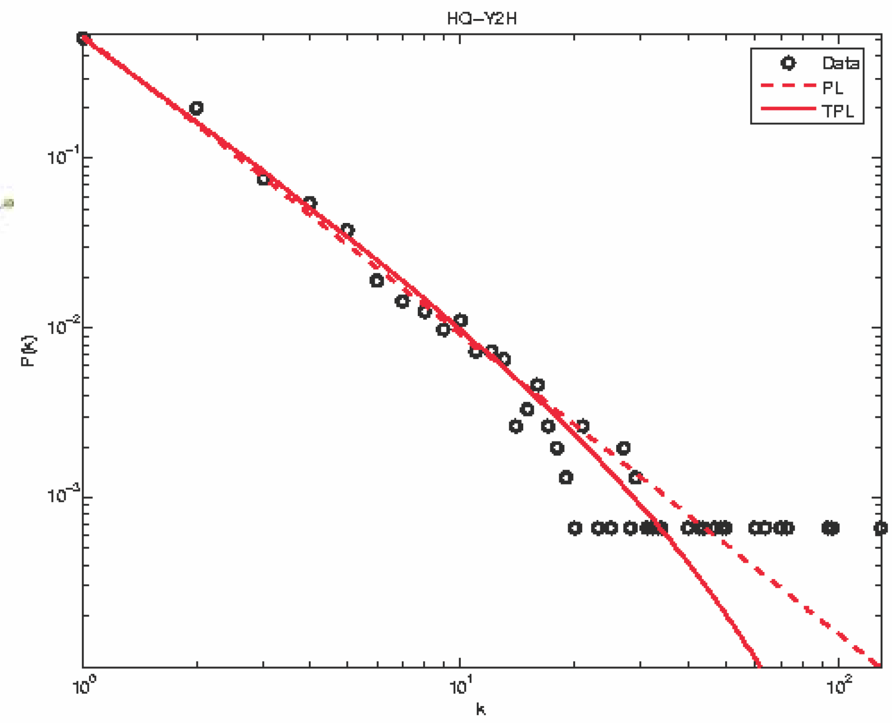
(ultra small world)

Cohen,Havlin, PRL'03

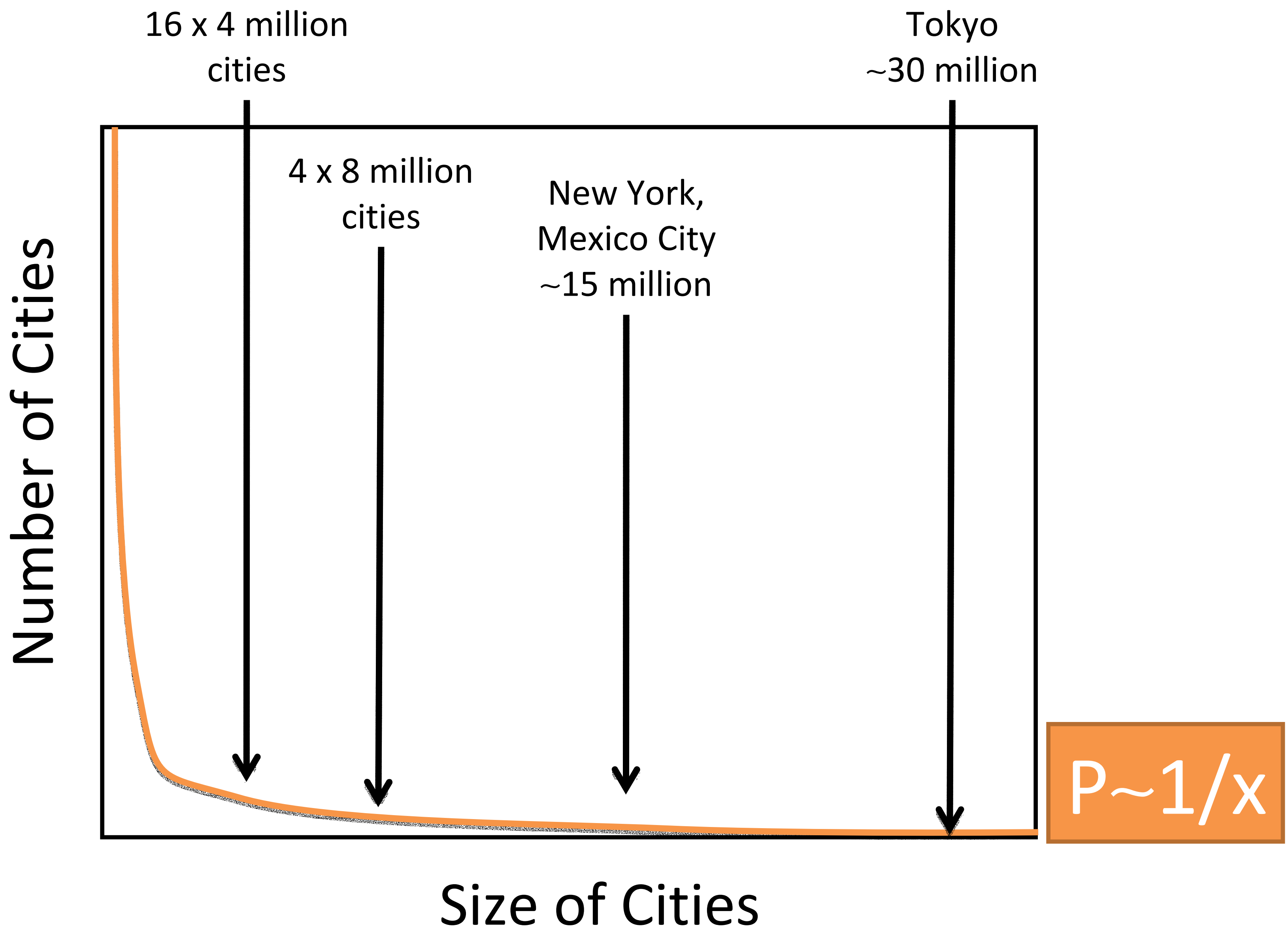
Human Interaction Network



2,800 Y2H interactions
4,100 binary LC interactions
(HPRD, MINT, BIND, DIP, MIPS)

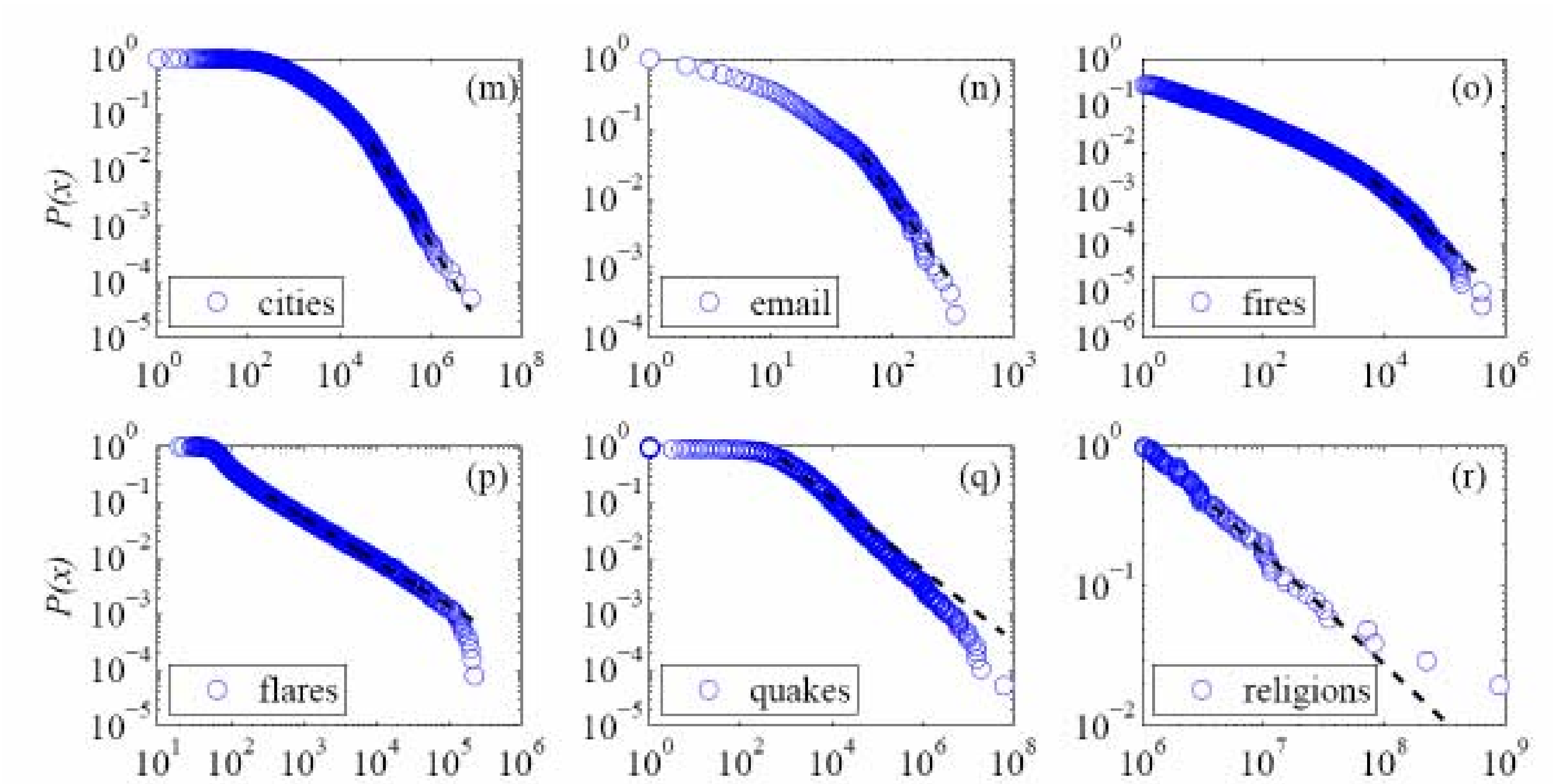


Rual *et al.* Nature 2005; Stelze *et al.* Cell 2005



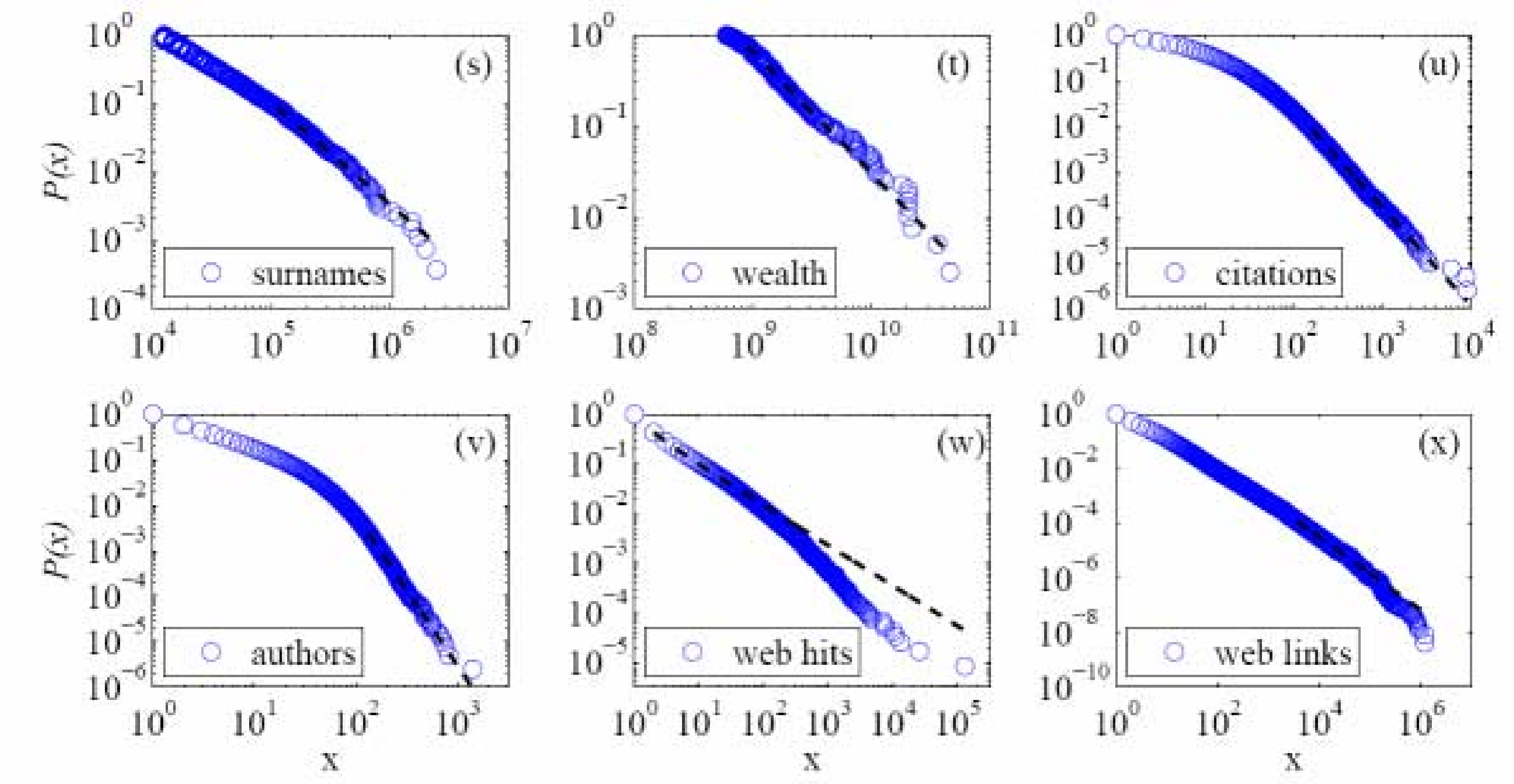
16 x 4 million cities
Tokyo ~30 million
4 x 8 million cities
New York, Mexico City ~15 million
Number of Cities
Size of Cities
P~1/x

Power laws everywhere



Power-law distributions in empirical data, Aaron Clauset, Cosma Rohilla Shalizi, and M. E. J. Newman, submitted to *SIAM Review*.

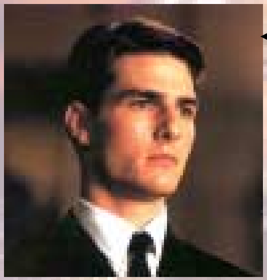
Power laws everywhere



Power-law distributions in empirical data, Aaron Clauset, Cosma Rohilla Shalizi, and M. E. J. Newman, submitted to *SIAM Review*.

ACTOR CONNECTIVITIES

Nodes: actors
Links: cast jointly

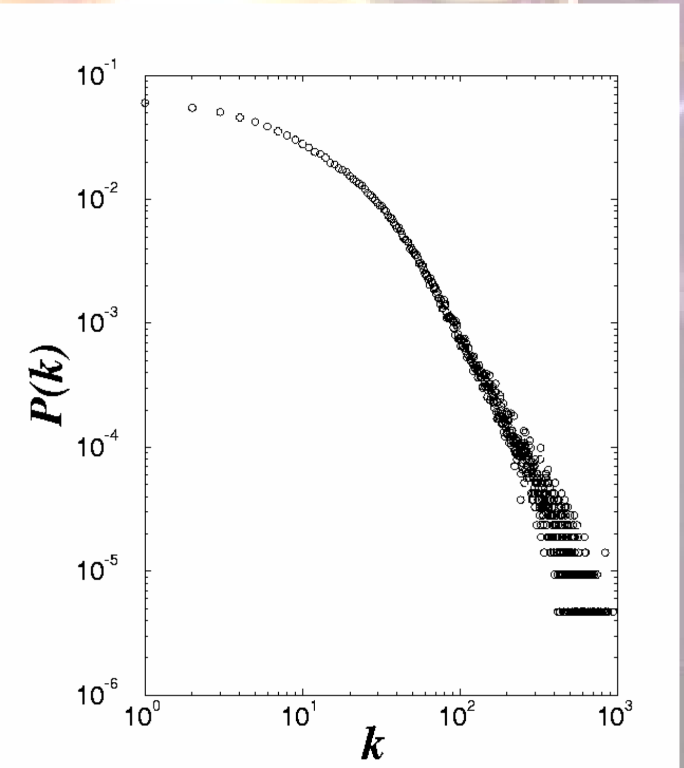


Days of Thunder (1990)
Far and Away (1992)
Eyes Wide Shut (1999)

N = 212,250 actors
⟨k⟩ = 28.78

P(k) ~$k^{-\gamma}$

γ=2.3



Scale free networks

- In scale free networks most nodes have few connections, whereas a few ones have very many nodes.
- They are called the HUBS
- Despite their clustered structure, scale free networks are small worlds because the hubs provide the long-range connections
- Is Kevin Bacon an Hub ?

Austin Powers:
The spy who
shagged me

Robert Wagner

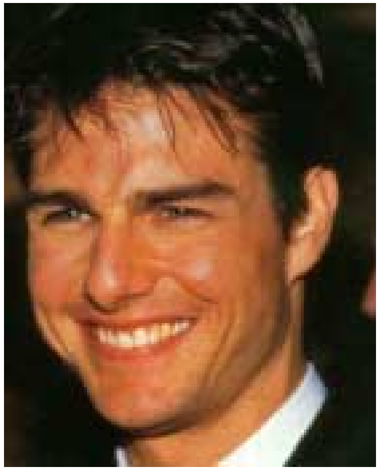
Let's make
it legal

Wild Things

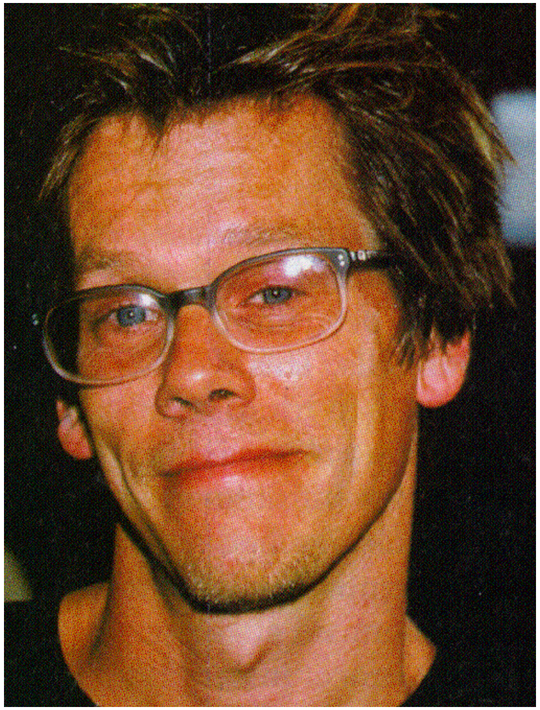
What Price Glory

Barry Norton

Monsieur
Verdoux



A Few
Good Man



Nothing special about Kevin Bacon

Kevin Bacon

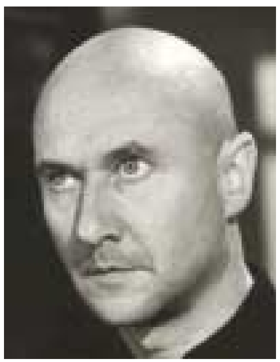
No. of movies : 46 No. of actors : 1811
Average separation: 2.79

Is Kevin Bacon the most connected actor?

NO!

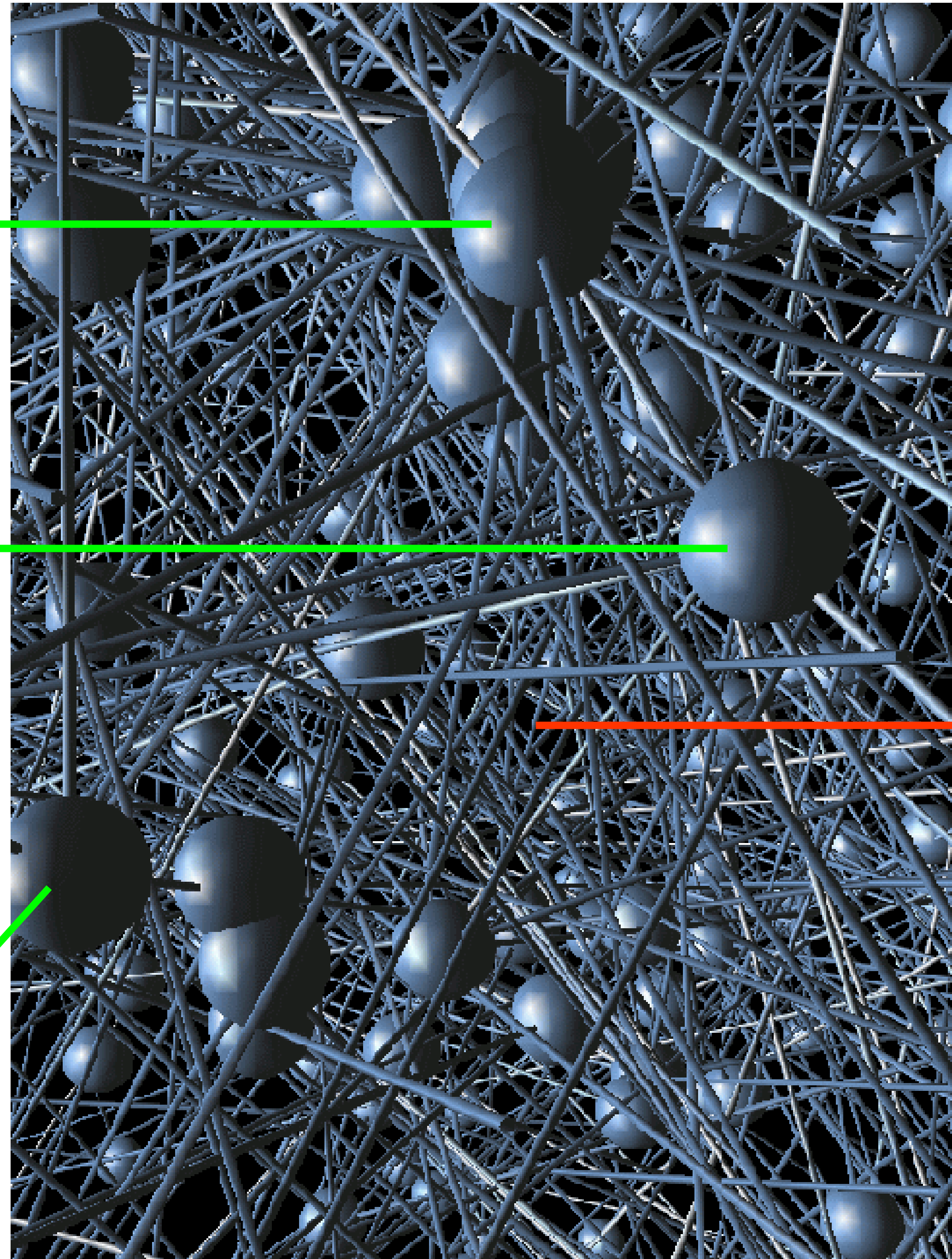
Rank	Name	Average distance	# of movies	# of links
1	Rod Steiger	2.537527	112	2562
2	Donald Pleasence	2.542376	180	2874
3	Martin Sheen	2.551210	136	3501
4	Christopher Lee	2.552497	201	2993
5	Robert Mitchum	2.557181	136	2905
6	Charlton Heston	2.566284	104	2552
7	Eddie Albert	2.567036	112	3333
8	Robert Vaughn	2.570193	126	2761
9	Donald Sutherland	2.577880	107	2865
10	John Gielgud	2.578980	122	2942
11	Anthony Quinn	2.579750	146	2978
12	James Earl Jones	2.584440	112	3787
…				
876	**Kevin Bacon**	**2.786981**	**46**	**1811**
…				

#1 Rod Steiger



#2 Donald Pleasence

#3 Martin Sheen



#876 Kevin Bacon

Why are networks scale-free?

- **DYNAMICS**
  Dynamics on networks
  Dynamics of networks

- **Dynamics of networks**
  Preferential attachment
  Fitness model
  Gene duplication

Growth versus preferential attachment

(1) Networks continuously expand by the addition of new nodes

WWW : addition of new documents
Citation : publication of new papers

(2) New nodes prefer to link to highly connected nodes.

WWW : linking to well known sites
Citation : citing again highly cited papers

GROWTH:
add a new node with m links

PREFERENTIAL ATTACHMENT: the probability that a node connects to a node with k links is proportional to k.

$$\Pi(k_i) = \frac{k_i}{\Sigma_j k_j}$$

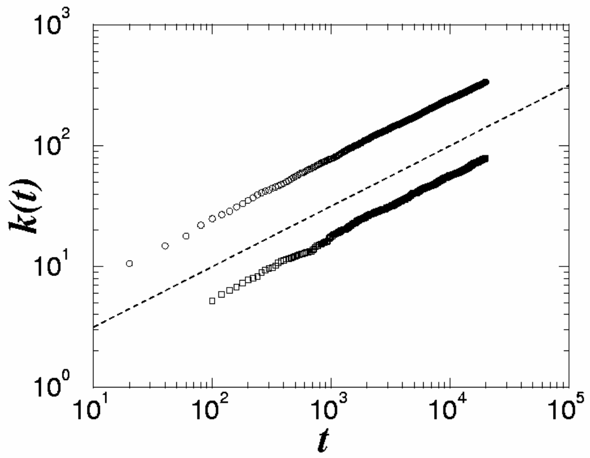
Barabási & Albert, *Science* 286, 509 (1999)

Mean Field Theory (Preferential attachment)

$$\frac{\partial k_i}{\partial t} \propto \Pi(k_i) = A\frac{k_i}{\sum_j k_j} = \frac{k_i}{2t}$$

, with initial condition $k_i(t_i) = m_0$

$$\boxed{k_i(t) = m\sqrt{\frac{t}{t_i}}}$$



$$P(k_i(t) < k) = P_t(t_i > \frac{m^2 t}{k^2}) = 1 - P_t(t_i \le \frac{m^2 t}{k^2}) = 1 - \frac{m^2 t}{k^2(m_0 + t)}$$

$$\therefore P(k) = \frac{\partial P(k_i(t) < k)}{\partial k} = \frac{2m^2 t}{m_o + t}\frac{1}{k^3} \sim k^{-3}$$

$$\boxed{\gamma = 3}$$

A.-L.Barabási, R. Albert and H. Jeong, *Physica* A 272, 173 (1999)

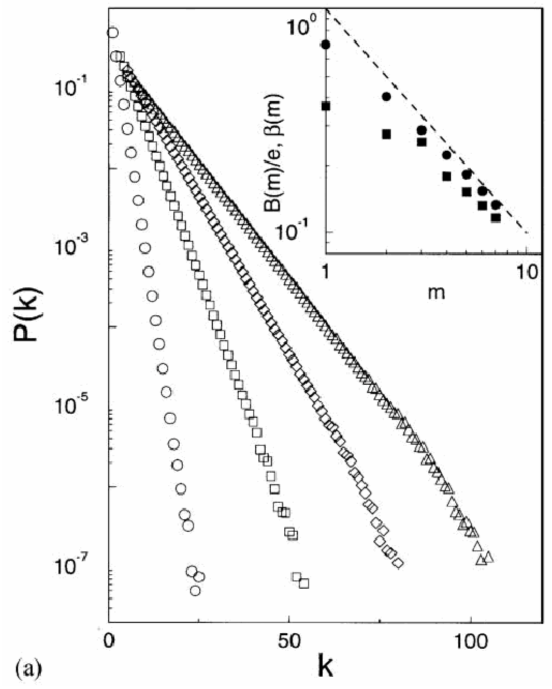
Growth model

$\Pi(k_i)$: uniform

$$\frac{\partial k_i}{\partial t} = A\Pi(k_i) = \frac{m}{m_0 + t - 1}$$

$$k_i(t) = m\left(\ln\left(\frac{m_0 + t - 1}{m + t_i - 1}\right) + 1\right)$$

$$P(k) = \frac{e}{m}\exp\left(-\frac{k}{m}\right) \sim e^{-k}$$



Can Latecomers Make It? Fitness Model

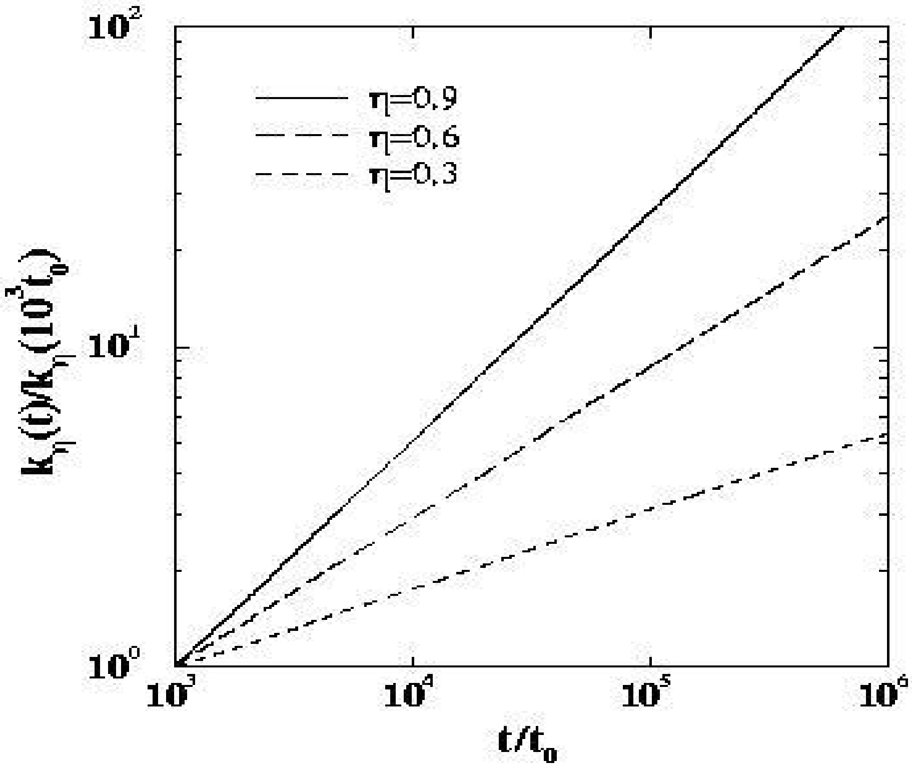
SF model: *k(t)~t* $^{1/2}$ (first mover advantage)

Real systems: nodes compete for links

Fitness Model: fitness (η)

$$\boxed{\Pi(k_i) \cong \frac{\eta_i k_i}{\sum_j \eta_j k_j}}$$

$$k(\eta,t) \sim t^{\beta(\eta)}$$

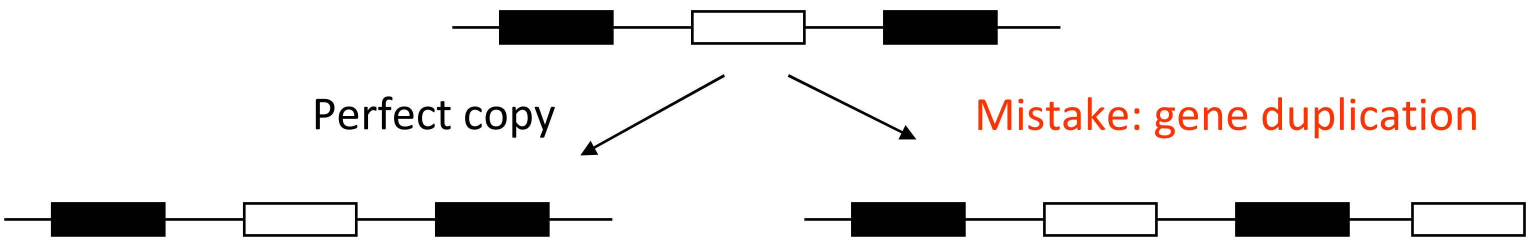


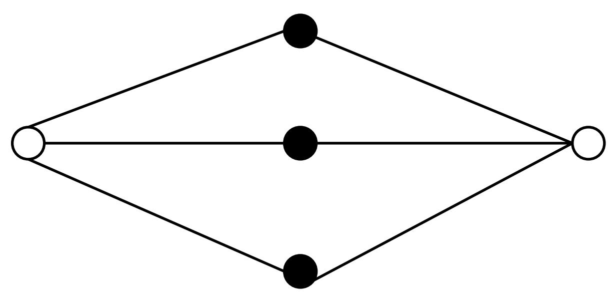
G. Bianconi and A.-L. Barabási, *Europhyics Letters.* **54**, 436 (2001).

Other Models

New concept or mechanism	Limits of γ	Reference
Linear growth, linear pref. attachment	$\gamma=3$	Barabási and Albert, 1999
Nonlinear preferential attachment $\Pi(k_i)\sim k_i^{\alpha}$	no scaling for $\alpha\neq 1$	Krapivsky, Redner, and Leyvraz, 2000
Asymptotically linear pref. attachment $\Pi(k_i)\sim a_\infty k_i$ as $k_i\to\infty$	$\gamma\to 2$ if $a_\infty\to\infty$ $\gamma\to\infty$ if $a_\infty\to 0$	Krapivsky, Redner, and Leyvraz, 2000
Initial attractiveness $\Pi(k_i)\sim A+k_i$	$\gamma=2$ if $A=0$ $\gamma\to\infty$ if $A\to\infty$	Dorogovtsev, Mendes, and Samukhin, 2000a, 2000b
Accelerating growth $\langle k\rangle\sim t^{\theta}$ constant initial attractiveness	$\gamma=1.5$ if $\theta\to 1$ $\gamma\to 2$ if $\theta\to 0$	Dorogovtsev and Mendes, 2001a
Accelerating growth $\langle k\rangle=at+2b$	$\gamma=1.5$ for $k\ll k_c(t)$ $\gamma=3$ for $k\gg k_c(t)$	Barabási *et al.*, 2001 Dorogovtsev and Mendes, 2001c
Internal edges with probab. p Rewiring of edges with probab. q	$\gamma=2$ if $q=\frac{1-p+m}{1+2m}$ $\gamma\to\infty$ if $p,q,m\to 0$	Albert and Barabási, 2000
c internal edges or removal of c edges	$\gamma\to 2$ if $c\to\infty$ $\gamma\to\infty$ if $c\to -1$	Dorogovtsev and Mendes, 2000c
Gradual aging $\Pi(k_i)\sim k_i(t-t_i)^{-\nu}$	$\gamma\to 2$ if $\nu\to-\infty$ $\gamma\to\infty$ if $\nu\to 1$	Dorogovtsev and Mendes, 2000b
Multiplicative node fitness $\Pi_i\sim\eta_i k_i$	$P(k)\sim\frac{k^{-1-C}}{\ln(k)}$	Bianconi and Barabási, 2001a
Additive-multiplicative fitness $\Pi_i\sim\eta_i(k_i-1)+\zeta_i$	$P(k)\sim\frac{k^{-1-m}}{\ln(k)}$ $1\leq m\leq 2$	Ergün and Rodgers, 2001
Edge inheritance	$P(k_{in})=\frac{d}{k_{in}^{\sqrt{2}}}\ln(ak_{in})$	Dorogovtsev, Mendes, and Samukhin, 2000c
Copying with probab. p	$\gamma=(2-p)/(1-p)$	Kumar *et al.*, 2000a, 2000b
Redirection with probab. r	$\gamma=1+1/r$	Krapivsky and Redner, 2001
Walking with probab. p	$\gamma=2$ for $p>p_c$	Vázquez, 2000
Attaching to edges	$\gamma=3$	Dorogovtsev, Mendes, and Samukhin, 2001a
p directed internal edges $\Pi(k_i,k_j)\propto(k_i^{in}+\lambda)(k_j^{out}+\mu)$	$\gamma_{in}=2+p\lambda$ $\gamma_{out}=1+(1-p)^{-1}+\mu p/(1-p)$	Krapivsky, Rodgers, and Redner, 2001
$1-p$ directed internal edges Shifted linear pref. activity	$\gamma_{in}=2+p$ $\gamma_{out}=2+3p$	Tadić, 2001a

Another origin of the scale-free topology: Gene Duplication





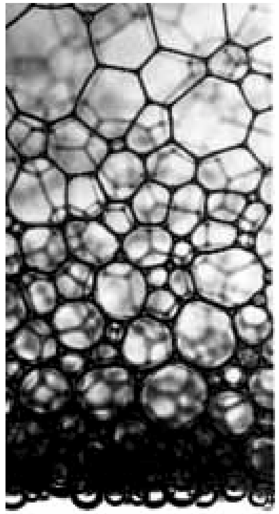
Proteins with more interactions are more likely to get a new link:
Π(k)~k
(preferential attachment).

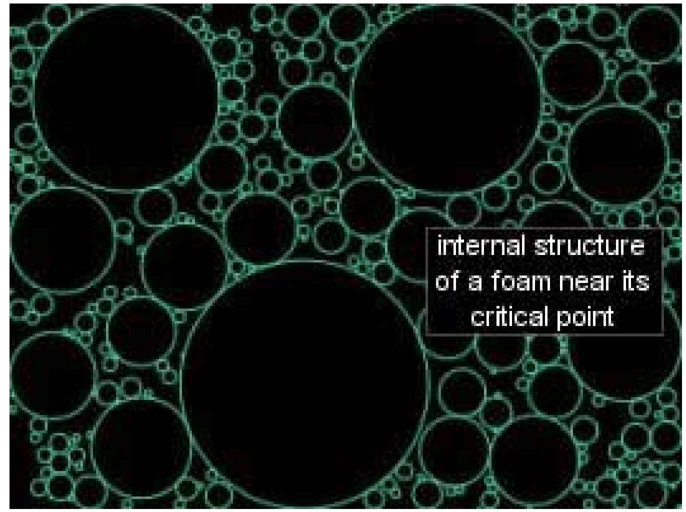
Wagner (2001); Vazquez *et al.* 2003; Sole *et al.* 2001; Rzhetsky & Gomez (2001); Qian et al. (2001); Bhan *et al.* (2002).

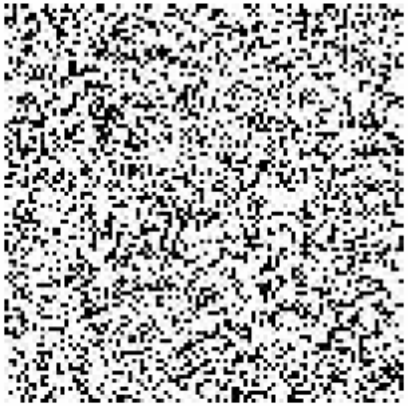
Power laws elsewhere
Is there a connection?

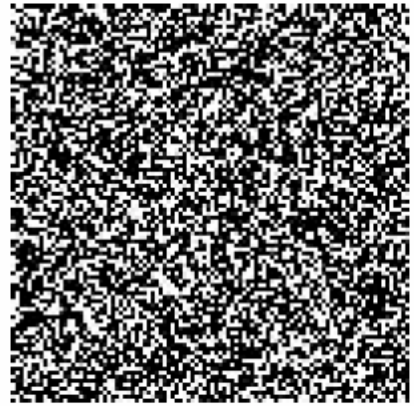
- Phase transitions
- Self-organized criticality
- Absence of a characteristic scale

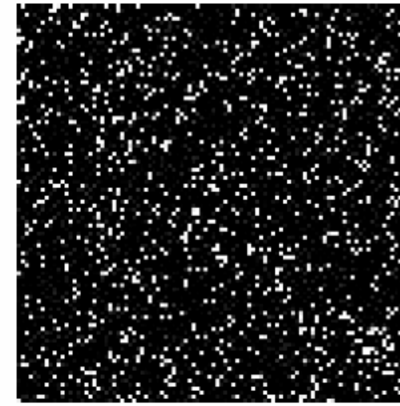
Phase transitions

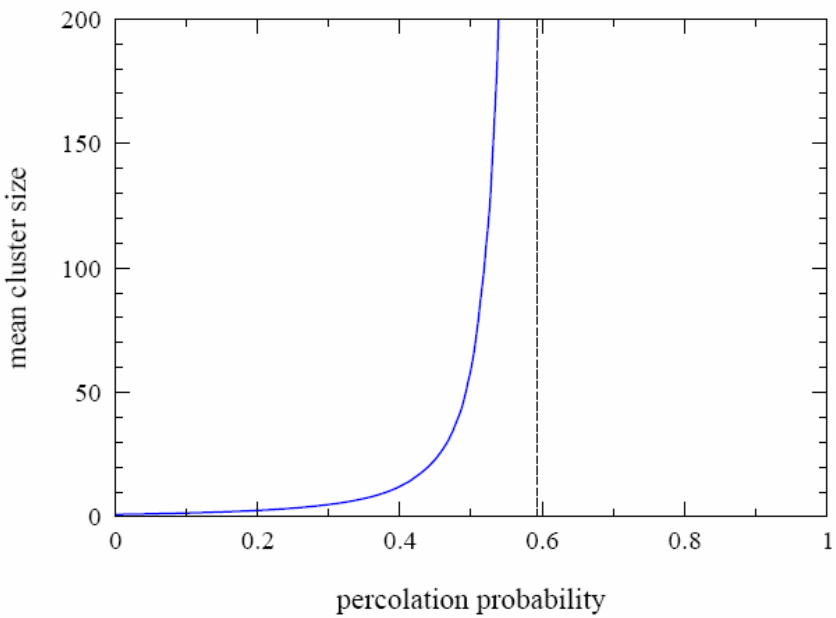




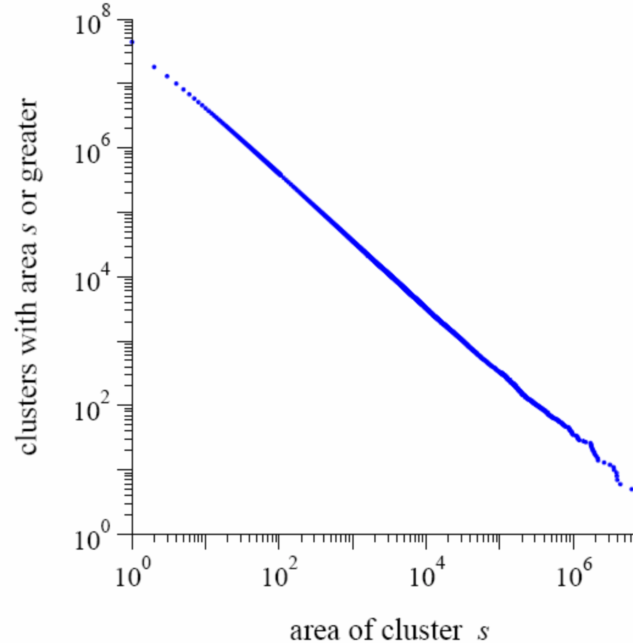




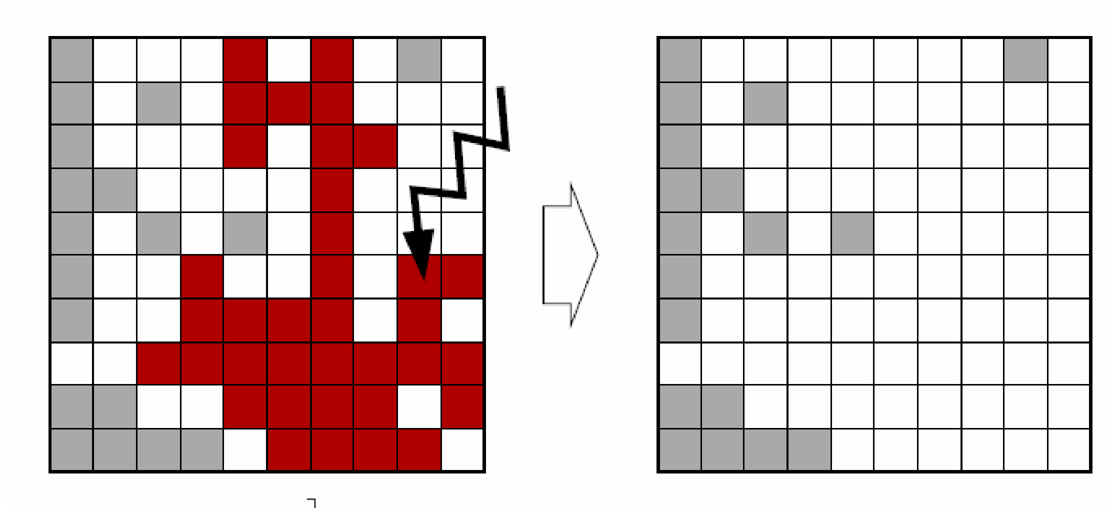


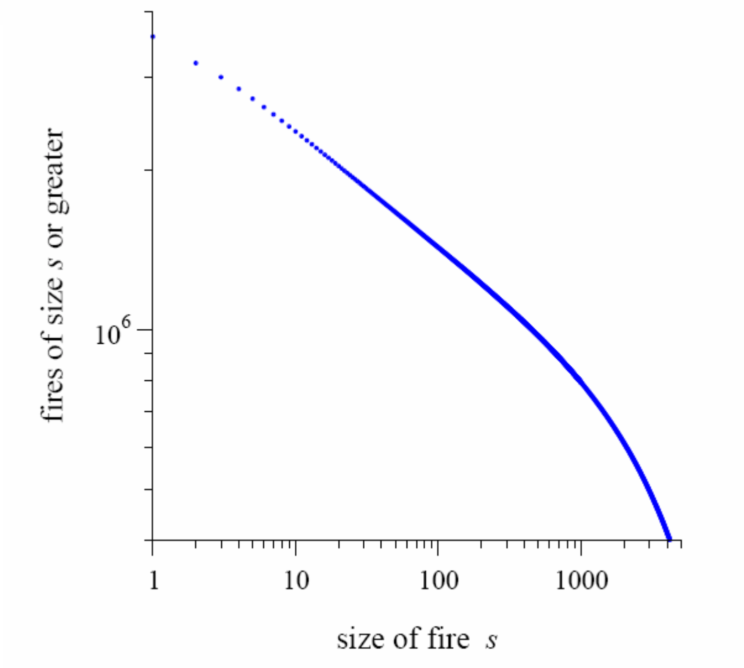


mean cluster size
200
150
100
50
0
0
0.2
0.4
0.6
0.8
1
percolation probability



clusters with area s or greater
10^8
10^6
10^4
10^2
10^0
10^0
10^2
10^4
10^6
area of cluster s





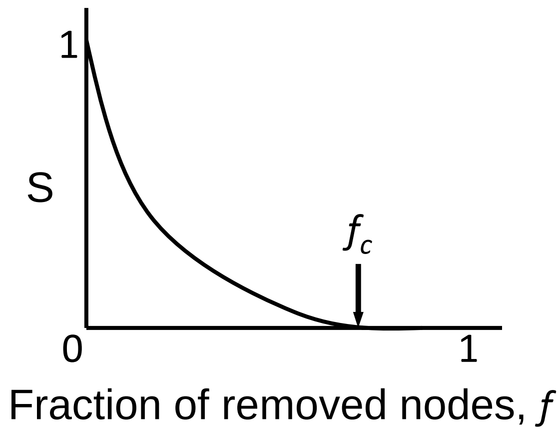
Self-Organized Criticality

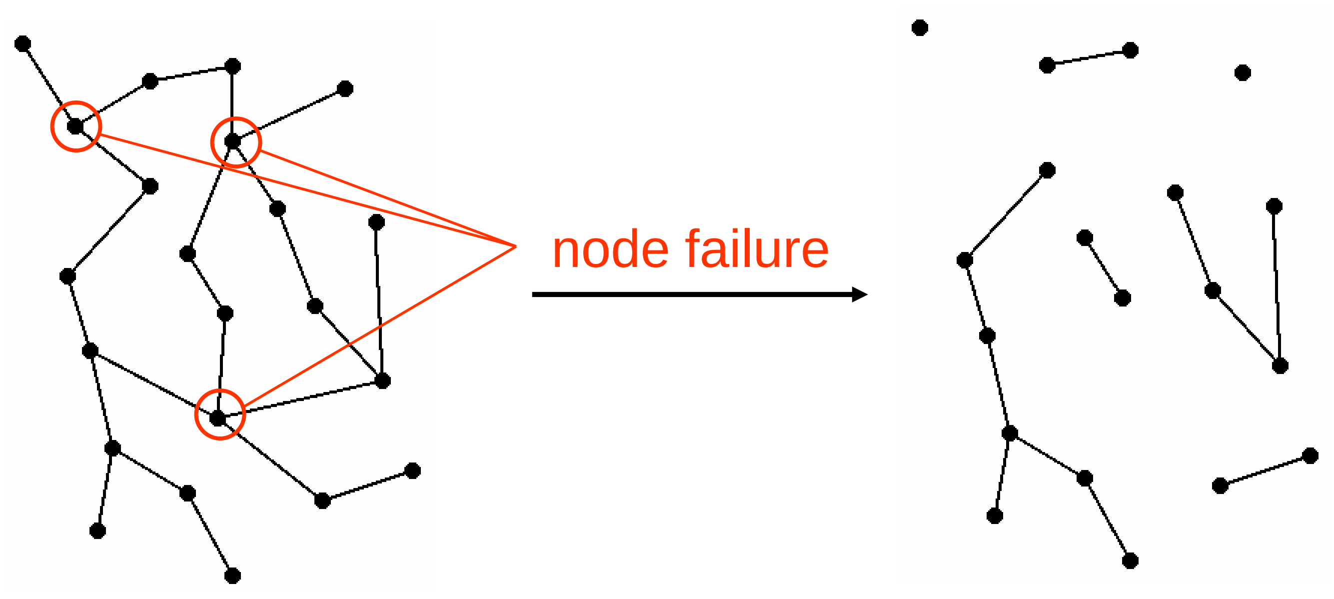
Bak, P., Tang, C. and Wiesenfeld, K. (1987). "Self-organized criticality: an explanation of 1 / *f* noise". *Physical Review Letters* **59**: 381–384.

Error and attack tolerance

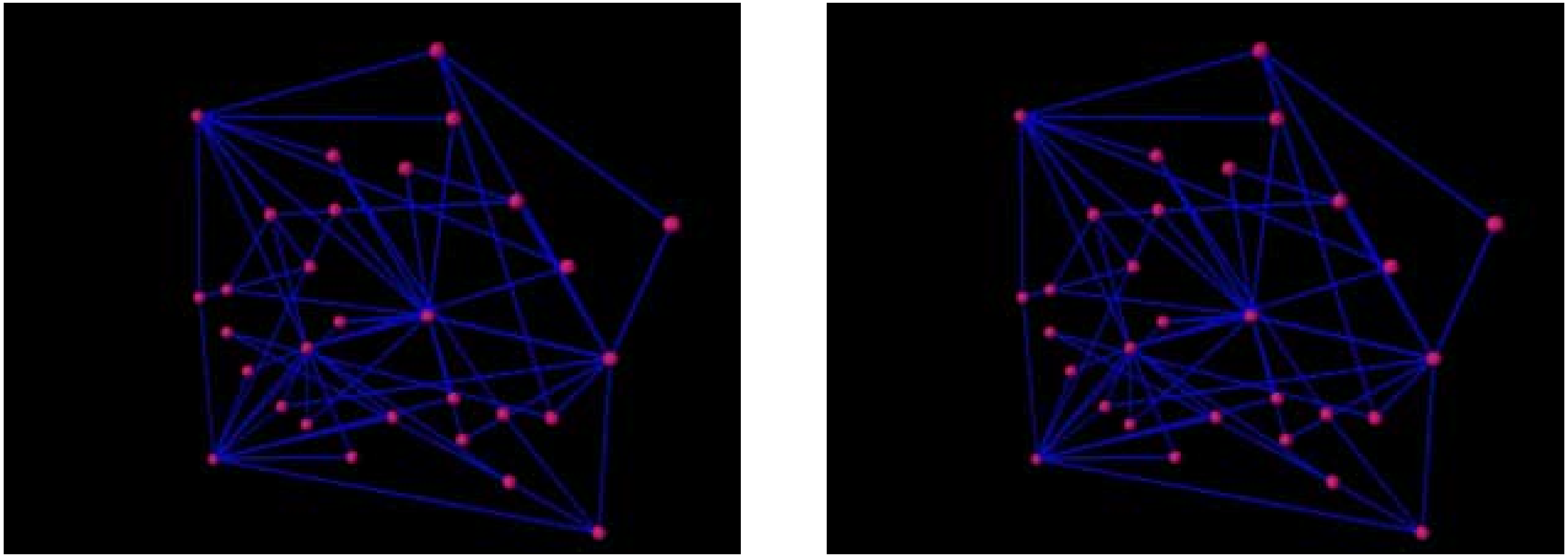
Robustness

Complex systems maintain their basic functions even under errors and failures
(cell $\rightarrow$ mutations; Internet $\rightarrow$ router breakdowns)



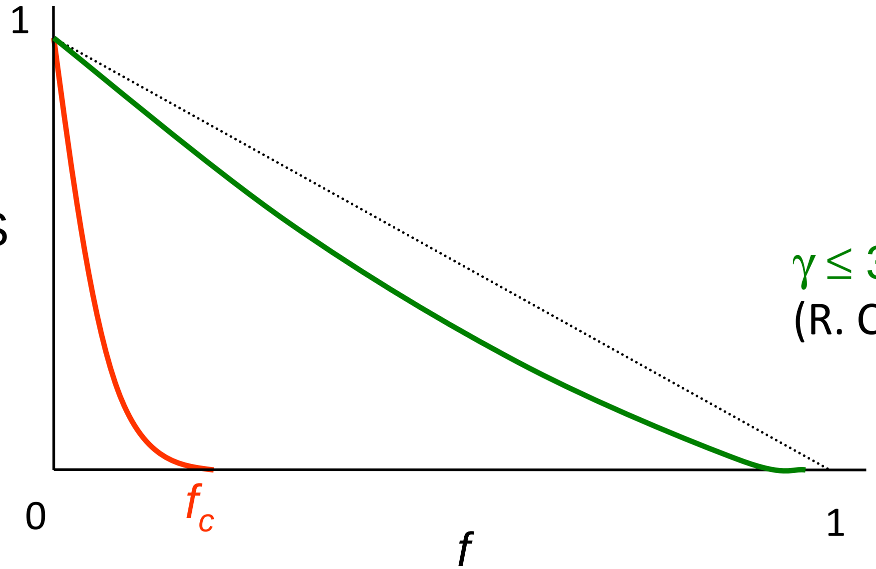


Robustness of scale-free networks



Attacks

Failures



$\gamma \leq 3$: f_c=1

(R. Cohen et al PRL, 2000)

C

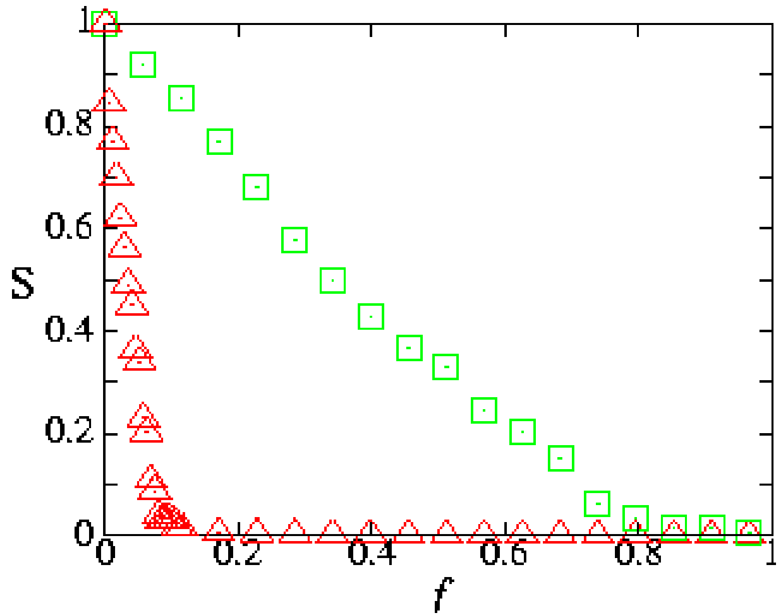
Albert, Jeong, Barabasi, *Nature* 406 378 (2000)

Achilles' Heel of complex networks

failure

attack

Internet



R. Albert, H. Jeong, A.L. Barabasi, *Nature* 406 378 (2000)

Scale free networks

- Natural networks are not random graphs
- Many networks in nature are Scale-Free (SF), meaning that just a few nodes have a disproportionately large number of connections.
- Power-law distributions are ubiquitous in nature.
- While power-laws are associated with critical points in nature, systems can self-organize to this critical state.
- Important dynamical implications of the Scale-Free topology:
  - SF Networks are robust to failures, yet vulnerable to targeted attacks
  - The SF structure has important implications for the dynamics of the agents on the networks (epidemics, etc.)