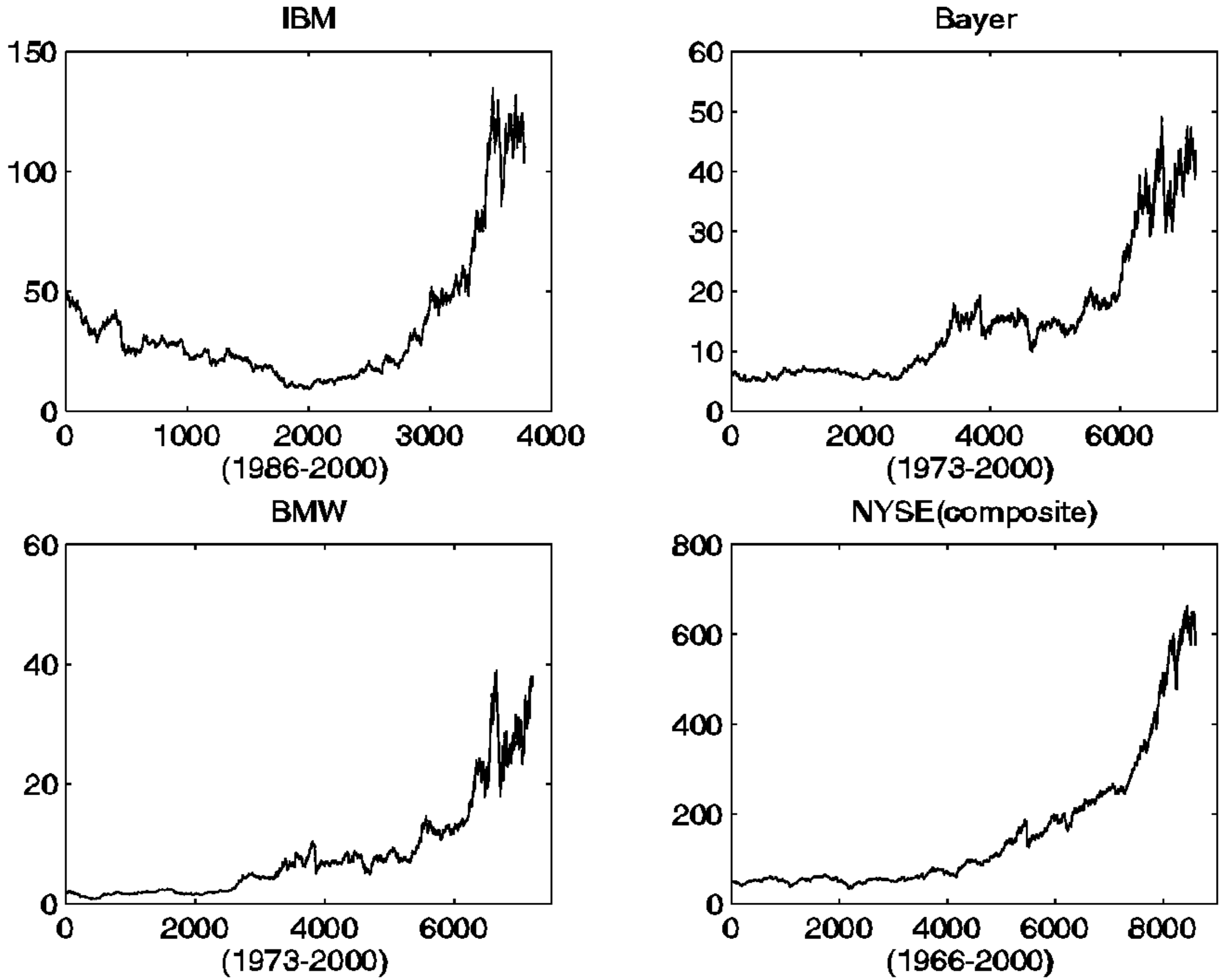
An empirical characterization of the market process

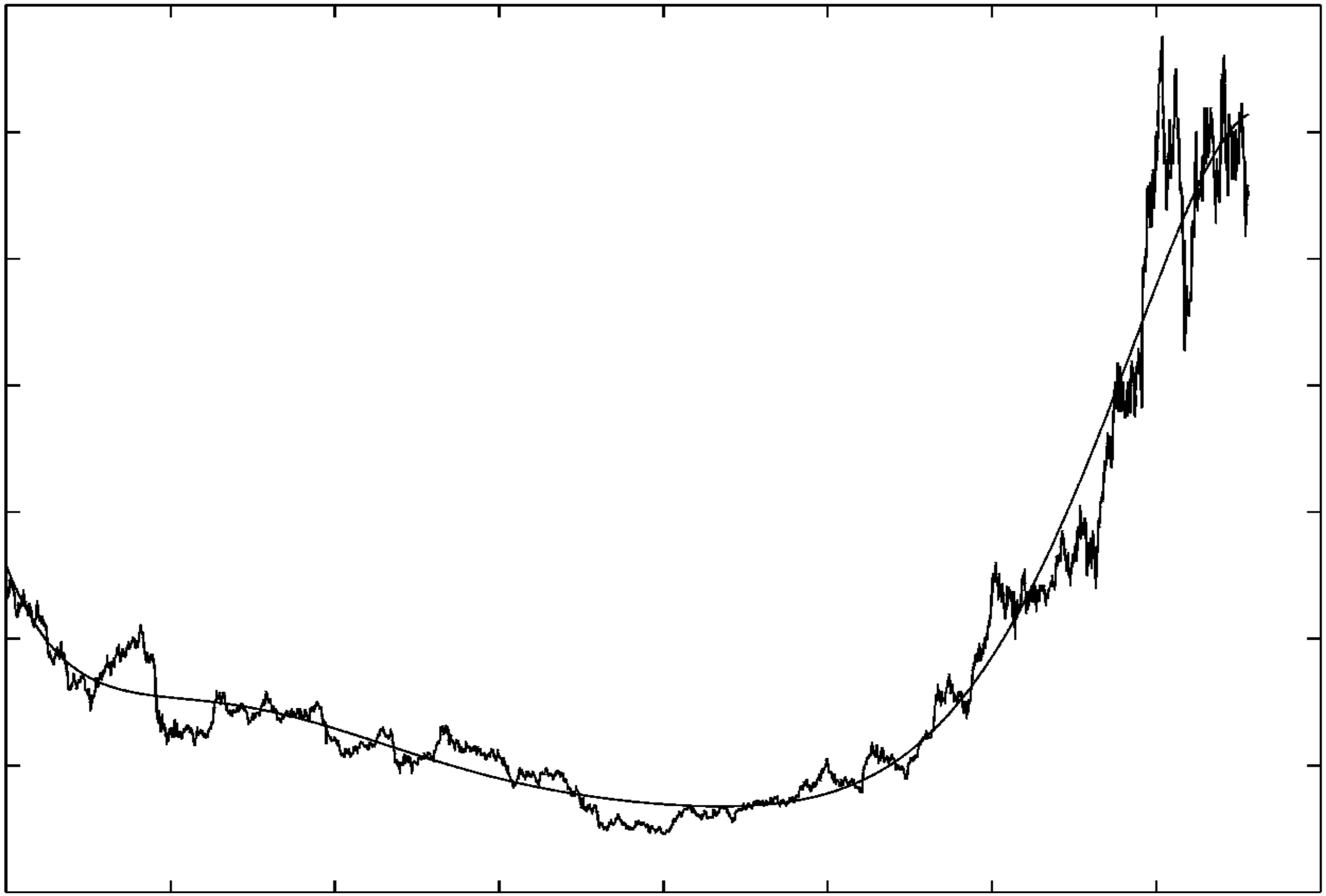
(The fractional volatility model)

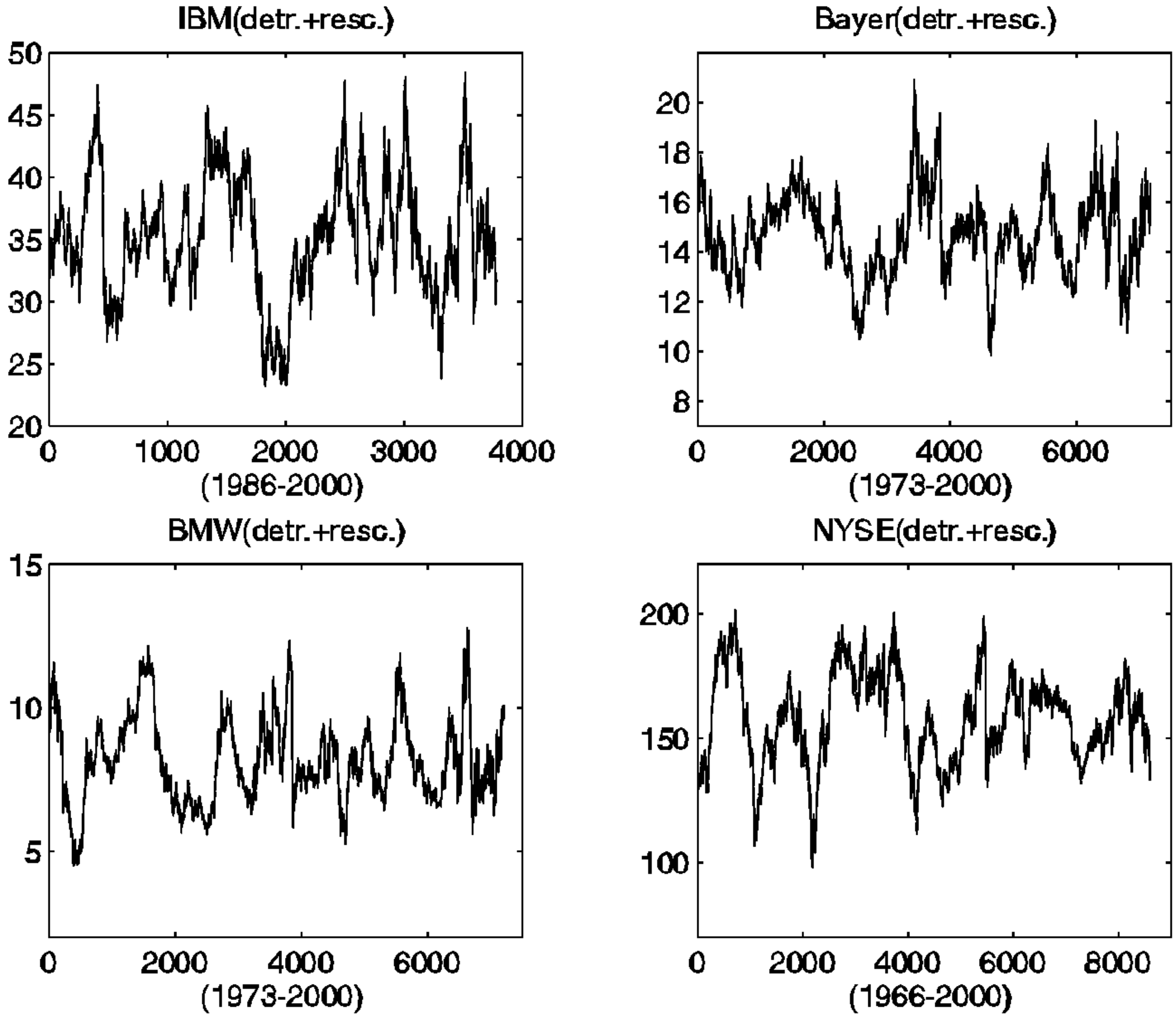
Contents

1. Market prices are non-differentiable



Detrending is needed for stationarity




IBM(detr.+resc.)
50
45
40
35
30
25
20
0
1000
2000
3000
4000
(1986-2000)
Bayer(detr.+resc.)
20
18
16
14
12
10
8
0
2000
4000
6000
(1973-2000)
BMW(detr.+resc.)
15
10
5
0
2000
4000
6000
(1973-2000)
NYSE(detr.+resc.)
200
150
100
0
2000
4000
6000
8000
(1966-2000)

The returns process

- r(t)=log(S(t+1))-log(S(t))

$$\frac{d}{dt}\log(S(t)) = \frac{1}{S(t)}\frac{d}{dt}S(t)$$

- Automatically detrended

2. Geometric Brownian motion ?

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW \qquad \textbf{?}$$

(a basis for most mathematical finance studies – Black-Scholes, etc.)

Consequences:

(i) $$p(\ln\frac{S_T}{S_t}) = \frac{1}{\sqrt{2\pi\sigma^2 (T-t)}} \exp\left(-\frac{\left[\ln\frac{S_T}{S_t} - (\mu - \frac{\sigma^2}{2})(T-t)\right]^2}{2\sigma^2 (T-t)}\right)$$

Price changes would be lognormal

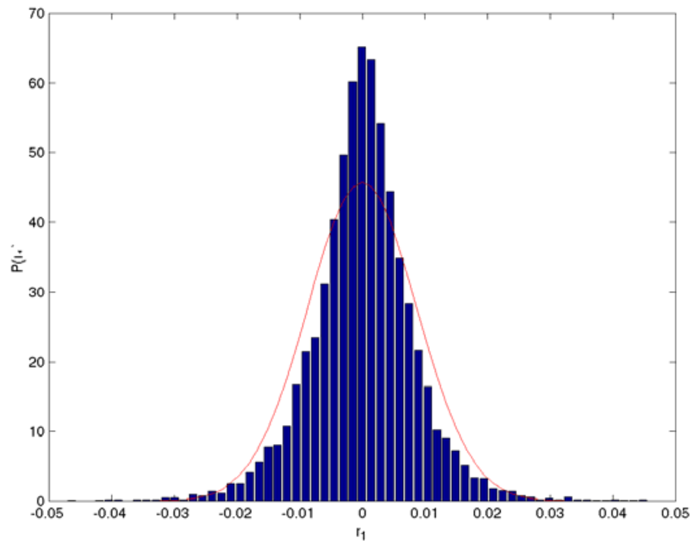
(ii) $$E\left|\frac{S(t+\Delta) - S(t)}{S(t)} - \mu\Delta\right| \approx \Delta^H$$

Self-similar (Law(X_{at})=Law($a^H X_t$)) with Hurst coefficient = 1/2

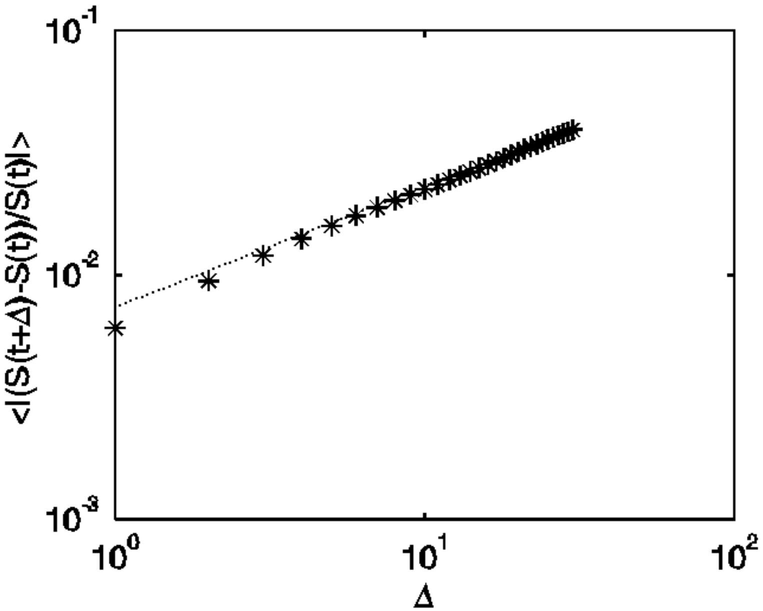
2. Geometric Brownian motion ?

- **Empirical tests :**
- **P(r_1) is not lognormal**

  r_1=log(S(t+1)/S(t))



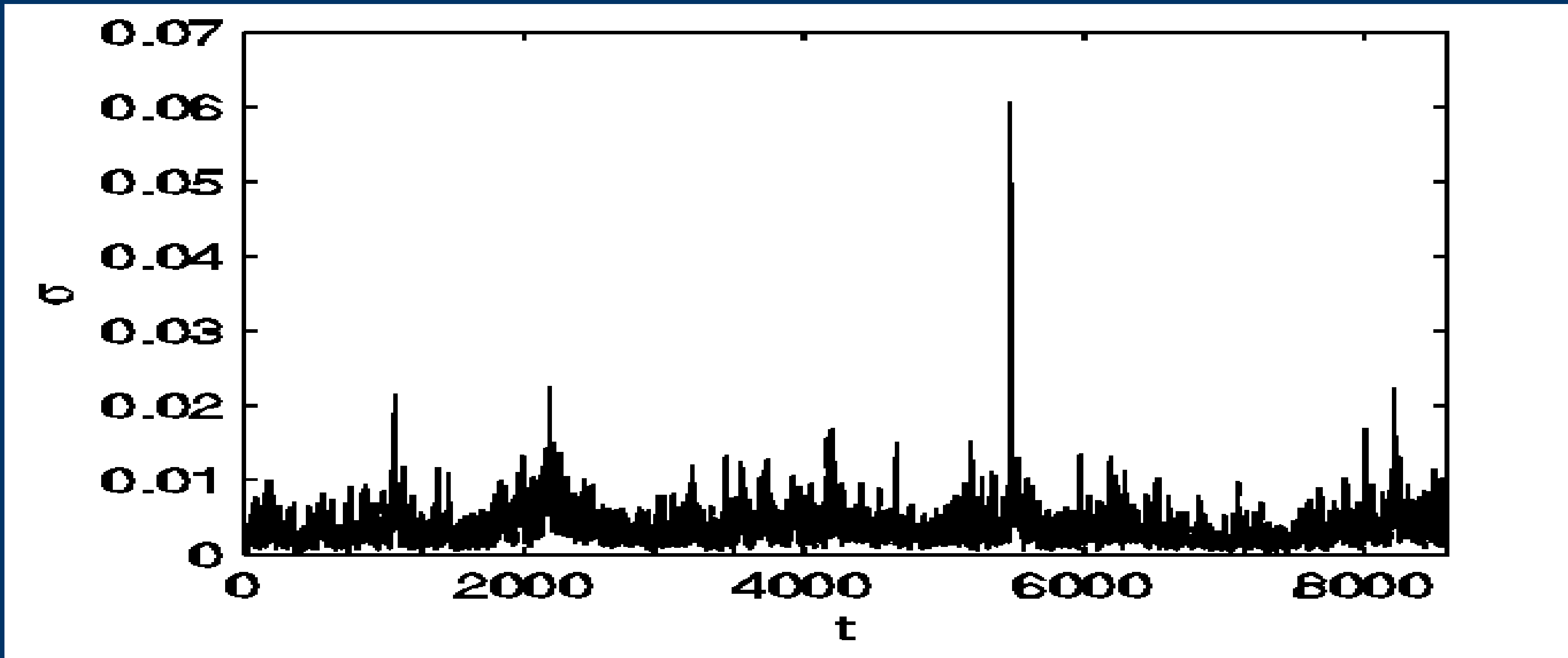
- **Deviations from scaling**
- **Larger deviations for high-frequency data**



σ is not constant

- **Return**

$$r_t(\Delta) = \log(S_t) - \log(S_{t-\Delta}) \qquad r_t(\Delta) \approx \frac{1}{S_t}\frac{dS_t}{dt}\Delta$$



- **Conclusion : Nor do returns follow geometric Brownian motion nor is σ constant (not even a smooth function of S and t)**

“Stilized” experimental facts

- (i) Returns have nearly **no autocorrelation**
- (ii) The autocorrelations of $|r_t|^d$ decline slowly with increasing lag. **Long memory** effect
- (iii) **Leptokurtosis** : asset returns have distributions with fat tails and excess peakedness at the mean
- (iv) Autocorrelations of sign r_t are insignificant
- (v) **Volatility clustering** : tendency of large changes to follow large changes and small changes to follow small changes. Volatility occurs in bursts.
- (vi) **Volatility is mean-reversing** and the distribution is close to lognormal or inverse gamma
- (vii) **Leverage effect** : volatility tends to rise more following a large price fall than following a price rise
- (viii) Why volatility is important : Uncertainty and risk are the driving factors for investors’ behavior

3. Volatility as a process

- When the future is uncertain investors are less likely to invest. Therefore uncertainty (volatility) would have to be changing over time. "... build a forecasting model for variance and make it a well-defined process ..." (Robert Engle – 1982)
- Structural model

$$\beta_1, \beta_2, \beta_3 \ldots \quad factors$$

$$u \quad error\ term$$

$$y = \beta_1 + \beta_2 x_2 + \beta_3 x_3 + \ldots + u$$

- Conditional variance

$$\sigma_t^2 = E\left[u_t^2 \middle| u_{t-1}, u_{t-2}, \ldots\right]$$

- Homoscedasticity = variance of errors is constant
- Heteroscedasticity = variance of errors is not constant

2. Volatility models

- **ARCH(q) (Autoregressive conditionally heteroscedastic)**

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \alpha_2 u_{t-2}^2 + \ldots + \alpha_q u_{t-q}^2$$

- **GARCH (1,1) (Generalized ARCH)**

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \beta \sigma_{t-1}^2$$

- **IGARCH (Integrated GARCH)**

$$\alpha_1 + \beta = 1$$

- **Leverage :**
  GJR (Glosten, Jagannathan, Runkle)

$$\sigma_t^2 = \alpha_0 + (\alpha_1 + \gamma I_{t-1}) u_{t-1}^2 + \beta \sigma_{t-1}^2$$

$$I_{t-1} = 1 \quad if \ u_{t-1} < 0 \quad ; = 0 \quad otherwise$$

EGARCH (exponential GARCH)

$$\ln(\sigma_t^2) = \omega + \beta \ln(\sigma_{t-1}^2) + \gamma \frac{u_{t-1}}{\sqrt{\sigma_{t-1}^2}} + \alpha \left[\frac{|u_{t-1}|}{\sqrt{\sigma_{t-1}^2}} - \sqrt{\frac{2}{\pi}} \right]$$

Stochastic volatility models

- **In GARCH models, the conditional volatility is a deterministic function of past quantities. In Stochastic Volatility models it is itself a random process.**
- **Heston model**

$$dS_t = S_t(\mu dt + \sigma_t dW) \qquad \langle dWdW' \rangle = \rho < 0$$
$$d(\sigma_t^2) = -\Omega(\sigma_t^2 - \sigma_0^2)dt + \gamma\sigma_t dW'$$

- **Two-time scales model (Perello, Masoliver)**

$$dS_t = S_t(\mu dt + e^{\xi_t} dW) \qquad \langle dWdW' \rangle = \rho < 0$$
$$d\xi_t = -\Omega(\xi_t - \xi_{0t})dt + \gamma dW' \qquad \langle dWdW'' \rangle = 0$$
$$d\xi_{0t} = -\Omega_0(\xi_{0t} - \xi_{00})dt + \gamma_0 dW''$$

- **Comte, Renault**

$$dS_t = S_t(\mu dt + \sigma_t dW)$$
$$d(\ln \sigma_t) = k(\theta - \ln \sigma_t)dt + \gamma dW'$$

W' is fractional Brownian motion

4. The induced volatility process

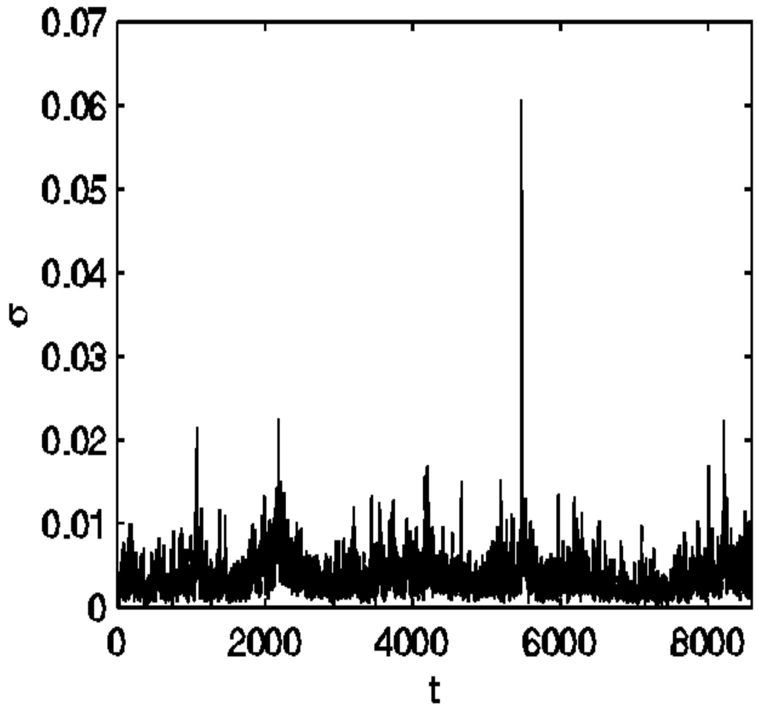
- **Let $\log S_t$ be a stochastic process defined on a tensor product probability space $\Omega \otimes \Omega'$**
- **$\log S_t(\omega,\omega')$ with Ω being Wiener space**
- **(M1) Then, if $\log S_t(\omega,\omega')$ is square integrable in Ω**

$$\frac{dS_t(\bullet,\omega')}{S_t} = \mu dt + \sigma(\bullet,\omega')dB_t$$ **for fixed ω'**

- **$\sigma_t(\omega, \omega')$ is called the "Induced volatility"**

  (E1) Obtained from the data $\sigma_t^2(\cdot, \omega') \approx \mathrm{var}(\log S_t)/(T_0 - T_1)$

  (μ=0)

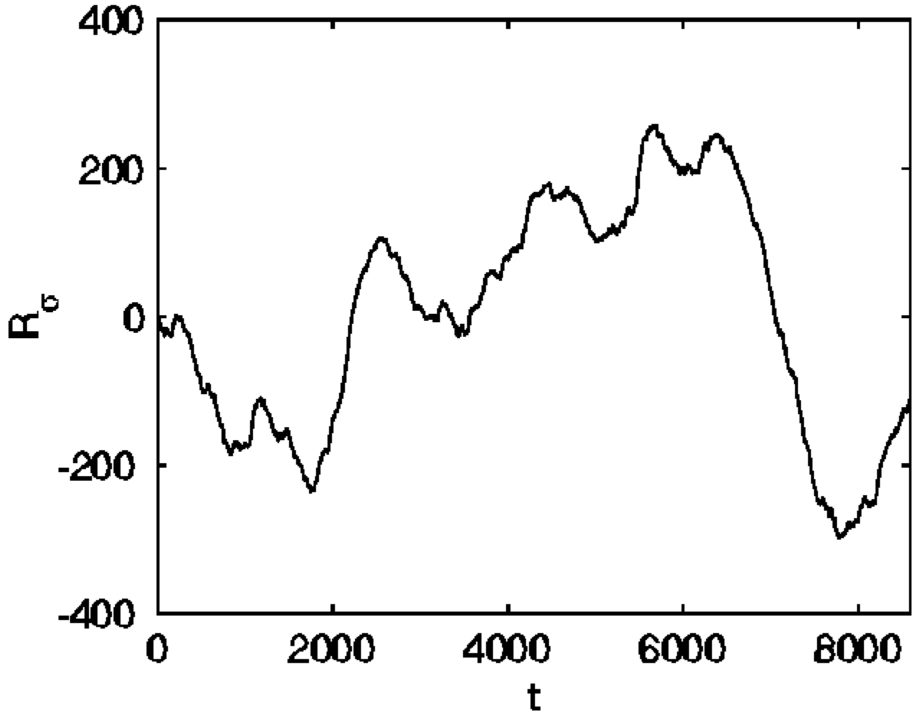


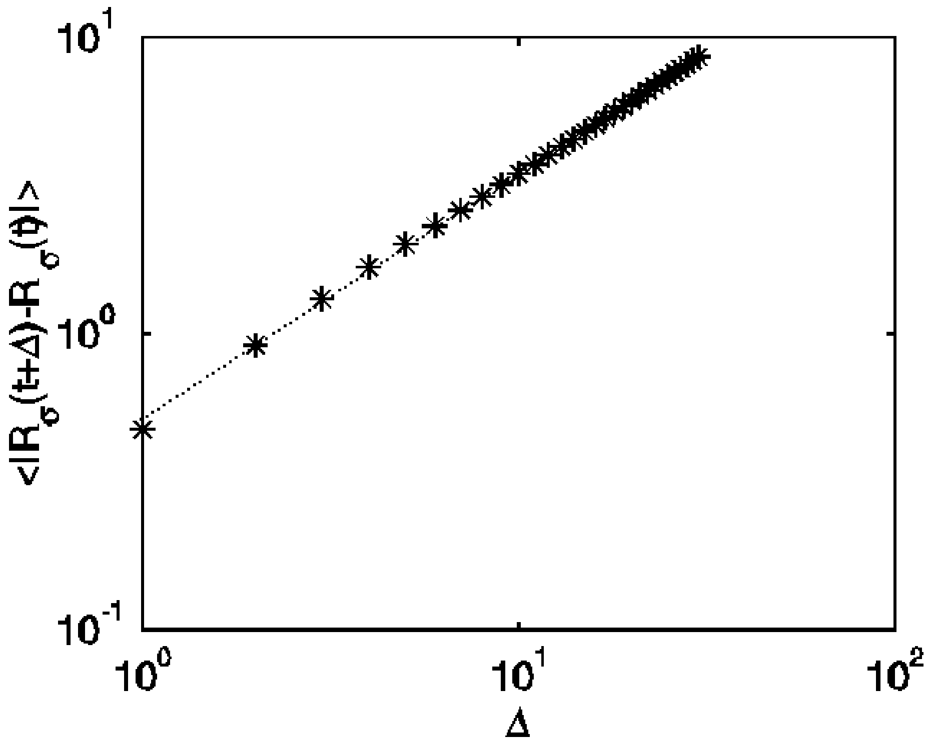
What does the data suggest for σ_t ?

- **σ_t is not self similar**
- **However $R_\sigma(t)$ is**
  $\Sigma \log \sigma(n\delta) = \beta t + R_\sigma(t)$
  $H \approx 0.8 - 0.9$

$$E\left|\frac{\sigma(t+\Delta)-\sigma(t)}{\sigma(t)}\right| \neq \Delta^H$$

$$E\left|\frac{R_\sigma(t+\Delta)-R_\sigma(t)}{R_\sigma(t)}\right| = \Delta^H$$





What does the data suggest for σ_t ?

- **Recall:**
  If a process X_t has finite variance, stationary increments and is self-similar, then
  $Cov(X_{s,}, X_t)=(|s|^{2H}+|t|^{2H}-|s-t|^{2H})E(X_1^2)$
  (M2) The simplest such process is a zero-mean Gaussian process, Fractional Brownian motion $B^H{}_t$ with long-range dependence for H>1/2
- **Conclusion :**
  $\log \sigma_t = \beta + (k/\delta) (B^H{}_t - B^H{}_{t-\delta})$
  σ_t modeled by a stochastic exponential of fractional noise

The fractional volatility model (FVM)

- **Two coupled processes :**

$$dS_t = \mu\, S_t\, dt + \sigma_t\, S_t\, dB_t$$

$$\log \sigma_t = \beta + (k/\delta)\,(\, B^H_t - B^H_{t-\delta}\,)$$

- **log σ_t driven by fractional noise, not by fractional Brownian motion**

5. Time scales and pdf's

- **From**

$$\ln \sigma_t = \beta + \frac{k}{\delta}\left(B_H(t) - B_H(t-\delta)\right)$$

$$\sigma_t = \theta \exp\left[\frac{k}{\delta}\left(B(t) - B(t-\delta)\right) - \frac{1}{2}\left(\frac{k}{\delta}\right)^2 \delta^{2H}\right]$$

- **log σ_t is a Gaussian process with mean β and covariance**

$$\psi(s,u) = \frac{k^2}{2\delta^2}\left\{\left|s-u+\delta\right|^{2H} + \left|u-s+\delta\right|^{2H} - 2\left|s-u\right|^{2H}\right\}$$

- **Then**

$$p_\delta(\sigma) = \frac{1}{\sqrt{2\pi}\sigma k \delta^{H-1}} \exp\left\{-\frac{(\log\sigma - \beta)^2}{2k^2\delta^{2H-2}}\right\}$$

5. Time scales and pdf's

- **and for the returns**

$$P_\delta(\log\frac{S_{t+\Delta}}{S_t}) \cong \int_0^\infty d\sigma\, p_\delta(\sigma)\, p_\sigma(\log\frac{S_{t+\Delta}}{S_t})$$

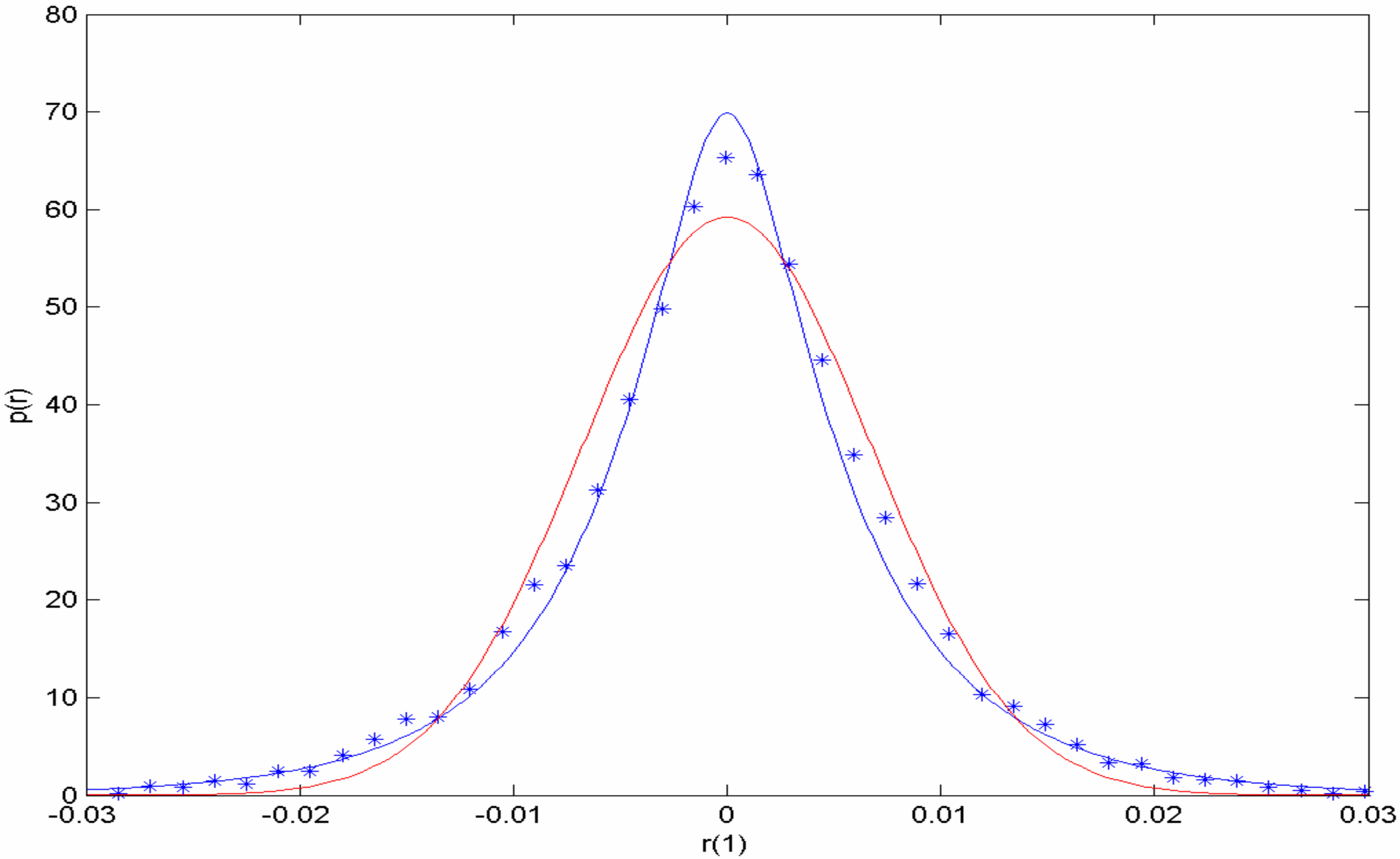
with

$$p_\sigma(\log\frac{S_{t+\Delta}}{S_t}) = \frac{1}{\sqrt{2\pi\sigma^2\Delta}}\exp\left\{\frac{\left(\log\frac{S_{t+\Delta}}{S_t} - \left(\mu - \frac{\sigma^2}{2}\right)\Delta\right)^2}{2\sigma^2\Delta}\right\}$$

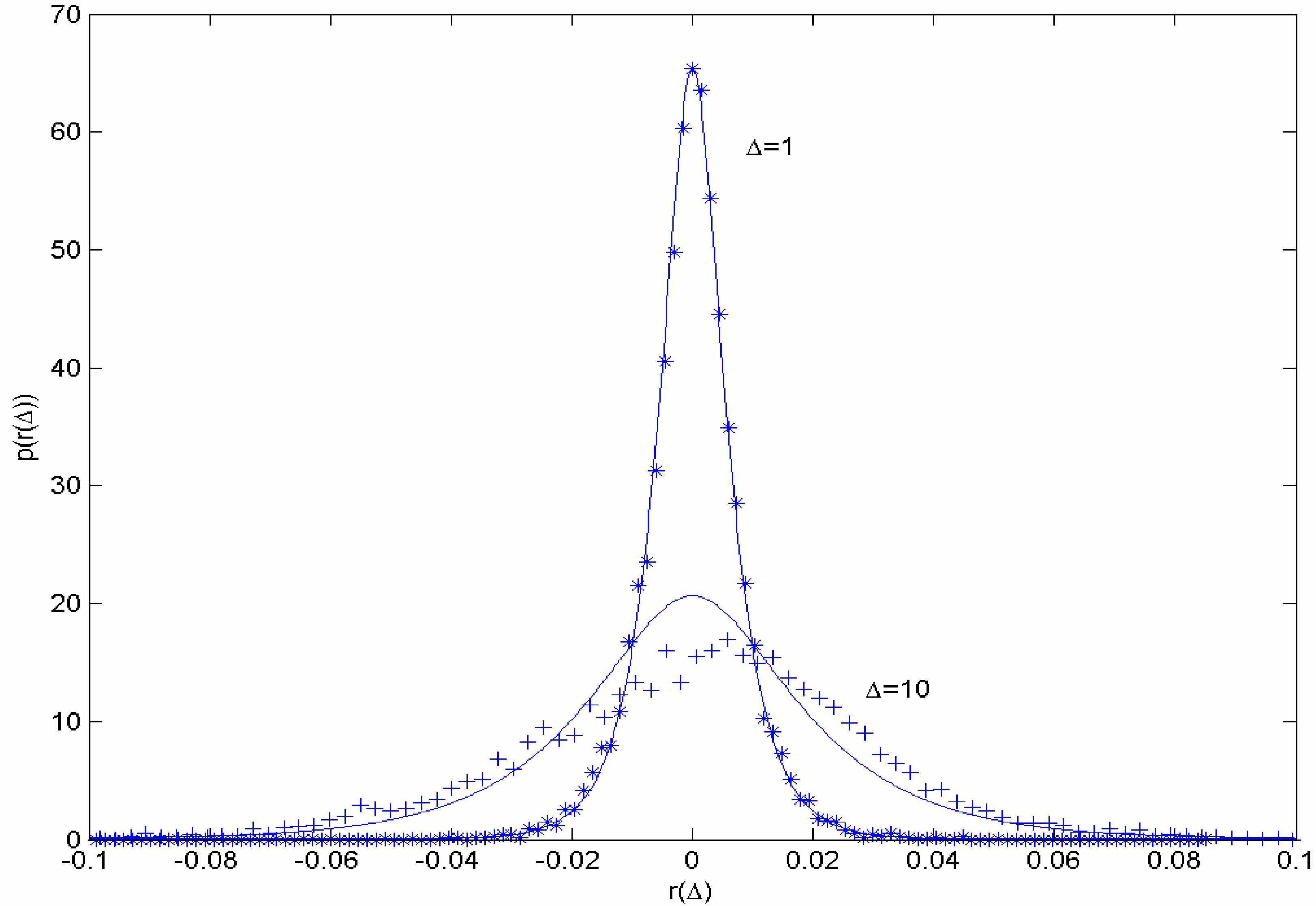
- **The probability distribution of the returns depends on the observation time scale δ**

5. Time scales and pdf's (NYSE 1973-2000)

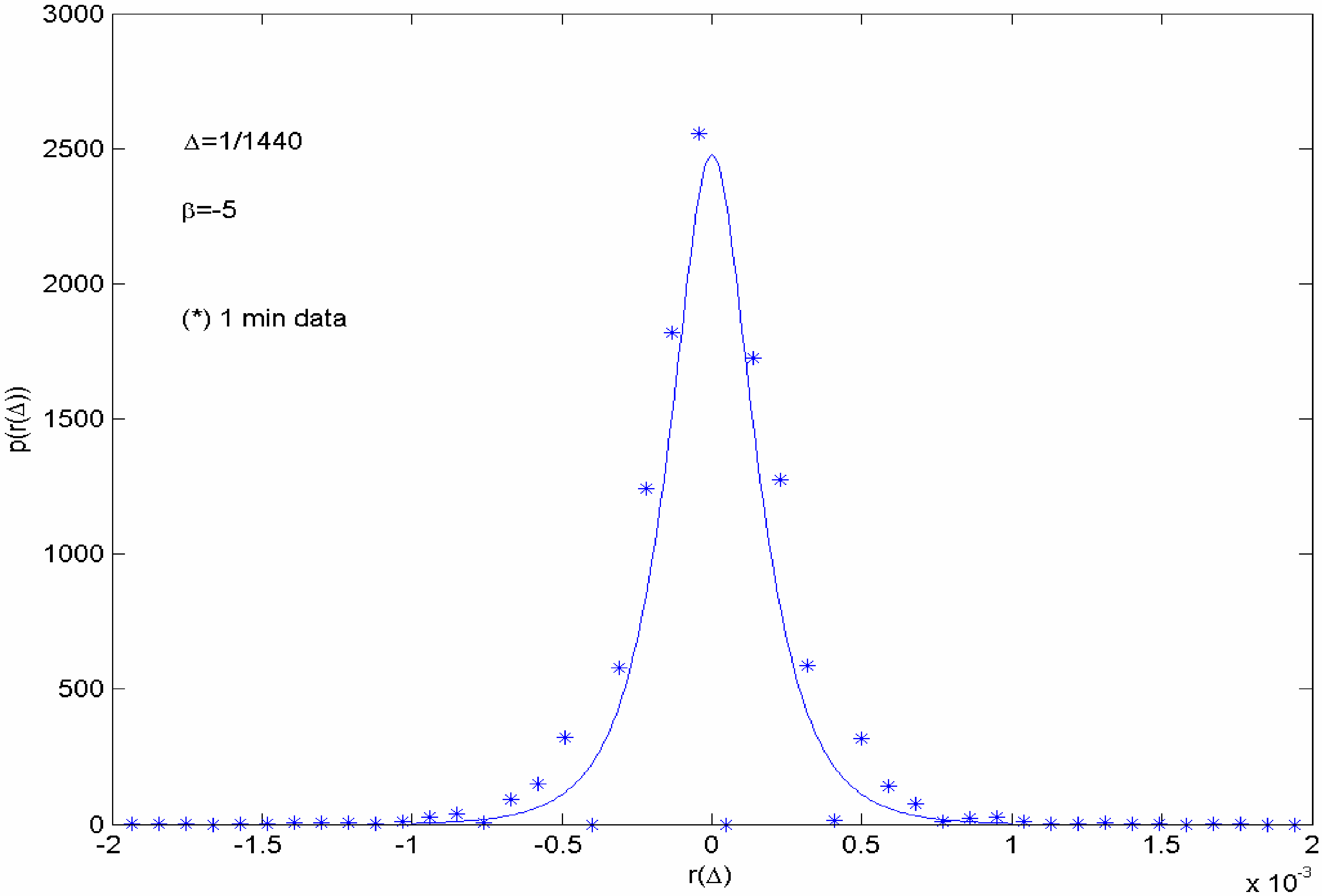
- **H=0.83 k=0.59 β= - 5 δ=1 Δ=1**



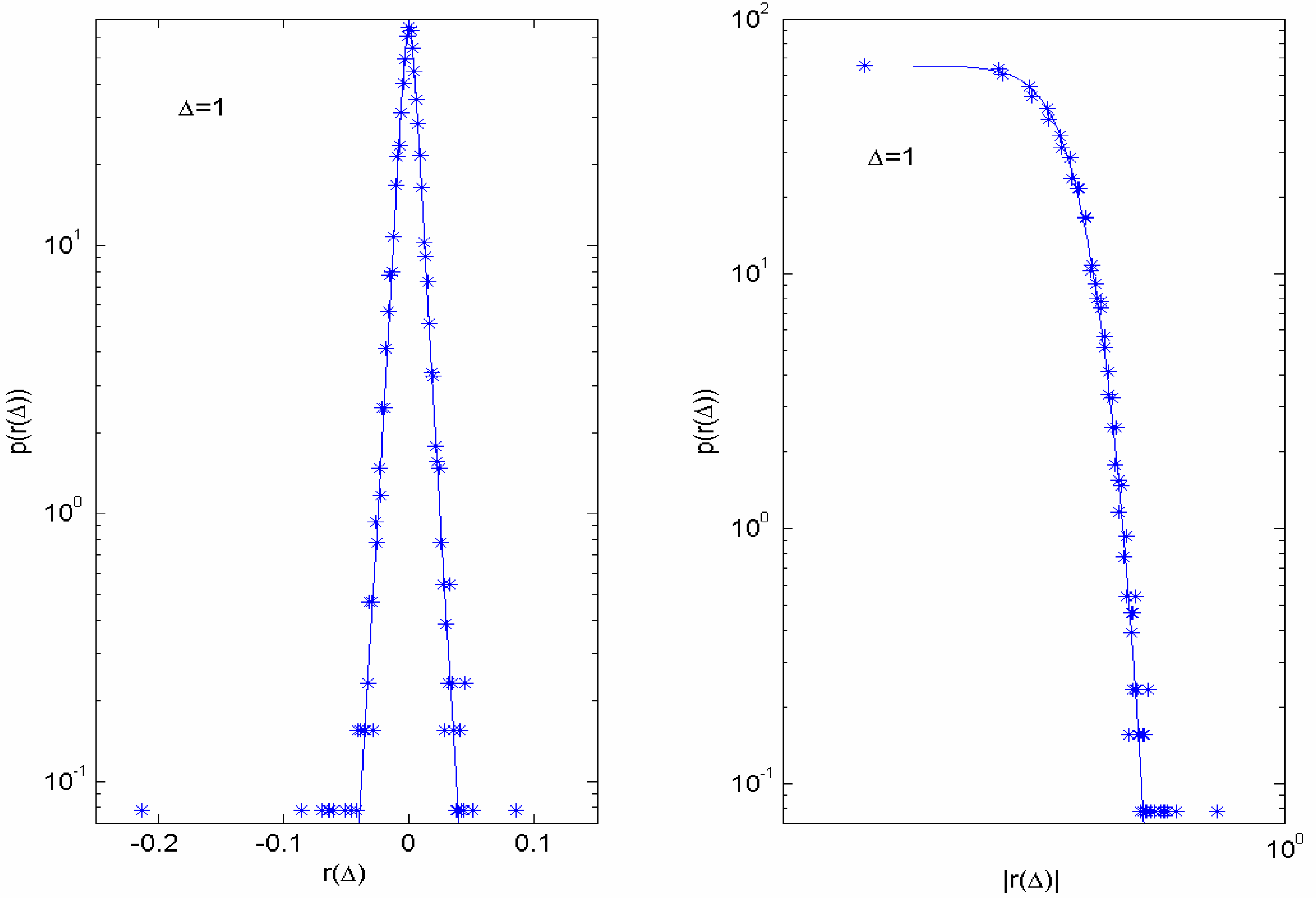
6. Time scales and pdf's (NYSE 1973-2000)



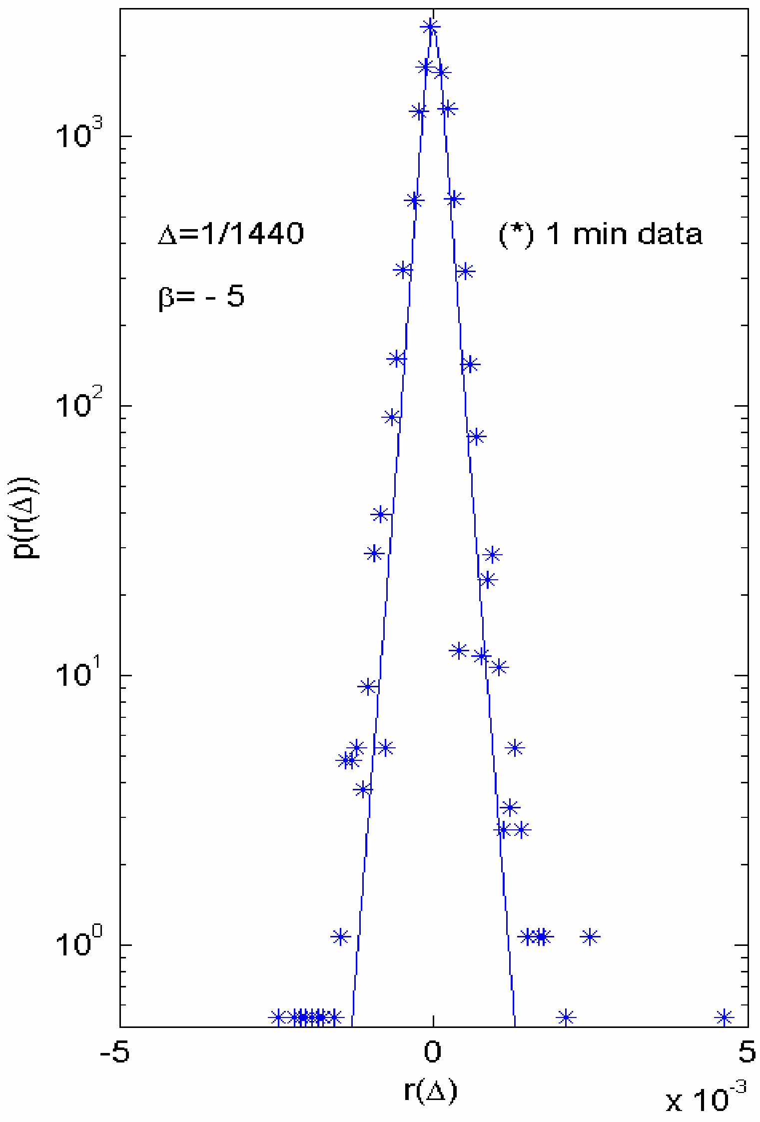
6. Time scales and pdf's (USD-Euro 05-06 2001)

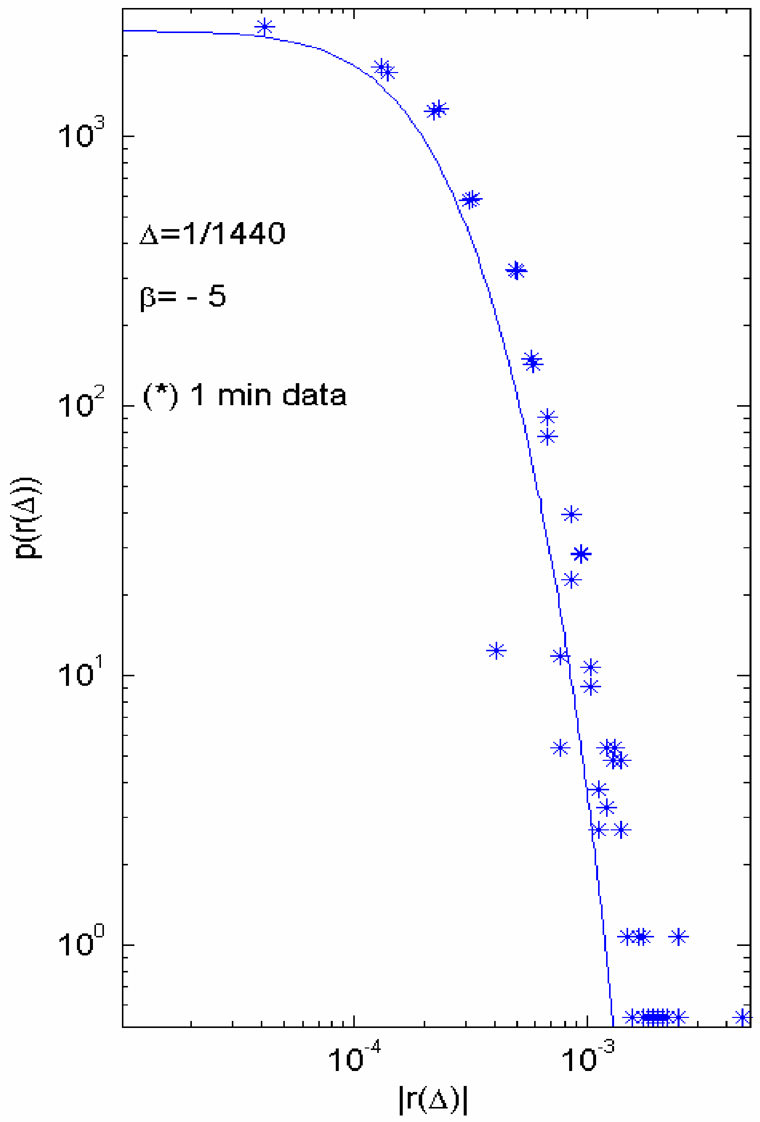


6. Time scales and pdf's (Scaling ??)



6. Time scales and pdf's (Scaling ??)





Closed form and return asymptotics

- **From**

$$P_\delta(r(\Delta)) = \int_0^\infty d\sigma\, p_\delta(\sigma)\, p_\sigma(r(\Delta))$$

one obtains

$$P_\delta(r(\Delta)) = \frac{1}{4\theta\pi k\delta^{H-1}\sqrt{\Delta}} \frac{1}{\sqrt{\lambda}} e^{-\frac{1}{C}\left(\log\lambda - \frac{d}{dz}\right)^2} \Gamma(z)\Big|_{z=\frac{1}{2}}$$

$$\theta = e^\beta \quad C = 8k^2\delta^{2H-2} \quad \lambda = \frac{\left(r(\Delta)-r_0\right)^2}{2\Delta\theta^2}$$

Asymptotic behavior :

$$P_\delta(r(\Delta)) \approx \frac{1}{\sqrt{\Delta}} \frac{1}{\sqrt{\lambda}} e^{-\frac{1}{C}(\log\lambda)^2}$$

Appendix:
Derivation of the Black-Scholes formula

- Assumptions:
  1) The price of the **underlying instrument** S_t follows a **geometric Brownian motion** defined by

  $$dS_t = \mu S_t\,dt + \sigma S_t\,dW_t$$

  where W_t is a **Wiener process** with **constant** drift μ and **volatility** σ.
  2) It is possible to **short sell** the underlying stock.
  3) There are no **arbitrage** opportunities.
  4) Trading in the stock is continuous.
  5) There are no **transaction costs** or **taxes**.
  6) All securities are perfectly divisible (*i.e.* it is possible to buy any fraction of a share).
  7)It is possible to borrow and lend cash at a constant **risk-free interest rate**.
  8) The stock does not pay a dividend

- Let $V(S,\sigma)$ be the value of a call option. By **Itō's lemma** we have

$$dV = \left(\mu S \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \sigma S \frac{\partial V}{\partial S} dW.$$

- Now consider a trading strategy under which one holds one option and continuously trades in the stock in order to hold

$$-\frac{\partial V}{\partial S}$$

shares. At time *t*, the value of these holdings will be

$$\Pi = V - S\frac{\partial V}{\partial S}.$$

- The composition of this portfolio, called the **delta-hedge** portfolio, will vary from time-step to time-step. Let *R* denote the accumulated profit or loss from following this strategy. Then over the time period [*t*, *t* + *dt*], the instantaneous profit or loss is

$$dR = dV - \frac{\partial V}{\partial S} dS.$$

- Substituting dV and dS from the equations above we are left with

$$dR = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt.$$

- This last equation contains no *dW* term. That is, it is entirely riskless (**delta neutral**). Thus, given that there is no arbitrage, the rate of return on this portfolio must be equal to the rate of return on any other riskless instrument. Assuming the risk-free rate of return to be *r* we must have over the time period [*t*, *t* + *dt*]

$$r\Pi\, dt = dR = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt.$$

- If we now insert the expression for Π and divide through by *dt* we obtain the **Black–Scholes PDE**:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

- This is the law of evolution of the value of the option. With the assumptions of the Black–Scholes model, this partial differential equation holds whenever *V* is twice differentiable with respect to *S* and once with respect to *t*.

Solution of the Black-Scholes equation

- For a call option the PDE above has the boundary condition

$$V(S,T) = \max(S-K,0).$$

- Introduce the change-of-variables

$$x = \ln(S/K) + (r - \sigma^2/2)(T-t)$$

$$\tau = T - t$$

$$u = Ve^{r(T-t)}.$$

- Then the Black–Scholes PDE becomes a diffusion equation

$$\frac{\partial u}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}.$$

- After some algebra we obtain

$$V(S,t) = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)$$

- where

$$d_1 = \frac{x + \sigma^2\tau}{\sigma\sqrt{\tau}} \qquad d_2 = \frac{x}{\sigma\sqrt{\tau}}$$

- and Φ is the **standard normal cumulative distribution function**.
- The formula for the price of a **put option** follows from this via **put-call parity**

$$V(t) + K \cdot B(t,T) = P(t) + S(t)$$

$$P(S,T) = Ke^{-rT}\Phi(-d_2) - S\Phi(-d_1).$$

◆ **The Greeks** under Black–Scholes:

$$C(t) = V(t) \qquad C(t) = P(t)$$

	What	Calls	Puts
delta	$\frac{\partial C}{\partial S}$	$\Phi(d_1)$	$-\Phi(-d_1) = \Phi(d_1) - 1$
gamma	$\frac{\partial^2 C}{\partial S^2}$	$\frac{\varphi(d_1)}{S\sigma\sqrt{T}}$	
vega	$\frac{\partial C}{\partial \sigma}$	$S\varphi(d_1)\sqrt{T}$	
theta	$-\frac{\partial C}{\partial t}$	$-\frac{S\varphi(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}\Phi(d_2)$	$-\frac{S\varphi(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT}\Phi(-d_2)$
rho	$\frac{\partial C}{\partial r}$	$KTe^{-rT}\Phi(d_2)$	$-KTe^{-rT}\Phi(-d_2)$

Option pricing in FVM. "Risk-neutral approach"

Let the value $V(S_t, \sigma_t, t)$ of an option be the expected terminal value discounted at the risk-free rate

$$V(S_t, \sigma_t, t) = e^{-r(T-t)} \int V(S_T, \sigma_T, T)\, p(S_T | S_t, \sigma_t)\, dS_T$$

$$V(S_T, \sigma_T, T) = \max[0, S - K]$$

Use of the relation between conditional probabilities of related variables,

$$p(S_T | S_t, \sigma_t) = \int p(S_T | S_t, \langle \log \sigma \rangle)\, p(\langle \log \sigma \rangle \,|\, \log \sigma_t)\, d(\langle \log \sigma \rangle)$$

$\langle \log \sigma \rangle$ being the random variable

$$\langle \log \sigma \rangle = \frac{1}{T-t} \int_t^T \log \sigma_s ds$$

Option pricing in FVM. "Risk-neutral approach"

Then

$$V\left(S_t,\sigma_t,t\right)=\int C\left(S_t,e^{\langle\log\sigma\rangle},t\right)p\left(\langle\log\sigma\rangle\,|\,\log\sigma_t\right)d\left(\langle\log\sigma\rangle\right)$$

where

$$C\left(S_t,e^{\langle\log\sigma\rangle},t\right)=\int e^{-r(T-t)}V\left(S_T,\sigma_T,T\right)p\left(S_T|S_t,\langle\log\sigma\rangle\right)dS_T$$

$C\left(S_t,e^{\langle\log\sigma\rangle},t\right)$ = Black-Scholes price for an option with average volatility $e^{\langle\log\sigma\rangle}$

$$C\left(S_t,\sigma,t\right)=S_t\left(a+b\right)N\left(a,b\right)-Ke^{-r(T-t)}\left(a-b\right)N\left(a,-b\right)$$

with

$$\begin{aligned}a&=\frac{1}{\sigma}\left(\frac{\log\frac{S}{K}}{\sqrt{T-t}}+r\sqrt{T-t}\right)\\ b&=\frac{\sigma}{2}\sqrt{T-t}\end{aligned}$$

$$N\left(a,b\right)=\frac{1}{\sqrt{2\pi}}\int_{-1}^{\infty}dye^{-\frac{y^2}{2}(a+b)^2}$$

Option pricing in FVM. "Risk-neutral approach"

To compute the conditional probability $p\left(\langle\log\sigma\rangle \mid \log\sigma_t\right)$

$$\langle\log\sigma\rangle = \log\sigma_t + \frac{1}{T-t}\int_t^T \frac{k}{\delta}ds\int_t^s \left(dB_H(\tau) - dB_H(\tau-\delta)\right)$$

$$E\left\{\langle\log\sigma\rangle \mid \log\sigma_t\right\} = \log\sigma_t$$

$$\begin{aligned}\alpha^2 &= E\left\{\left(\langle\log\sigma\rangle - \log\sigma_t\right)^2\right\} \\ &= \frac{k^2}{\delta^2(T-t)}\left\{\frac{1}{2(T-t)}I_1 + I_2\right\} + k^2\delta^{2H-2}\end{aligned}$$

with

$$I_1 = \frac{2}{(2H+1)(2H+2)}\left\{\begin{array}{c}(T-t+\delta)^{2H+2} + (T-t-\delta)^{2H+2} \\ -2(T-t)^{2H+2} - 2\delta^{2H+2}\end{array}\right\}$$

$$I_2 = \frac{1}{2H+1}\left\{2(T-t)^{2H+1} - (T-t+\delta)^{2H+1} - (T-t-\delta)^{2H+1}\right\}$$

Option pricing in FVM. “Risk-neutral approach”

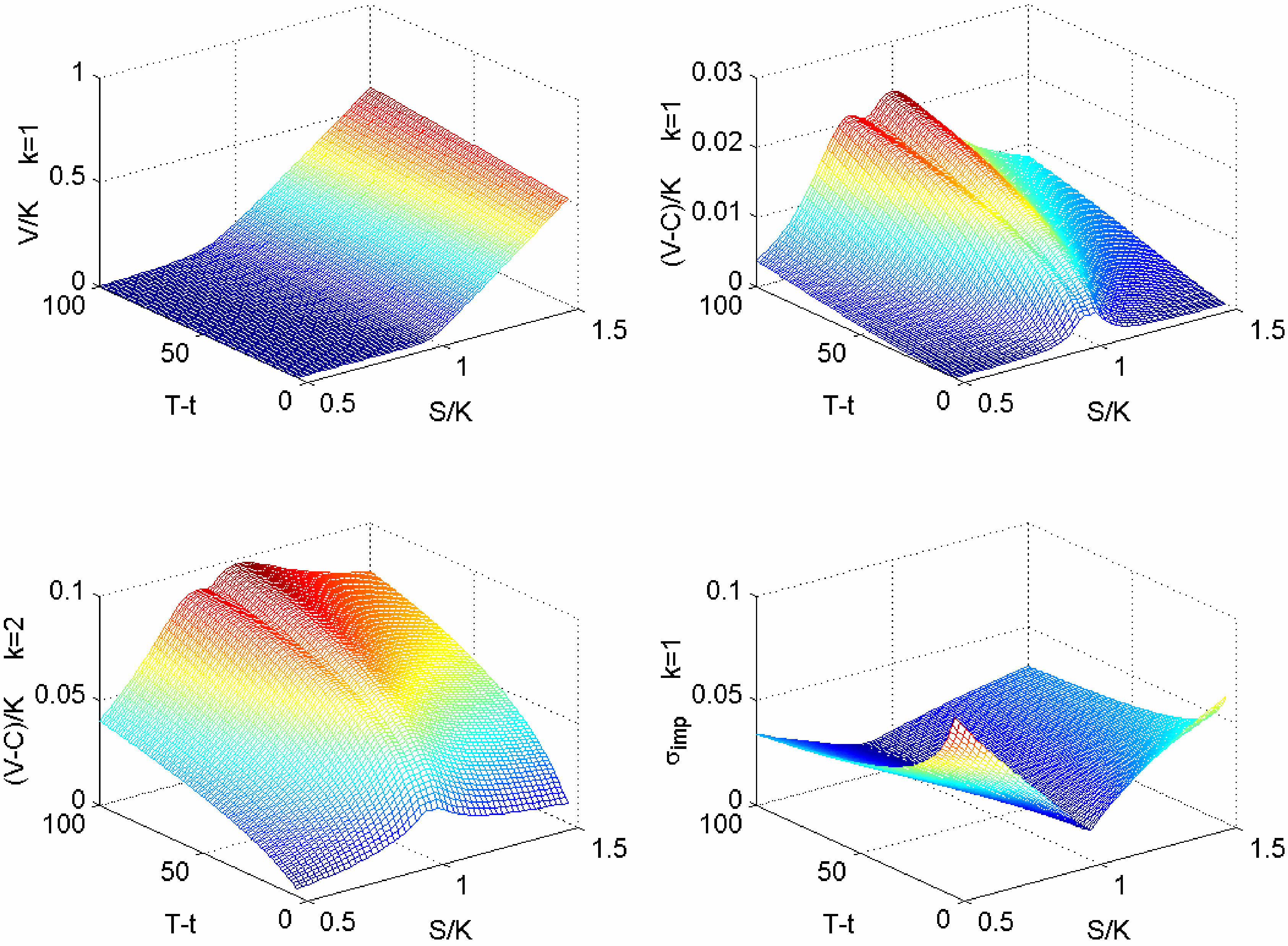
Finally

$$p\left(\langle \log \sigma \rangle \mid \log \sigma_t\right) = \frac{1}{\sqrt{2\pi}\alpha} \exp\left\{ \frac{-\left(\langle \log \sigma \rangle - \log \sigma_t\right)^2}{2\alpha^2} \right\}$$

one obtains

$$\begin{aligned} V\left(S_t, \sigma_t, t\right) &= S_t \left[aM\left(\alpha, a, b\right) + bM\left(\alpha, b, a\right)\right] \\ &\quad -Ke^{-r(T-t)} \left[aM\left(\alpha, a, -b\right) - bM\left(\alpha, -b, a\right)\right] \end{aligned}$$

$$\begin{aligned} M\left(\alpha, a, b\right) &= \frac{1}{2\pi\alpha} \int_{-1}^{\infty} dy \int_{0}^{\infty} dx e^{-\frac{\log^2 x}{2\alpha^2}} e^{-\frac{y^2}{2}\left(ax + \frac{b}{x}\right)^2} \\ &= \frac{1}{4\alpha} \sqrt{\frac{2}{\pi}} \int_0^{\infty} dx \frac{e^{-\frac{\log^2 x}{2\alpha^2}}}{ax + \frac{b}{x}} erfc\left(-\frac{ax}{\sqrt{2}} - \frac{b}{\sqrt{2}x}\right) \end{aligned}$$



1
0.5
0
V/K k=1
100
50
0
T-t
0.5
1
1.5
S/K
0.03
0.02
0.01
0
(V-C)/K k=1
100
50
0
T-t
0.5
1
1.5
S/K
0.1
0.05
0
(V-C)/K k=2
100
50
0
T-t
0.5
1
1.5
S/K
0.1
0.05
0
σimp k=1
100
50
0
T-t
0.5
1
1.5
S/K

The option pricing equation in FVM

Form a portfolio

$$\Pi(t) = V(S,\sigma,t) - \Delta(S,\sigma,t)\, S_t$$

Choosing $\Delta(S,\sigma,t) = \frac{\partial V}{\partial S}$ we obtain

$$\begin{aligned} d\Pi(t) &= \left\{ \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 \right\} dt \\ &\quad + \sigma \frac{\partial V}{\partial \sigma}\frac{k}{\delta}\left(dB_H(t) - dB_H(t-\delta)\right) \\ &\quad + \left(\sigma^2 \frac{\partial^2 V}{\partial \sigma^2} + \sigma \frac{\partial V}{\partial \sigma}\right) H \frac{k^2}{\delta^2}\delta^{2H-1} dt \end{aligned}$$

The option pricing equation in FVM

The fractional Itô formula

if $X_t = \left(X_t^{(1)}, X_t^{(2)}, \cdots, X_t^{(n)}\right)$

with $dX_t^{(i)} = c_i(t,\omega)\, dB_H^{(i)}(t)$, then

$$df(t, X_t) = \frac{\partial f}{\partial t} dt + \sum_i \frac{\partial f}{\partial X^{(i)}} dX_t^{(i)} + \sum_i \frac{\partial^2 f}{\partial X^{(i)2}} c_i(t,\omega)\, D_{i,t}^{\phi}(X_t)$$

$D_{i,t}^{\phi}(X_t)$ is the $\phi-$Malliavin derivative corresponding to the $X_t^{(i)}-$process

$$\begin{aligned} D_{i,f}^{\phi} X_t(\omega_i) &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \{ X\left(\omega_i + \varepsilon \int_0^{\bullet} ds \int_0^{\infty} \phi(s,u) f(u)\, du\right) \\ &\quad - X(\omega_i) \} \\ &= \int_0^{\infty} D_{i,u}^{\phi}(X_t) f(u)\, du \end{aligned}$$

and $\phi(s,u)$ the kernel

$$\phi(s,u) = H_i (2H_i - 1) |s-u|^{2H_i - 2}$$

The option pricing equation in FVM

Dealing with the stochastic term

$\sigma \frac{\partial V}{\partial \sigma} \frac{k}{\delta} \left(dB_H(t) - dB_H(t-\delta) \right)$

Volatility is not a tradable security.

Cannot be eliminated by a portfolio choice.

Instead, equate the deterministic term in $d\Pi(t)$ to

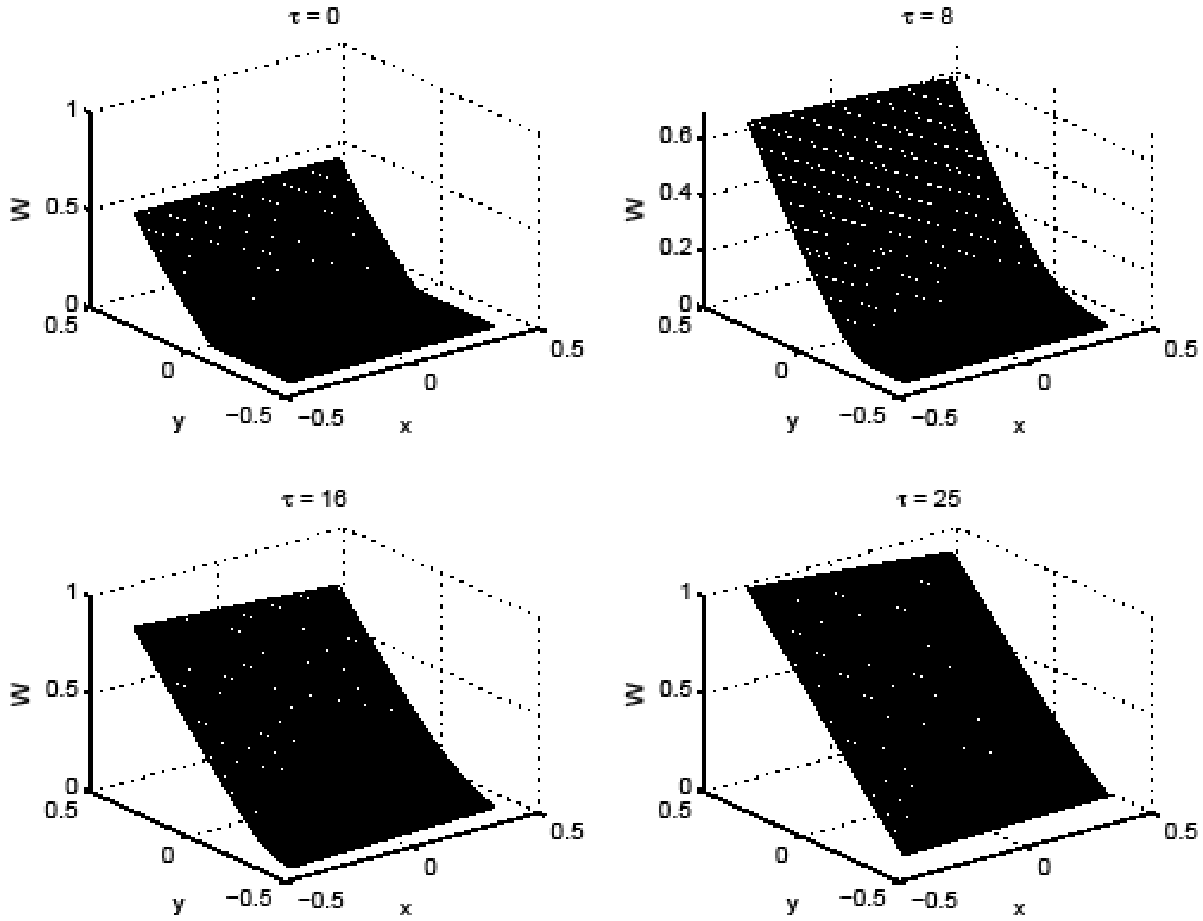
$$\left(r\Pi(t) + \nu \frac{k}{\delta} \sigma \frac{\partial V}{\partial \sigma} \right) dt$$

The second term is a measure of the market price of volatility risk

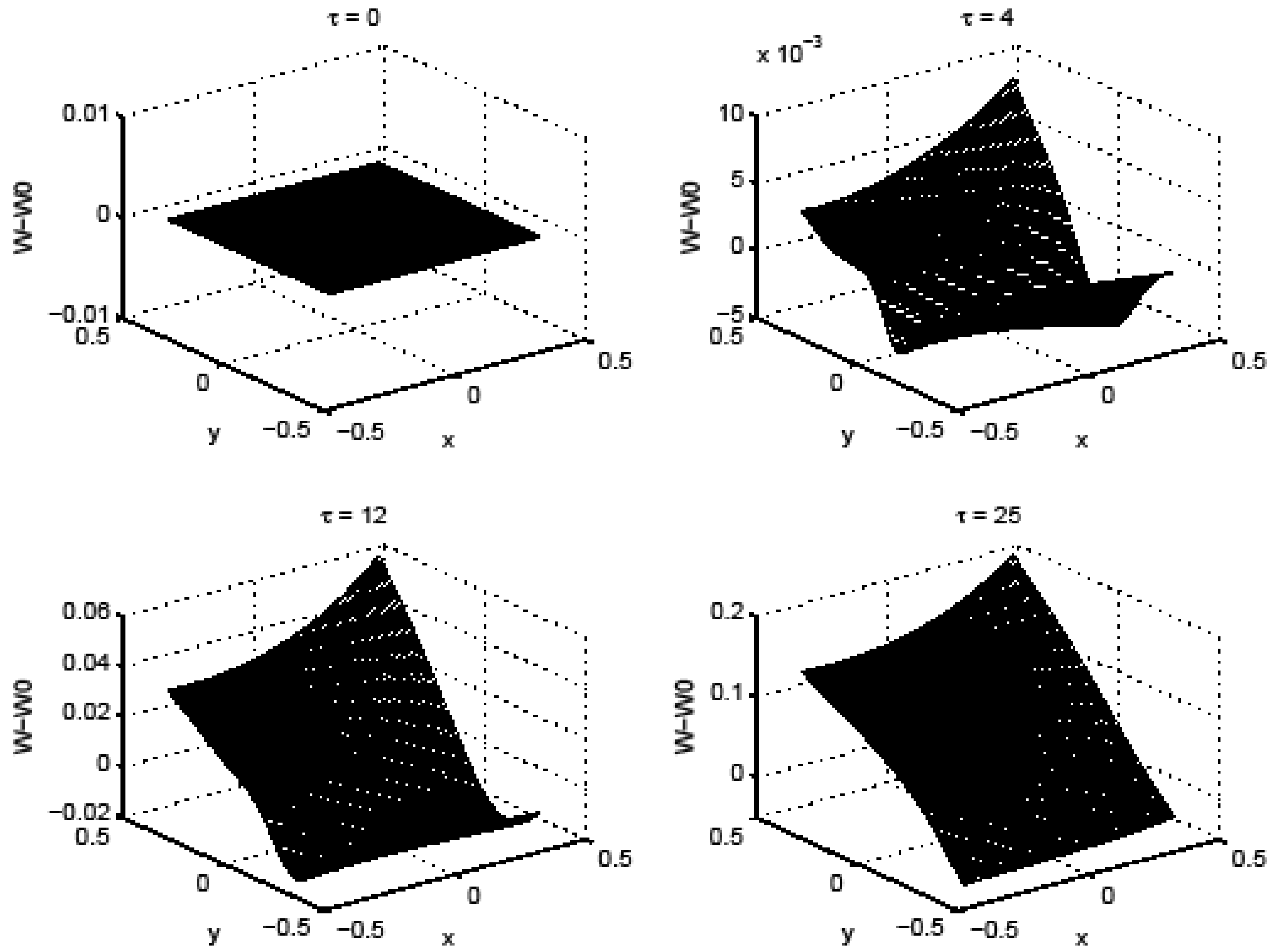
Option pricing equation

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + \frac{k}{\delta}\left(kH\delta^{2H-2} - \nu \right)\sigma\frac{\partial V}{\partial \sigma}$$
$$+ Hk^2\delta^{2H-3}\sigma^2\frac{\partial^2 V}{\partial \sigma^2} = rV$$

Option pricing in FVM. Numerical solutions



Option pricing in FVM. Numerical solutions



The option pricing in FVM. Analytical solution

$x = \log \frac{S}{K}$
$V(t,x,\sigma) = \int\int d\phi d\rho F(\phi,\rho,\sigma) e^{i(\phi t+\rho x)}$
we obtain
$Hk^2\delta^{2H-3}\sigma^2 \frac{\partial^2 F}{\partial\sigma^2} + \frac{k}{\delta}\left(kH\delta^{2H-2} - \nu\right)\sigma\frac{\partial F}{\partial\sigma}$
$+ \left(i\left(\phi + \rho r - \frac{\sigma^2\rho}{2}\right) - \frac{\sigma^2\rho^2}{2} - r\right) F = 0$
Define new constants

$$\begin{aligned} \chi(\rho) &= \frac{\nu}{2Hk\delta^{2H-2}} \\ \xi^2(\rho,\phi) &= \chi^2(\rho) - \frac{r - i(\phi+\rho r)}{Hk^2\delta^{2H-3}} \\ \zeta^2(\rho) &= -\frac{i\rho+\rho^2}{2Hk^2\delta^{2H-3}} \end{aligned}$$

and
$F(\sigma) = \sigma^\chi Z_\xi(\zeta\sigma)$
The solution is

$$V(t,x,\sigma) = \int\int d\rho d\phi e^{i(\phi t+\rho x)} \sigma^{\chi(\rho)} Z_{\xi(\rho,\phi)}(\zeta(\rho)\sigma)$$

$Z_\xi(\zeta\sigma)$ being a Bessel function.
Linear combination $Z_\xi(\zeta\sigma) = c_1 J_\xi(\zeta\sigma) + c_2 N_\xi(\zeta\sigma)$
Coefficients c_1 and c_2 fixed by the boundary conditions,
$V(T,x,\sigma) = \max(x,0)$ (call option)

8. Leverage and the fractional calculus interpretation

- **A nonlinear correlation of returns**

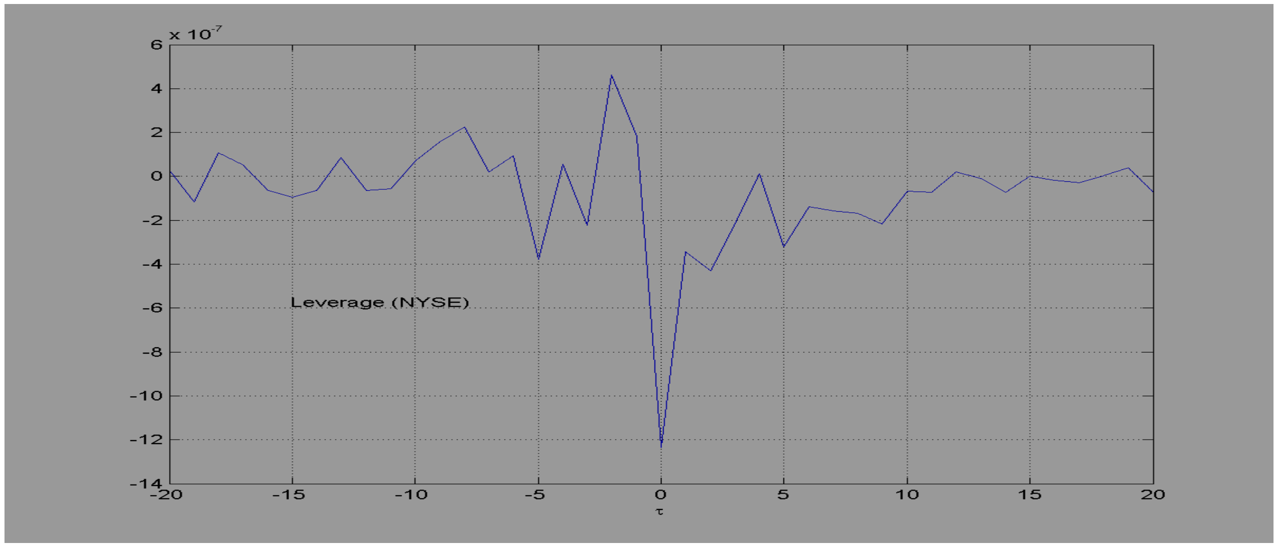
$$L(\tau) = \left\langle \left| r(t+\tau) \right|^2 r(t) \right\rangle - \left\langle \left| r(t+\tau) \right|^2 \right\rangle \langle r(t) \rangle$$

- **The leverage effect**

  For $\tau > 0$ $L(\tau)$ negative and decays to zero

  For $\tau < 0$ $L(\tau)$ is negligible

- **Leverage in the NYSE index (1966-2000 one-day data)**



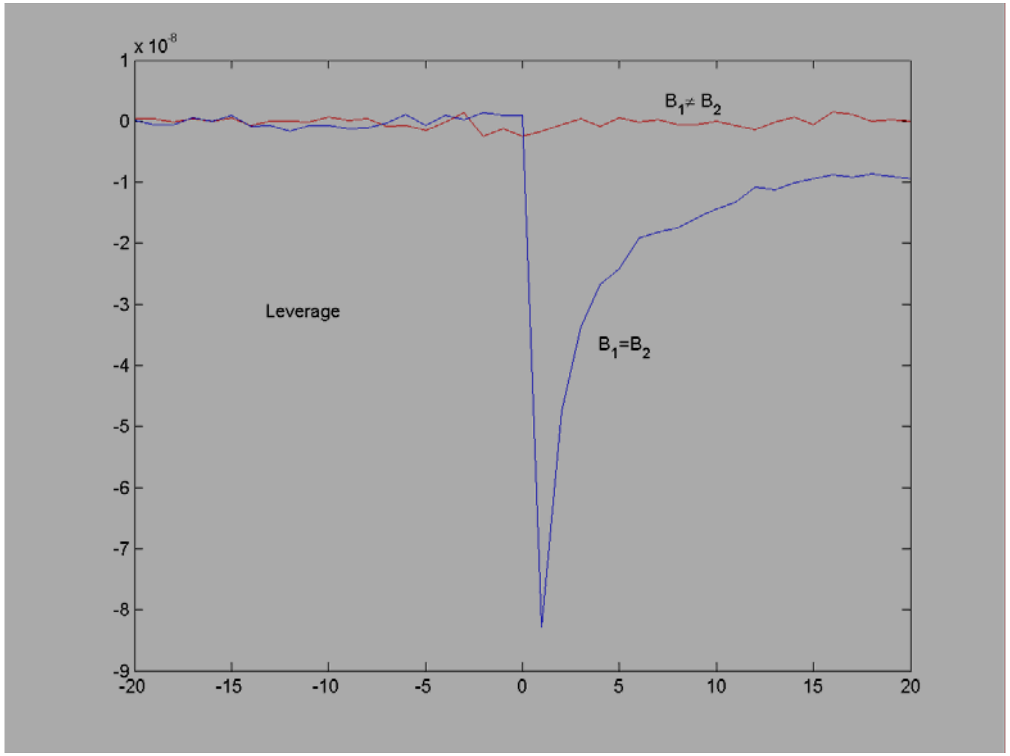
- **Leverage and the fractional volatility model**

- Use an integral representation of fractional Brownian motion

$$B_H(t)=\frac{1}{\Gamma\left(H+\frac{1}{2}\right)}\left\{\int_{-\infty}^{0}\left[(t-u)^{H-1/2}-(-u)^{H-1/2}\right]dB(u)+\int_{0}^{t}(t-u)^{H-1/2}dB(u)\right\}$$

- Then

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma_t S_t dB^{(1)}(t) \\ \log\sigma_t &= \beta + k' \int_{-\infty}^{t} (t-s)^{H-\frac{3}{2}} dB^{(2)}(s) \end{aligned}$$



- **Conclusion**

  No leverage for $B^{(1)} \neq B^{(2)}$

  At least part of the leverage effect described by the fractional volatility model when $B^{(1)} = B^{(2)}$

Fractional Brownian motion and fractional calculus

- Integral representation of fBm

$$B_H(t) = \frac{1}{C(H)} \int_{\mathbb{R}} \left\{ (t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}} \right\} dB(s)$$

$$C(H) = \frac{\Gamma\left(H+\frac{1}{2}\right)}{\left(\Gamma(2H+1)\sin \pi H\right)^{1/2}}$$

- Equality in distribution

- Riemann-Liouville fractional integrals

$$\left(I_-^\alpha \phi\right)(s) = \frac{1}{\Gamma(\alpha)} \int_s^\infty \phi(u)\,(u-s)^{\alpha-1}\,du = \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}} \phi(u)\,(u-s)_+^{\alpha-1}\,du$$

$$\left(I_+^\alpha \phi\right)(s) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^s \phi(u)\,(s-u)^{\alpha-1}\,du = \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}} \phi(u)\,(s-u)_+^{\alpha-1}\,du$$

- Fractional integration by parts

$$\int_{\mathbb{R}} \phi(s)\left(I_+^\alpha \psi\right)(s)\,ds = \int_{\mathbb{R}} \left(I_-^\alpha \phi\right)(s)\,\psi(s)\,ds$$

- From

$$\left(I_{-}^{\alpha}\mathbf{1}_{[a,b)}\right)(s) = \frac{1}{\Gamma(\alpha+1)}\left\{(b-s)_{+}^{\alpha} - (a-s)_{+}^{\alpha}\right\}$$

$$B_H(t) = \frac{\Gamma\left(H+\frac{1}{2}\right)}{C(H)}\int_{\mathbb{R}}\left(I_{-}^{H-\frac{1}{2}}\mathbf{1}_{[o,t)}\right)(s)\,dB(s)$$

- In the framework of WNA

$$B(t) = \int^{t} \dot{B}(s)\,ds$$

$$B_H(t) = \frac{\Gamma\left(H+\frac{1}{2}\right)}{C(H)}\int_{0}^{t}\left(I_{+}^{H-\frac{1}{2}}\dot{B}\right)(s)\,ds$$

The fractional calculus interpretation of the FVM

- **The fractional volatility model**

  $dS_t = \mu \, S_t \, dt + \sigma_t \, S_t \, dB_t$

  $\log \sigma_t = \beta + (k/\delta) \, (B^H_t - B^H_{t-\delta})$

- **becomes**

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma_t S_t dB^{(1)}(t) \\ \log \sigma_t &= \beta + \frac{k}{\delta} \frac{\Gamma\left(H+\frac{1}{2}\right)}{C(H)} \int_{\mathbb{R}} \left(I_-^{H-\frac{1}{2}} \mathbf{1}_{[t-\delta,t)} \right)(s) \, dB^{(2)}(s) \end{aligned}$$

- In the small δ limit

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sigma_t S_t \overset{\bullet}{B}^{(1)} dt \\
\log \sigma_t &= \beta + \frac{\Gamma\left(H+\frac{1}{2}\right)}{C(H)} \left(I_+^{H-\frac{1}{2}} \overset{\bullet}{B}^{(2)} \right)(t)
\end{aligned}
$$

- The random parts of both log-price and log-volatility are driven by integrals of white noise, a regular integral for log-price and a fractional one for log-volatility

Conclusions

- (i) The fractional volatility model provides a fairly accurate mathematical description of the bulk market data
- (ii) A small modification of the original model, identifying the random generator of the log-price process and the integrator of the volatility process, also describes, at least, part of the leverage effect
- (iii) The model may be formulated in terms of fractional stochastic integration with the volatility memory represented by a fractional integration of white noise

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