Agent-based models: The market and other stories
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Economics and agent-based models

- Economics is an human science. Therefore it studies the effect of human actions.
- Mathematical Finance stochastic models are simply parametrizations of time series which could as well be of the fluctuations of the wind on Mars.
- To find the actions that might lead to such time series one has to desaggregate the process, that is, to find a set of elementary actions leading to the aggregate effect of the price fluctuations.
- Two alternatives:
  1. An exhaustive study of all the elementary actions in the market (investors psychology, trader room dynamics, financial institutions, etc.)
  2. To use surrogate models (agent-based models) to isolate the dominant effects.
2. Agent-based models and fractional volatility

- The fractional volatility model:

\[ dS_t = \mu S_t \, dt + \sigma_t S_t \, dB_t \]

\[ \log \sigma_t = \beta + \left( \frac{k}{\delta} \right) \left( B^H_t - B^H_{t-\delta} \right) \]

- Returns driven by Brownian motion with stochastic volatility \( \sigma_t \) driven by fractional noise
2 - Two agent-based models

2.1 – A model with evolutive agent strategies

- The collective variable \( z_t = \log(p_t) \)
- Investor’s i payoff

\[
\Delta_{t}^{(i)} = \left( m_t^{(i)} + p_t \times s_t^{(i)} \right) - \left( m_0^{(i)} + p_0 \times s_0^{(i)} \right)
\]

\( \omega_t \) = sum of buying and selling orders
- Dynamics of the collective variable

\[
z_{t+1} = f\left( z_t, \omega_t \right) + \eta_t
\]
2.1 - A model with evolutive agent strategies

- Strategies and market impact

- The loglinear law

\[ z_{t+1} - z_t = \frac{\omega_t}{\lambda} + \eta_t \]

\[ p(p(p_0, \omega^{(1)}), \omega^{(2)}) = p(p_0, \omega^{(1)} + \omega^{(2)}) \]
2.1 - A model with evolutive agent strategies

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- Refined to

\[ z_{t+1} - z_t = \frac{\omega_t}{\lambda_0 + \lambda_1 |\omega_t|^\alpha} + \eta_t \]
2.1 - A model with evolutive agent strategies

- **Misprice and trend**

  \[ \xi_t - z_t = \log(v_t) - \log(p_t) \]
  \[ z_t - z_{t-1} = \log(p_t) - \log(p_{t-1}) \]

- **Strategies**

  \[ f_1(x) = \theta(x) \]
  \[ f_2(x) = \frac{1}{1 + \exp(-\beta x)} \]

  \[ \gamma_t = \begin{pmatrix}
  f(\xi_t - z_t) f(z_t - z_{t-1}) \\
  f(\xi_t - z_t) (1 - f(z_t - z_{t-1})) \\
  (1 - f(\xi_t - z_t)) f(z_t - z_{t-1}) \\
  (1 - f(\xi_t - z_t)) (1 - f(z_t - z_{t-1}))
  \end{pmatrix} \]

- **Labelling**

  \[ n^{(i)} = \sum_{k=0}^{3} 3^k \left( \alpha_{k}^{(i)} + 1 \right) \]
2.1 - A model with evolutive agent strategies

- Measured quantities

\[
\sigma_t^2 = \frac{1}{|T_0 - T_1|} \text{var}(\log p_t), \\
\sum_{n=0}^{t/\delta} \log \sigma(n\delta) = \beta t + R_\sigma(t) \\
|R_\sigma(t + \Delta) - R_\sigma(t)|
\]
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\]

\[
|R_\sigma(t + \Delta) - R_\sigma(t)|
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2.2 - The dynamics of a limit order book with random agents

- Asks and bids of size $n$ arriving at random in a window $[p-w, p+w]$ around the current price $p$.
- Random buy and sell orders of size one, filled up by the closest limit order.
- An example, $n=2$, $2w+1=21$, $dp=0.1$
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2. Agent-based models and fractional volatility: Conclusions

- The market statistical properties in “business-as-usual” days seem to depend more on the nature of the price fixing institutions (the double auction process) than on the traders strategies.
- Traders strategies and psychology might however be important during market crisis and bubbles
- Two different market phases: (a) an agent dominated phase and (b) a financial institutions dominated phase
3. Structural characterization of the dynamics in agent-based models

Ergodic tools. Exponents and entropies

- Invariant measures and ergodic parameters
  \[ I_F(\mu) = \lim_{T \to \infty} \frac{1}{T} \sum_{n=1}^{T} F(f^n x_0) \]

- Lyapunov and conditional exponents
  
  From the k x k and (m-k)x(m-k) blocks of the Jacobian, obtain the conditional exponents as the eigenvalues of the limits:
  \[
  \lim_{n \to \infty} \left( D_{k} f^{n*}(x) D_{k} f^{n}(x) \right)^{\frac{1}{2n}} \\
  \lim_{n \to \infty} \left( D_{m-k} f^{n*}(x) D_{m-k} f^{n}(x) \right)^{\frac{1}{2n}}
  \]

  or
  \[
  \lim_{n \to \infty} \frac{1}{n} \log \| D_k f^n(x) u \| = \xi^{(k)}_i \\
  0 \neq u \in E^{i}_{x}/E^{i+1}_{x} \\
  E^{i}_{x} \text{ is the subspace spanned by the eigenstates corresponding to eigenvalues } \leq \exp(\xi^{(k)}_i)
  \]
Existence of the conditional exponents

- First proposed by Pecora and Carroll to study the phenomenology of synchronization of chaotic systems
  PRL 64 (1990) 821 ; PRA 44 (1991) 2374

- **Theor. 1** The existence of the conditional exponents is guaranteed under the same conditions as for the Lyapunov exponents

Existence of a measurable map from the dynamical space $V$ to $m \times m$ matrices

$$T : V \rightarrow M_m$$

and

$$\int \mu(dx) \log^+ \|T(x)\| < \infty$$

The proof follows the same steps as for the Oseledec’s theorem
PLA 248 (1998) 167

- Regular functionals of the exponents will also be well-defined ergodic parameters
Structures and self-organization

- **Structure index**

  \[ S = \frac{1}{N} \sum_{i=1}^{N+} \left( \frac{\lambda_0}{\lambda_i} - 1 \right) \]

  diverges whenever a Lyapunov exponent approaches zero from above (points where long time correlations develop)

- **Self-organization**  

  (partitions \( \Sigma_k = \mathbb{R}^k \times \mathbb{R}^{m-k} \))

\[
I_\Sigma(\mu) = \sum_{k=1}^{N} \left\{ h_k(\mu) + h_{m-k}(\mu) - h(\mu) \right\}
\]

\[
h_k(\mu) = \sum_{\xi_i^{(k)} > 0} \xi_i^{(k)}; h_{m-k}(\mu) = \sum_{\xi_i^{(m-k)} > 0} \xi_i^{(m-k)}; h(\mu) = \sum_{\lambda_i > 0} \lambda_i
\]
Self-organization concerns the dynamical relation of the whole to its parts. Therefore, $I_\Sigma(\mu)$ is a measure of dynamical self-organization.

It is a measure of apparent dynamical freedom (or apparent rate of information production), that each agent may infer from the local dynamics.

Self-organization occurs when local information is very different from global behavior.

These global parameters, besides providing information on structure formation and self-organization, may also be used to characterize the topology of the interactions (network connectivity).
4 - Some agent models:

- 4.1 - A fully coupled system

\[ x_i(t+1) = (1-c) f(x_i(t)) + \frac{c}{N-1} \sum_{k \neq i} f(x_k(t)) \]

\[ f(x) = 2x \pmod{1} \]

\[ c = 0.495 \quad \text{and} \quad c = 0.51 \]
Fully coupled system. Structure and self-organization index
Nearest-neighbor coupling

\[ x_i(t+1) = (1-c) f(x_i(t)) + (c/2) ( f(x_{i+1}(t) + f(x_{i-1}(t)) ) \]
4.2 - Synchronization and beyond

- Synchronous flashing of fireflies, cells, fads, ....
4 - Synchronization and beyond

- Synchronization
  (Classical mathematical example: the Kuramoto model)
  A similar, discrete-time oscillators model:

\[
x_i(t + 1) = x_i(t) + \omega_i + \frac{k}{N-1} \sum_{j=1}^{N} f_\alpha(x_j - x_i)
\]

\[
p(\omega) = \frac{\gamma}{\pi \left[ \gamma^2 + (\omega - \omega_0)^2 \right]}
\]

\[
f_\alpha(x_j - x_i) = \alpha (x_j - x_i) \mod 1
\]

- Order parameter

\[
r(t) = \frac{1}{N} \sum_{j=1}^{N} e^{i2\pi x_j(t)}
\]
The Lyapunov spectrum controls the dynamical self-organization of the system.

In this case

\[ \lambda_1 = 0 \text{ and } \lambda_i = \log(1 - \alpha \lambda_k (N/N-1)) \text{ (N-1) times} \]

N-1 contracting directions for \( k \neq 0 \)

“One-dimensional” system!

\[ \Rightarrow \text{ strong dynamical correlations even before synchronization} \]
4.3 Network structure and dynamics. The small world phase

\[ \beta = 0 \quad \text{Increasing randomness} \quad \beta = 1 \]

![Graph showing scaled length and clustering against \( \beta \)]
Define a dynamical system on the network nodes

- $x_i(t+1) = \sum_{k=1}^{N} W_{ik} f(x_k(t))$
  
  $f(x) = \alpha x \pmod{1}$ if $i \neq k$ and $k \in n_v(i)$

- $W_{ik} = \begin{cases} 
1 - \frac{n_v(i)}{2c} & \text{if } i = k \\
\frac{c}{2c} & \text{if } i \neq k \text{ and } k \in n_v(i) \\
0 & \text{otherwise}
\end{cases}$

- $D_\beta = -\sum_{\lambda_i < 0} \lambda_i$

- $D_\beta = cN(\beta - \beta_{c1})^\eta$ \quad $\beta_{c1} < 10^{-5}$ \quad $\eta = 1.01 \pm 0.06$
Define a dynamical system on the network nodes

\[ x_i(t+1) = \sum_{k=1}^{N} W_{ik} f(x_k(t)) \]

\[ f(x) = \alpha x \pmod 1 \]

\[ W_{ik} = \begin{cases} 
1 - \frac{n_v(i)}{2v} & \text{if } i = k \\
\frac{c}{2v} & \text{if } i \neq k \text{ and } k \in n_v(i) \\
0 & \text{otherwise}
\end{cases} \]

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\[ C_\beta = \begin{bmatrix} h_0^* - h_0^* \\ h_0^* - h_\beta \\ \end{bmatrix}; \quad h_\beta = \sum_{i=1}^{N} \left( \frac{1}{d_i} \sum_{\lambda_\beta>0} \lambda_\beta(j) \right); \quad h_\beta = \sum_{\lambda_\beta>0} \lambda_\beta(j) \]

\[ \beta_{c2} = 0.04 \quad C_\beta \sim |\beta - \beta_{c2}|^{-\delta} \quad \delta_1 = 1.14 \quad \delta_2 = 0.93 \]
4.4 - Self-organized criticality (SOC)

- A qualitative definition:
  SOC = *mechanism of slow energy accumulation and fast energy redistribution (avalanches) driving the system towards a critical state, where the distribution of avalanche sizes is a power law obtained without fine tuning, that is, there is no tunable parameter in the model.*

- Power law → no natural scale, excitations at all scales
- No tunable parameter ≠ usual critical points in phase transitions
- A critical point as an attractor?
- Ubiquity of SOC (geophysics, cosmology, evolutionary biology, ecology, economics, sociology, solar physics, ...)

- Objective: Characterize SOC by ergodic parameters
Real world manifestations

- The Gutenberg-Richter law
  Data from 1977-1995
Real world manifestations

- Electron temperature fluctuations in a magnetically confined plasma (ECE diagnostic)
  (Politzer, PRL 84 (2000) 1192)
Real world manifestations

- Avalanche "in living neurons"
- Magnetoencephalography data compared with models
  (de Arcangelis et al. PRL 96 (2006) 028107)
Real world manifestations

- Distribution of lengths of open spaces in urban environments
  (Carvalho and Penn, Physica A 332 (2004) 539)
Real world manifestations

(a) and (b) Plots showing the relationship between rank and line length for various cities.

(c) and (d) Diagrams illustrating the Levy Stable Region for different cities.

Legend:
- Bristol
- Hereford
- London
- Manchester
- Norwich
- Nottingham
- Winchester
- Wolverhampton
- York
- Athens
- Nicosia
- Dhaka
- Hong Kong
- Istanbul
- Milton Keynes
- Eindhoven
- Barcelona
- Denver
- Pensacola
- Seattle
- Hamedan
- Kerman
- Kermanshah
- Qazvin
- Shiraz
- Ahmenabad
- Bangkok
- Tokyo
- Amsterdam
- Hague
- Baltimore
- Chicago
- Las Vegas
- New Orleans
- Santiago
- Semnan

The plots show a power-law distribution, indicating a levy stable behavior for the line length as a function of rank.
Toy models

- Sand piles on the computer and on the lab

- However, the emergence of scaling laws on lab sand piles depend on grain size and shape
Toy models

- Springer – slider block mode
  (friction of the blocks on the fixed plate)
A mathematical model: Bak-Sneppen (BS)

- *Toy model for the evolution of species*

After a short transitory period the system self-organizes with most species having fitness above 0.667

- Avalanches show power-law behavior
Two features of most models and a mathematical result

- **Most SOC models display**:  
  - Instable behavior of the local dynamics  
  - Extremal dynamics

  If, in a N-agent model:  
  - The single-agent dynamics has positive Lyapunov exponents and  
  - The global dynamics is extremal with finite range  

  then, in the $N \to \infty$, the Lyapunov spectrum converges to $0^+$

- In the $T \to \infty$ limit, used to compute the Lyapunov spectrum, the tangent maps have only a nontrivial finite size block during an average time of order $(2r+1)T/N$

- With the Lyapunov spectrum converging to $0^+$ there are no dynamical scales. Thus, in the $N \to \infty$, the system is “tuned” to SOC

Head’s critique of parameter-independence in SOC

- “… SOC models do in fact require parameter tuning, but they had been defined in such a way that the tuning had been carried out implicitly.”

- To make his point, he modified the Bak-Sneppen model defining the probability of activation of an element by

\[
P_i = \frac{e^{-E_i / T}}{\sum_{k=1}^{N} e^{-E_k / T}}
\]

- Then he finds that it is only in the $T \to 0$ limit that power laws are obtained, that is, BS is a zero temperature limit of his model.
Head’s critique of parameter-independence in SOC
A deterministic version of B-S-Head’s model

\[ x_i(t + 1) = \Gamma_i(\tilde{x}) x_i(t) + \left(1 - \Gamma_i(\tilde{x})\right) f(x_i(t)) \quad (1) \]

\( \tilde{x} = \{x_i\} \) is the vector of agent coordinates

\[ f(x_i) = kx \mod 1 \]

\( k = 2, 3, \ldots \).

\( \Gamma_i(\tilde{x}) \) is nearly zero if \( i \) corresponds to the minimum \( x \) value or to one of its \( 2n_v \) neighbors and is nearly one otherwise.

\[ \Gamma_i^{(1)}(\tilde{x}) = \prod_{j=i-n_v}^{j=i+n_v} \left(1 - \prod_{k \neq j} \left(1 + e^{-\alpha(x_k-x_j)}\right)^{-1}\right) \quad (2) \]

for large \( \alpha \), satisfies the above conditions.

\[ \Gamma_i^{(2)}(\tilde{x}) = \prod_{j=i-n_v}^{j=i+n_v} \left(1 - \frac{e^{-x_i/T}}{\sum_{j=1}^{N} e^{-x_j/T}}\right) \quad (3) \]

a similar behavior for \( T \to 0^+ \).
A deterministic version of B-S-Head’s model

- The absence of power laws for non-zero $T$ is indeed related to the Lyapunov spectrum
A deterministic version of B-S-Head’s model

- Notice that at T=0 the Lyapunov spectrum does not reach zero because N=100.

- All this is expected from the proposition. However the deterministic model also allows to study a few other features:

  - What is the measure of the SOC state?
  - Is the SOC state an attractor?
  - Avalanches are return times to the SOC state. What is the prefactor in the return times (avalanches) distribution in the T=0 limit?
A deterministic version of B-S-Head’s model

Kac’s lemma (for an ergodic invariant measure $\mu$)
Average return time to a set $A$ of measure $\mu (A)$ is
$1 / \mu (A)$.
For a scaling law $\rho (\tau) \sim 1 / \tau^\alpha$, $\alpha \leq 2$ implies $\mu (A) = 0$.

The distance process $d$
$$d = \sum_i \max (b - x_i, 0)$$ (4)
A deterministic version of B-S-Head’s model

- The SOC state has zero measure, but its finite-dimensional projections have full measure.
- It is not an attractor, nor a repeller (not invariant)
- “Ghost weak repeller”
- The invariant measure is like a cloud around the SOC state.
Beyond the classical ergodic parameters

- Lyapunov and conditional exponents and derived quantities depend on the actual (or expected) average rates of expansion
- **Fluctuations** of the expansion rates along the trajectories

**Generalized Lyapunov exponents**

\[
\Lambda(\beta) = \lim_{N \to \infty} \frac{1}{\beta N} \log \int d\mu(x_0) \exp \left[ \beta \sum_{n=0}^{N-1} \log |f'(x_n)| \right]
\]

**Dynamical Rényi entropies**

\[
K(\alpha) = \lim_{N \to \infty} \frac{1}{1-\alpha} \frac{1}{N} \log \sum_{i_0 \ldots i_{N-1}} (p(i_0 \ldots i_{N-1}))^\alpha \quad \Lambda(\beta) = K(1-\beta)
\]

**Cumulants of the Lyapunov spectrum**

\[
K(\alpha) \equiv \sum_{s=1}^{\infty} \frac{c}{s} (1-\alpha)^s - 1
\]

**Traces of Hessian powers**

\[
\frac{1}{2} H_N = \delta_{\alpha,\beta} \delta_{j,k} (1-\delta_{k,N}) \delta_{j,k-1} \left( \frac{\partial^\alpha(x_k)}{\partial \alpha_k} - (1-\delta_{j,N}) \delta_{j,k-1} \left( \frac{\partial^\beta(x_j)}{\partial \beta_j} + (1-\delta_{j,N}) \delta_{j,k} \left( \frac{\partial^\gamma(x_j)}{\partial \gamma_j} + \frac{\partial^\gamma(x_j)}{\partial \gamma_j} \right) \right) \right)
\]
References